Mini-course

Packing & covering

Part I : Introduction

Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property

Recall the key definition:

A 0,1 matrix M is **Mengerian** if

$$Mx \geq 1, x \geq 0 \quad \text{TDI}$$

$\forall w \in \mathbb{Z}_+^n$:

$$\min\{w^T x : Mx \geq 1, x \geq 0, x \text{ integer}\} = \max\{1^T y : M^T y \leq w, y \geq 0, y \text{ integer}\}.$$

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We say $\mathcal{F}$ is Mengerian if $M(\mathcal{F})$ Mengerian

Examples of Mengerian matrices/clutters

Odd cycles

Odd cycles

Pr: Let $\mathcal{F}$ be clutter of odd cycles of signed graph (G, Σ). Then

1.

2.

3.

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1. $\mathcal{F}$ not Mengerian
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① $\mathcal{F}$ not Mengerian $\Longleftrightarrow$

② $\mathcal{F}$ has Q_6 minor $\Longleftrightarrow$

$$M(Q_6) = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & & & \\ 1 & & & 1 & 1 & \\ & 1 & & 1 & & 1 \\ & & 1 & & 1 & 1 \end{array}\right] \end{array}$$

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odd-K_4

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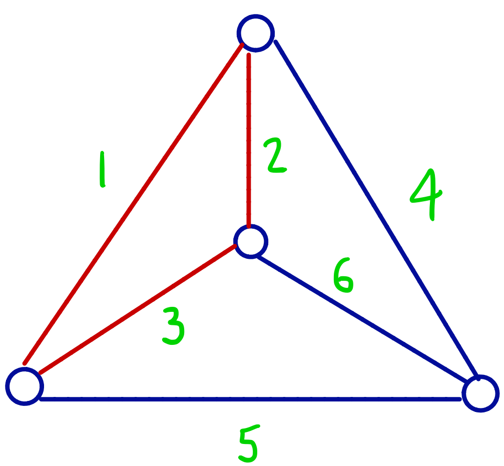
③ (G, Σ) has (K_4, EK_4) minor.



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Directed cuts / directed joins

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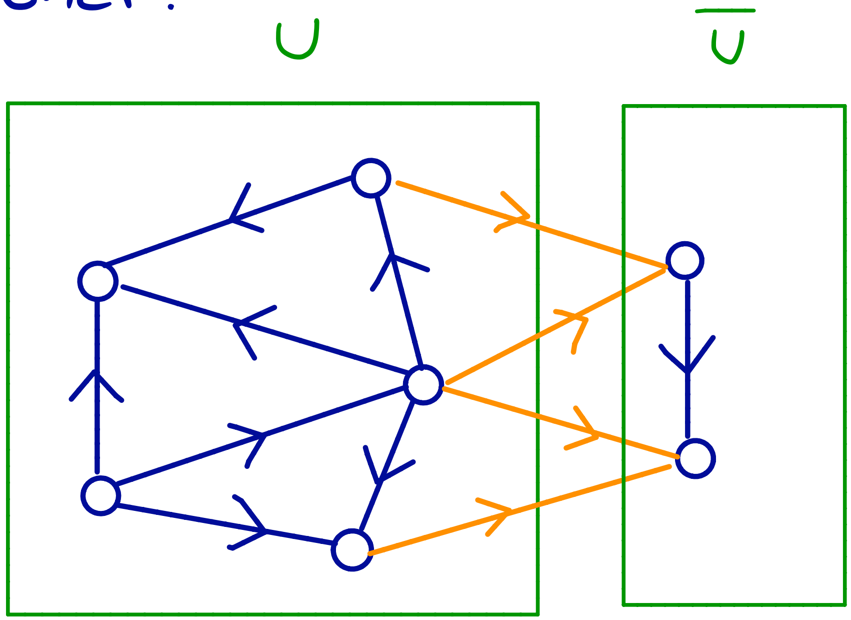

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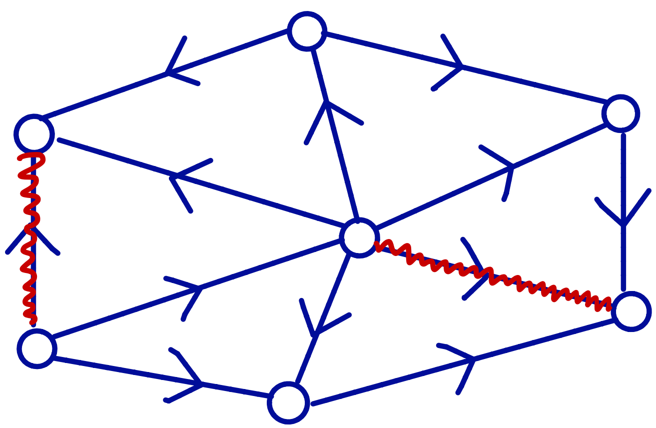

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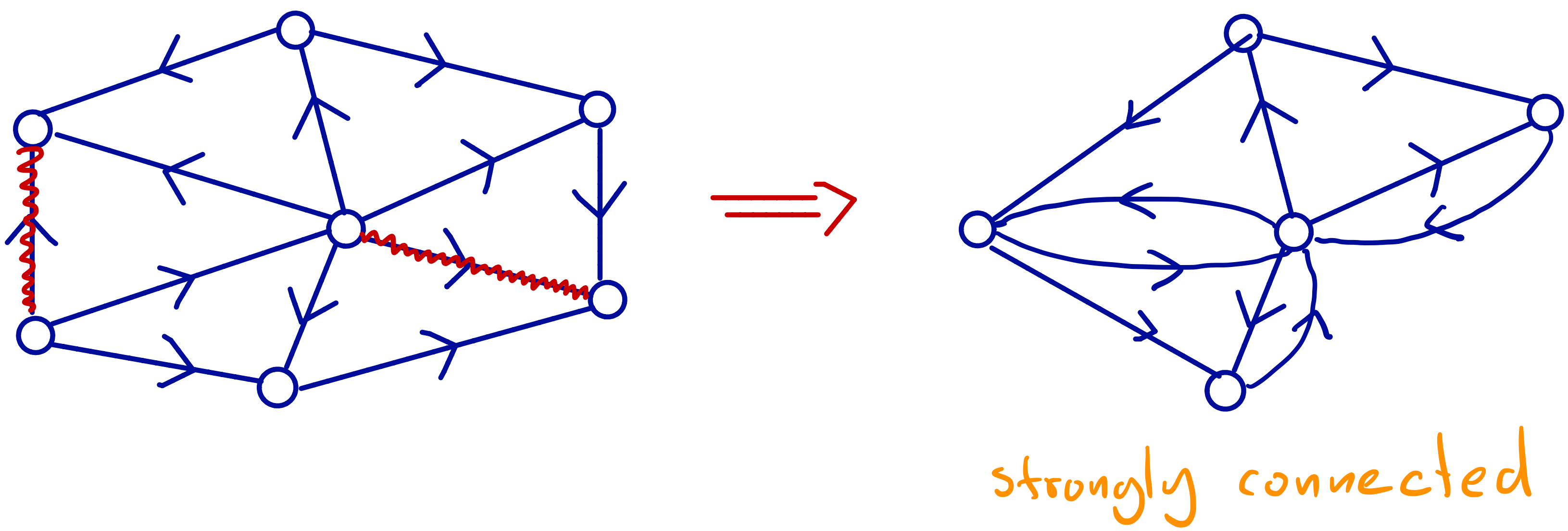

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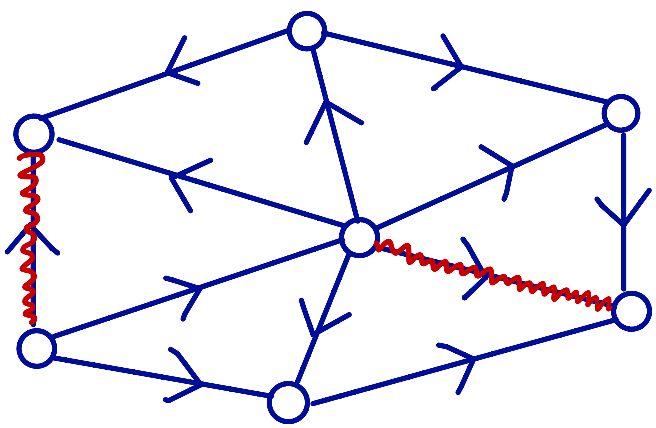
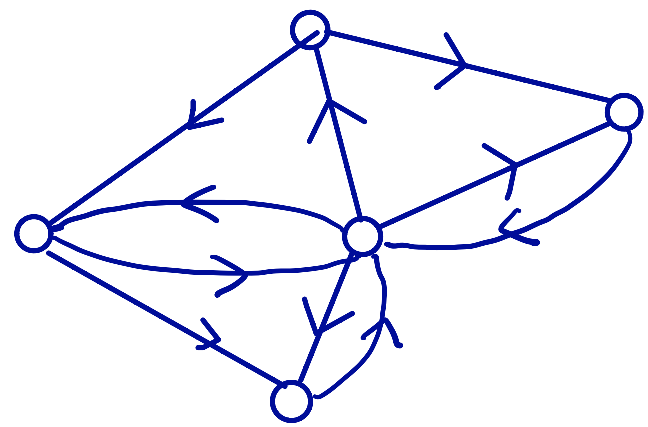


$\Longrightarrow$



strongly connected

Pr: dicuts & dijoins are blockers

Th [Lucchesi-Younger]

The clutter of dicuts is Mengerian

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Thus clutter of dicuts packs :

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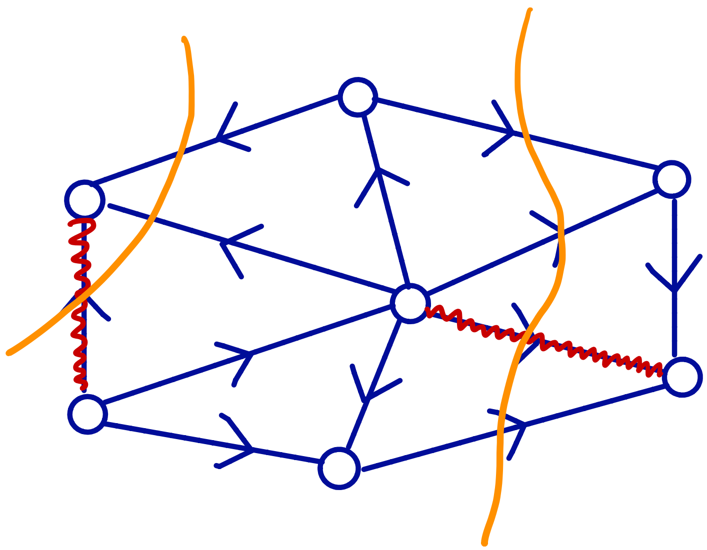

dicuts

dijoin

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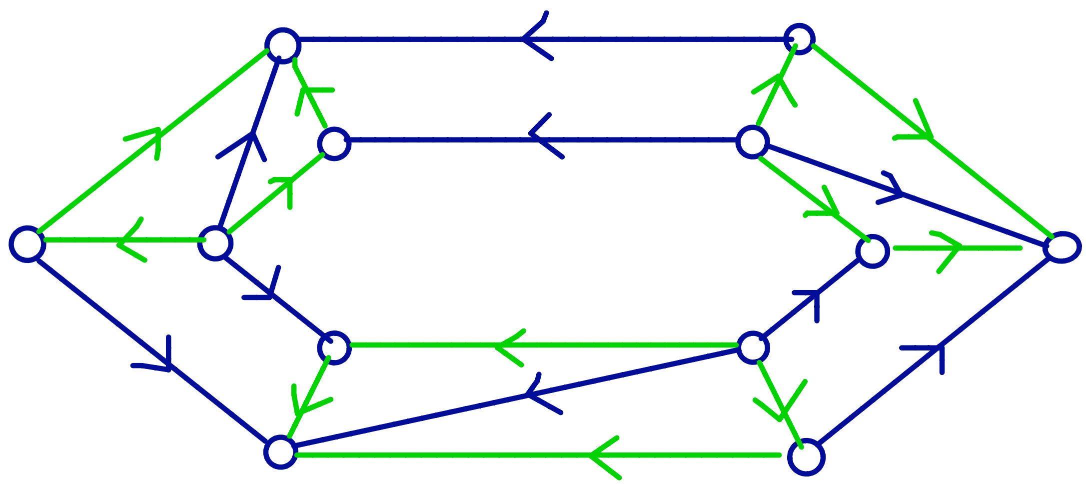

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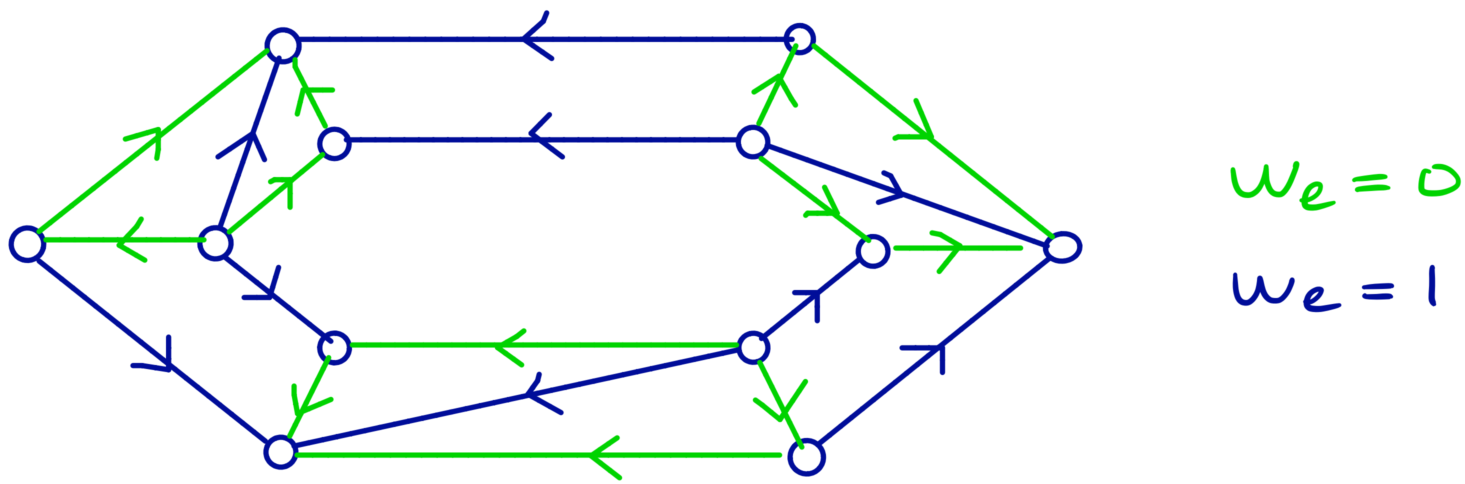

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$M = M(\mathcal{F})$, $\mathcal{F}$ dijoins

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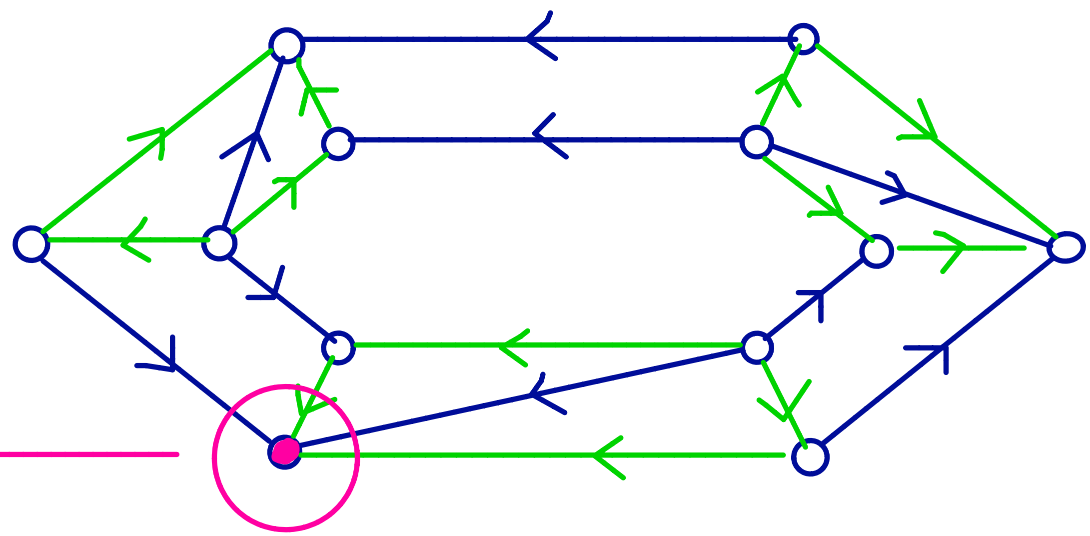

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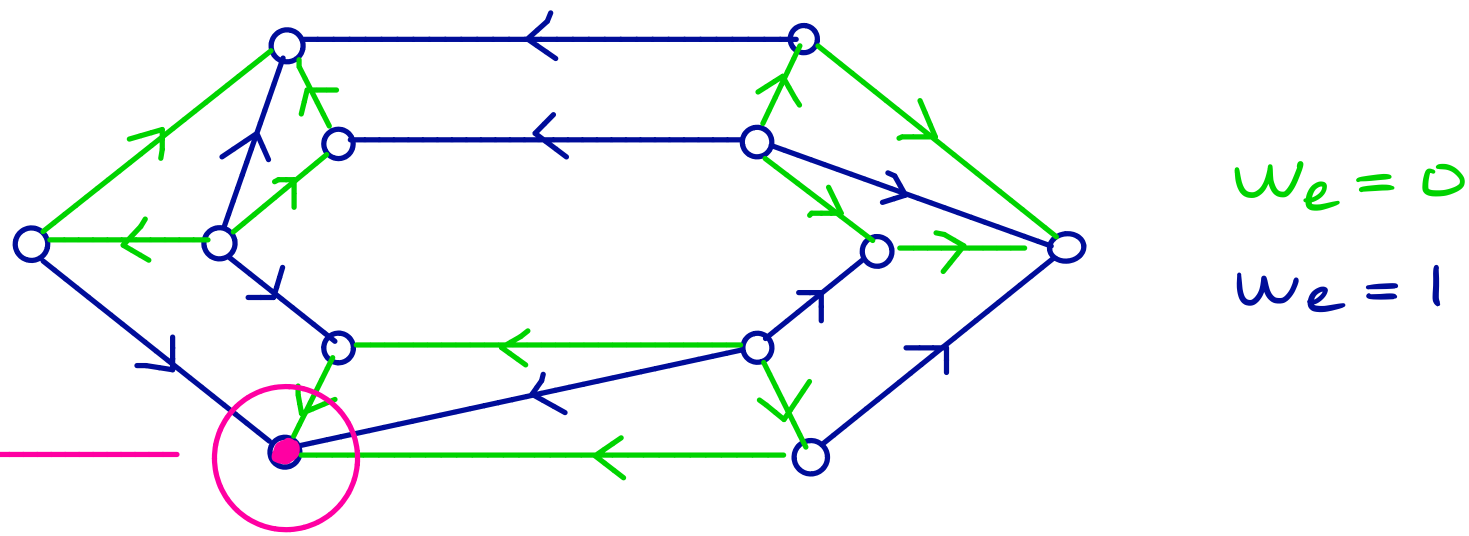

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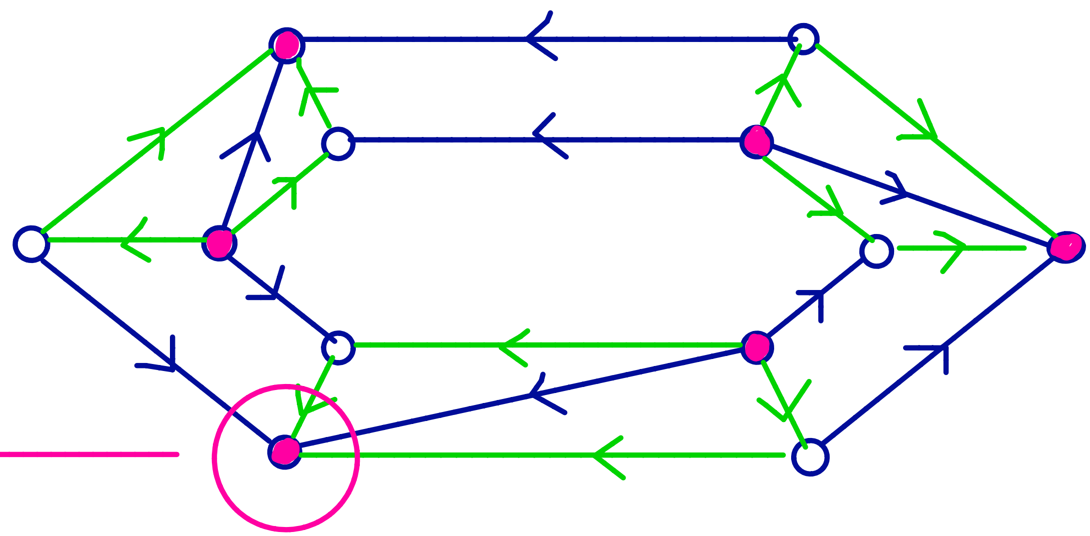

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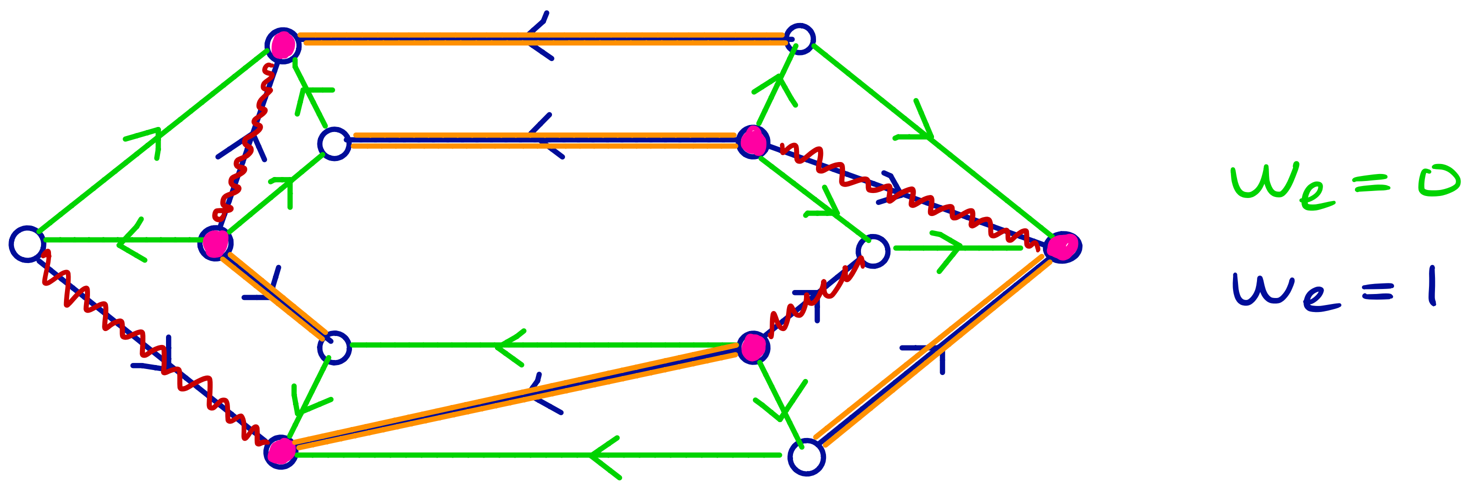

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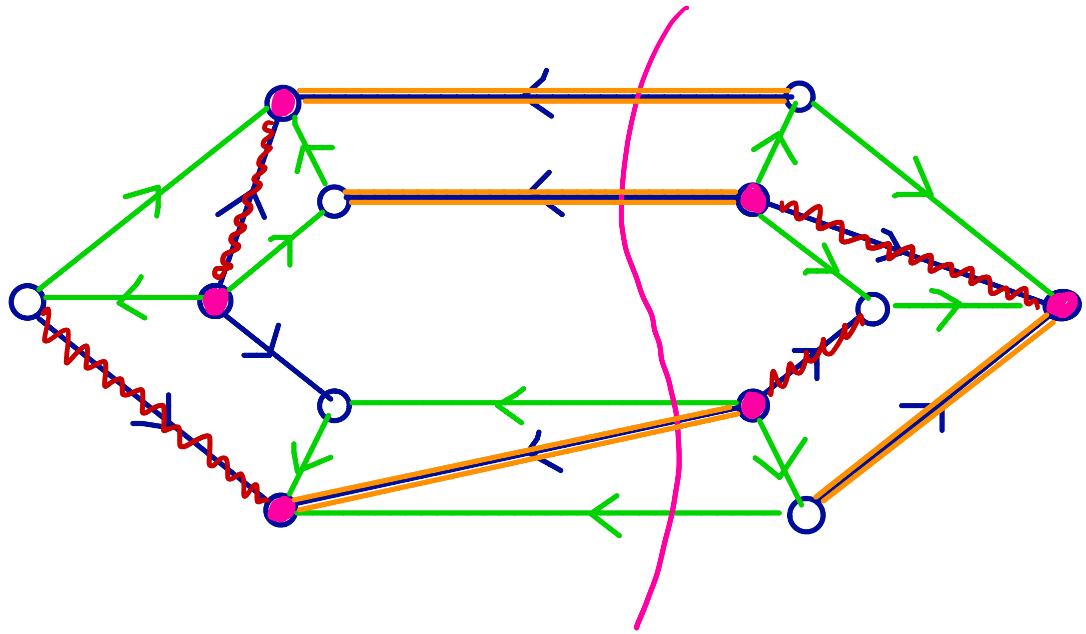

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Maybe one can do better:

Conj. [Guenin, Williams]
Let $\vec{G}$ be digraph with source r & sink s.
If $\exists$ dipath from r to every sink &
$\exists$ dipath from every source to s then
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Unweighted version of Lucchesi-Younger th:

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wide open (hard ?)

Replication & the packing property

Back to perfect graphs

Let G be graph & M stable-set matrix of G.

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Let us find analogue def. for <u>set covering</u>

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Packing property = analogue of perfect graphs

We proved

Pr: G perfect graph, M stable matrix of G
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The analogue for set-covering would be

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The following are equivalent for clutter $\mathcal{F}$:

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Note $2 \Rightarrow 1$ trivial.

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Then H is obtained from $\mathcal{F}$ by replicating e if

$$H = \{S : e \notin S \in \mathcal{F}\} \cup \{S, S - e \cup \bar{e} : e \in S \in \mathcal{F}\}$$

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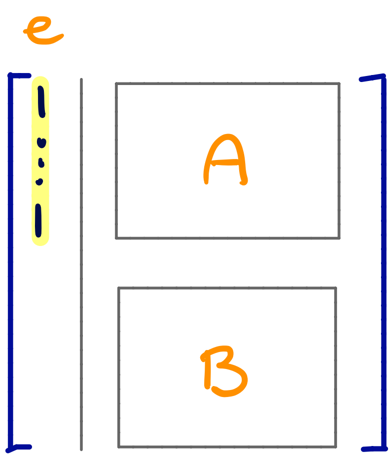
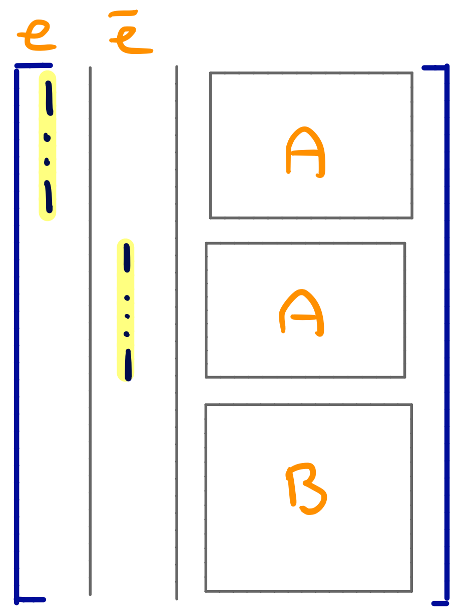
$M(\mathcal{F}) =$ 

$M(\mathcal{H}) =$ 

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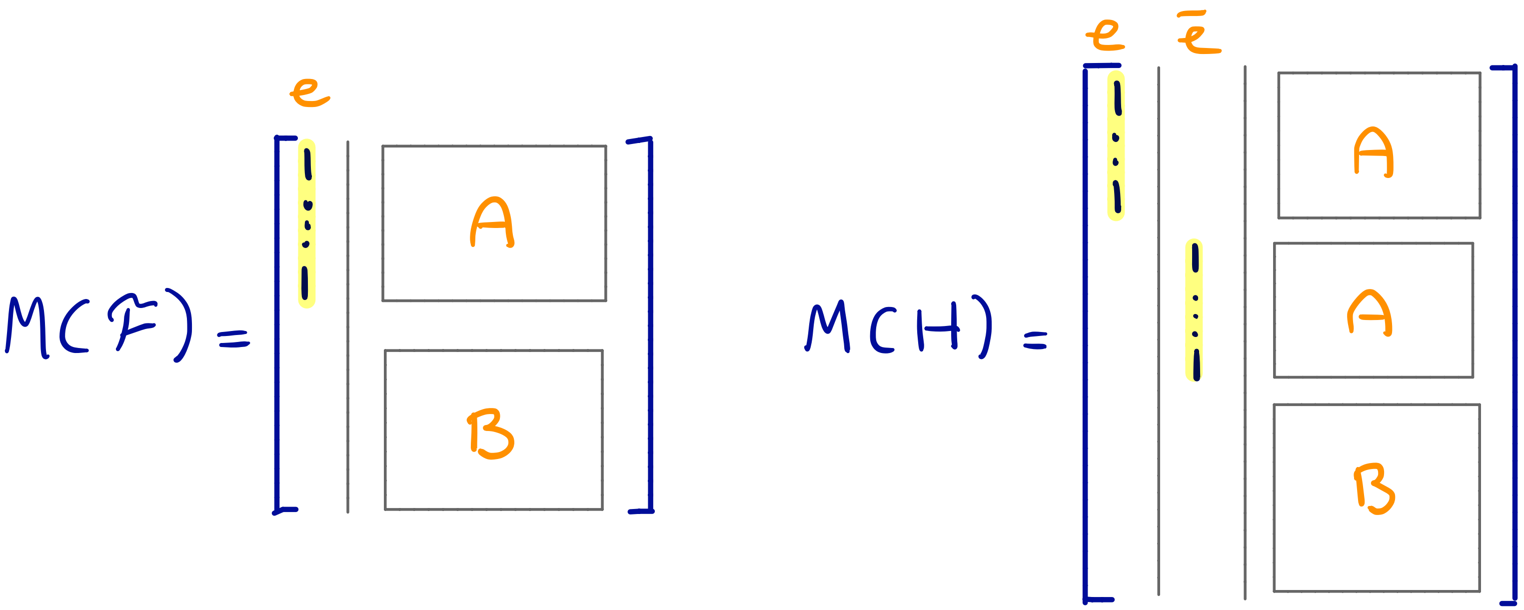

(This is what duplication does for stable set matrices)

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similar to use of duplication to prove TDI for set packing problem

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This follows from Lehman's theorem

pf:

Spse $\mathcal{F}$ not ideal

pf:

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If $H \approx \Delta_n$ then H does not pack ✓

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Otherwise by Lehman's th,

$\bar{x} = \frac{1}{r}\mathbb{1}$ fractional extreme point of $\{x \geq 0 \mid M(\mathcal{H})x \geq \mathbb{1}\}$

$$A = \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}, \quad B = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}, \quad A^T B = J + dI \text{ for } d \geq 1$$

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$nrs = \mathbb{1}^T A B^T \mathbb{1} = \mathbb{1}^T (J + dI)\mathbb{1} = n(n+d) \Longrightarrow n = rs - d.$

$\boxed{/\!/}$

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Excluded minors for the packing property

Question: why excluded minors for the packing property & not for Mengerian property?

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Clutters with packing property are ideal $\Longrightarrow$

A minimally non-packing clutter is

① minimally non-ideal <u>or</u>

② ideal

Minimally non-packing clutters that are mni

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Question: does the converse hold?

Minimally non-packing clutters that are ideal

A far-fetched conjecture ← but possibly true

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$\tau = 2$ Conjecture [Cornuejols, Guenin, Margot]
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Motivation for conjecture ?

Pr: $\tau = 2$ conj. $\Longrightarrow$ replication conj.

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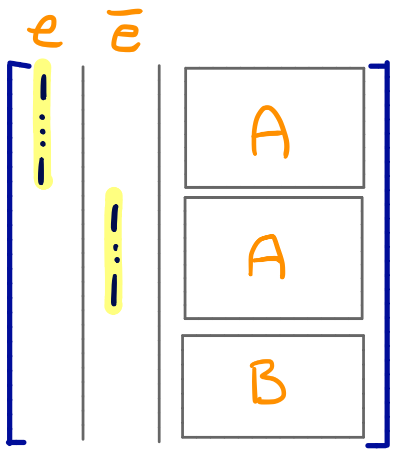
$M(\mathcal{F}) =$ 

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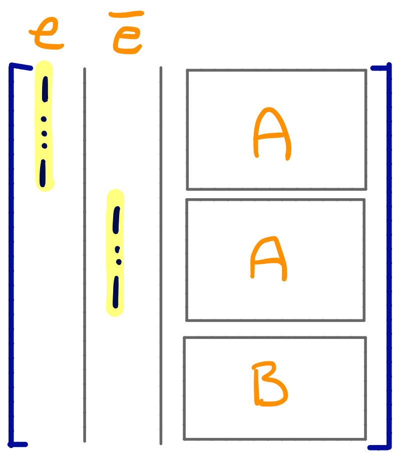
$M(\mathcal{F}) =$ 

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$\Longrightarrow S_1 - e \cup \bar{e} \in \mathcal{F}$ & $S_1 - e \cup \bar{e} \cap S_2 = \emptyset$ $\square$

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The following are equivalent for a binary clutter $\mathcal{F}$:

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The replication conj. holds for binary clutters.

Thank you for your attention

Keep safe !

I would like to thank Ahmad Abdi for numerous discussions on this topic.