Mini-course

Packing & covering

Part I : Introduction

Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property

What are perfect graphs?

Let $G = (V, E)$ be a graph.

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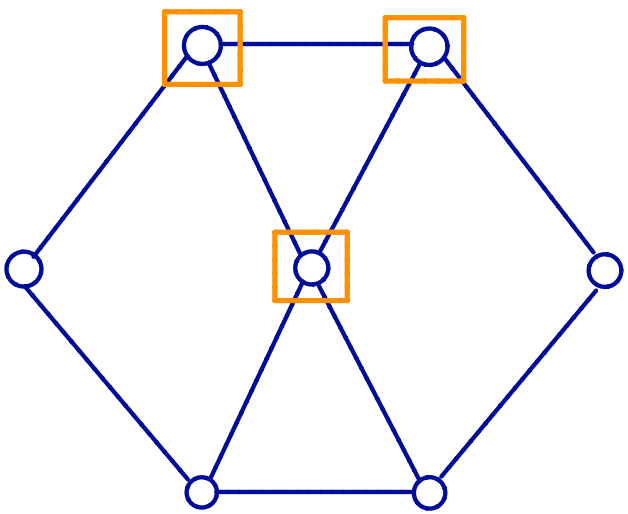
def: a complete subgraph is a clique



clique

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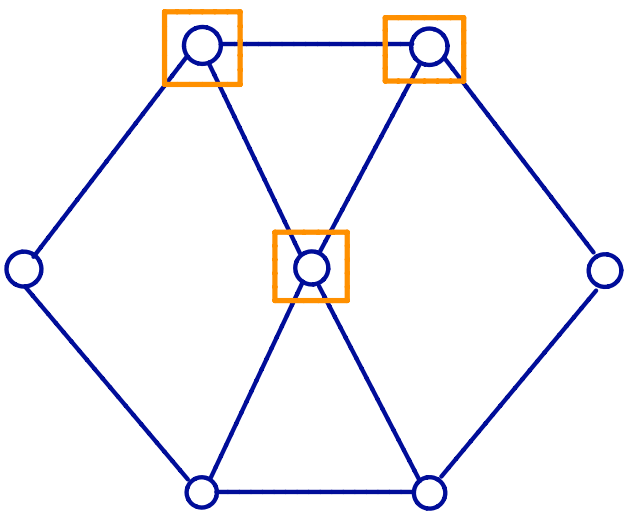
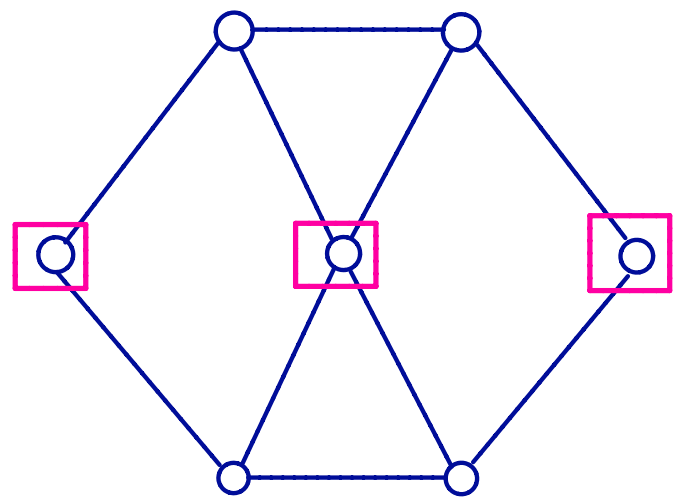
a stable set is a clique in complement



clique



stable set

def:

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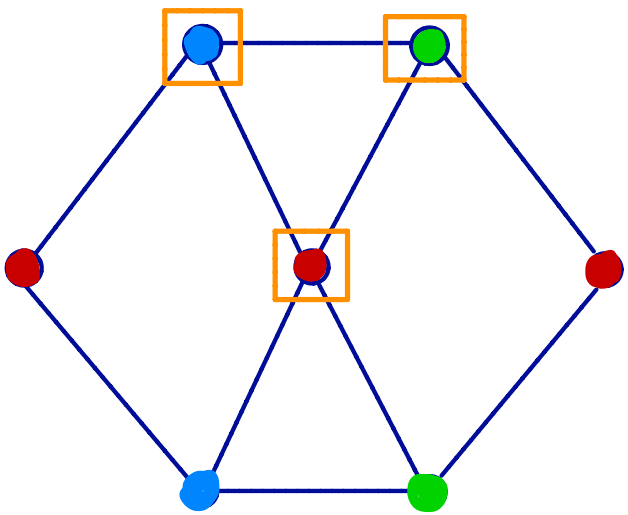

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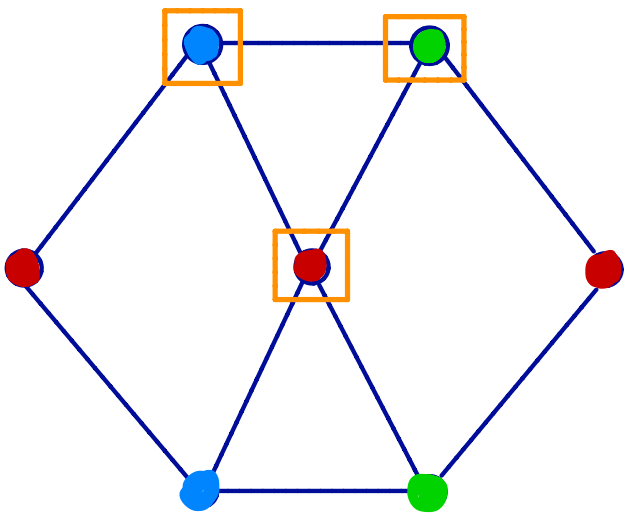

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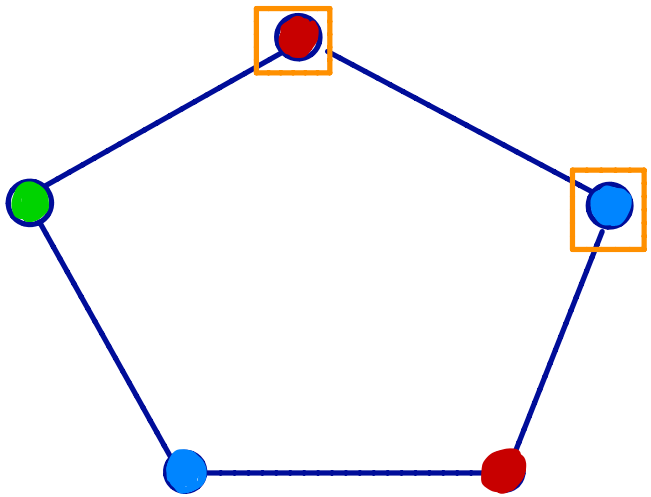

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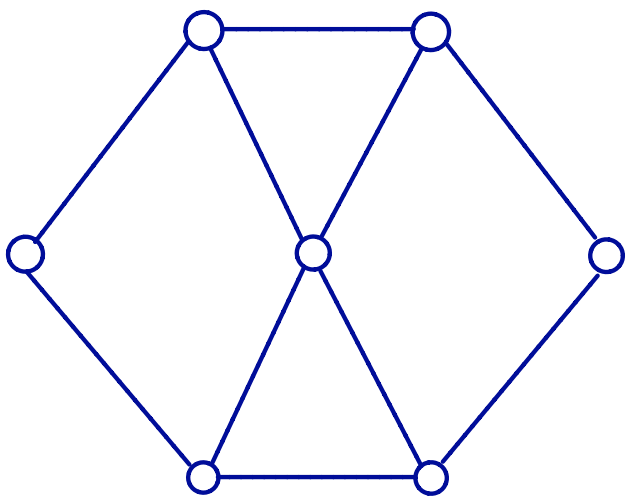

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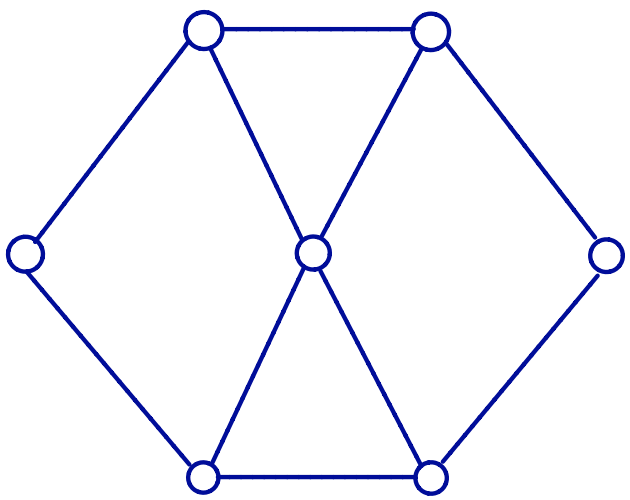

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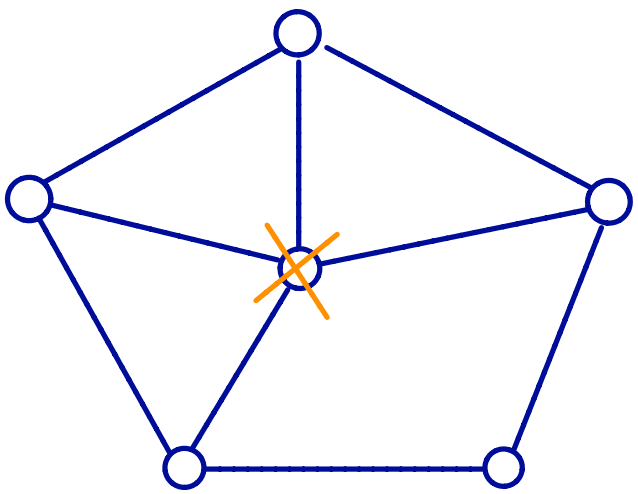

NOT perfect

From perfect matrices to perfect graphs

Pr [Fulkerson]

M perfect $\Longrightarrow$ $Mx \leq 1,\ x \geq 0$ TDI

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$\{x \geq 0 : Mx \leq 1\}$ integral.

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pf:

Pick $c \in \mathbb{Z}_+^n$

$$z(c) := \max\{c^T x : Mx \leq 1, x \geq 0\} \qquad (P_c)$$
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If $1^T c = 0$ pick $\bar{y} = 0$. ✓

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$\Longrightarrow$ $\hat{y} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is optimal sol of (D_c) $\blacksquare$

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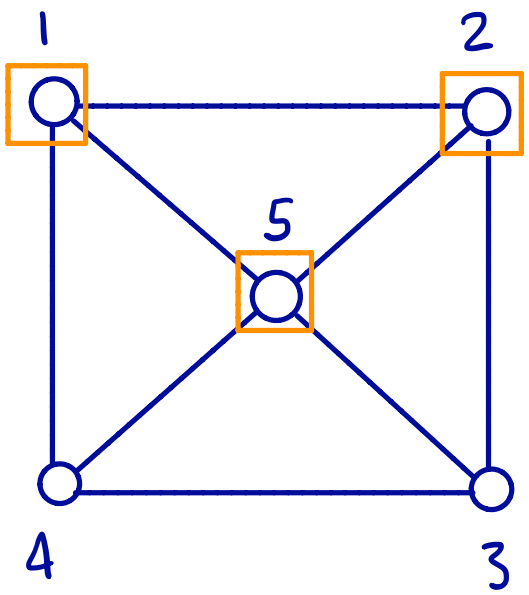

$$M = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{c|c|c|c|c} 1 & 1 & & & 1 \\ \hline & 1 & 1 & & 1 \\ \hline & & 1 & 1 & 1 \\ \hline 1 & & & 1 & 1 \end{array}\right] \end{array}$$

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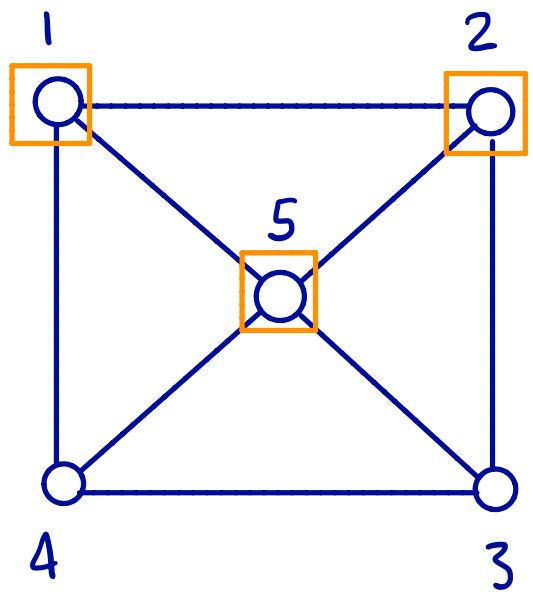

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pf: Define $G = (V, E)$ where

$V = \{1, \ldots, n\}$ &

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$\Longrightarrow$ rows of M are cliques of G

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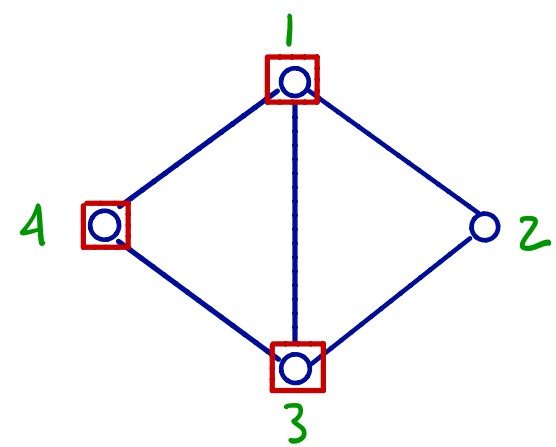

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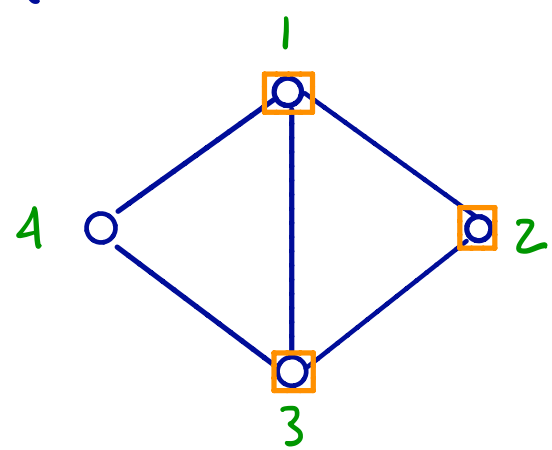

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To show:

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Suppose not true for clique S.

Let c be char. vector of S.

$$z_P := \max\{c^T x : Mx \leq 1, x \geq 0\} \qquad (P)$$

$$z_{IP} := \max\{c^T x : Mx \leq 1, x \geq 0, x \text{ integer}\} \qquad (IP)$$

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$$\bar{x}_j = \begin{cases} \frac{1}{|S|-1} & \text{if } j \in S \\ 0 & \text{if } j \notin S \end{cases}$$

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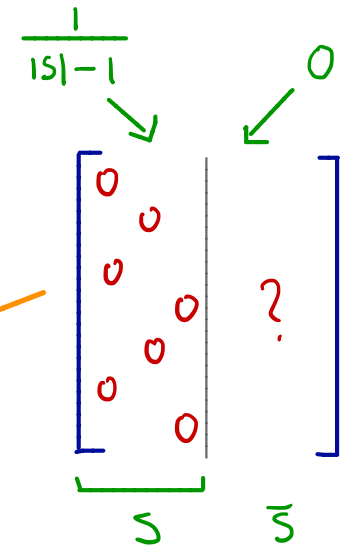

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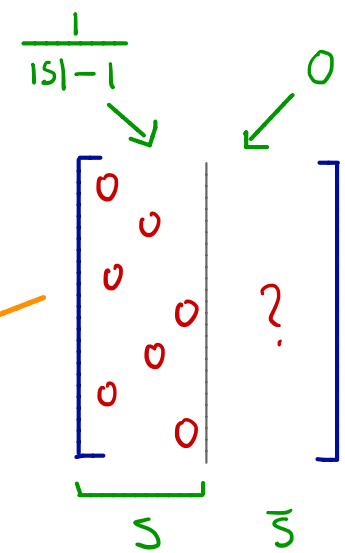

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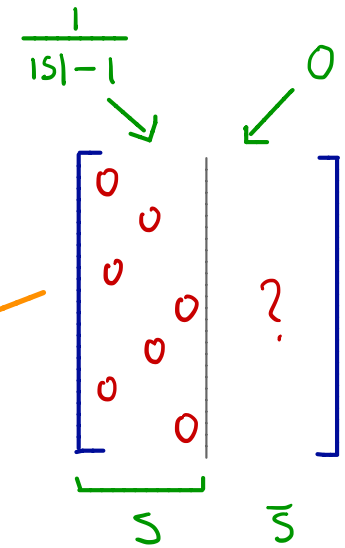

$\bar{x}$ feasible for (P) $\Longrightarrow$ $z_P \geq \frac{|S|}{|S|-1} > 1$

Let c be char. vector of S.

$$z_P := \max\{c^T x : Mx \leq 1, x \geq 0\} \qquad (P)$$

$$z_{IP} := \max\{c^T x : Mx \leq 1, x \geq 0, x \text{ integer}\} \qquad (IP)$$

S clique $\Longrightarrow$ $z_{IP} \leq 1$

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But $z_P = z_{IP}$ as M perfect $\lightning$ $\blacksquare$

Pr [Chvátal]

M perfect $\Longrightarrow$

M stable set matrix of perfect graph.

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stable set matrices of induced subgraphs of G.

By ① + ②, suffices to show:

$$\boxed{\omega(G) = \chi(G)}$$

Let

$$\max\{\mathbb{1}^T x : Mx \leq \mathbb{1},\ x \geq 0,\ x \text{ integer}\} \qquad \text{(IP)}$$

$$\min\{\mathbb{1}^T y : M^T y \geq \mathbb{1},\ y \geq 0,\ y \text{ integer}\} \qquad \text{(ID)}$$

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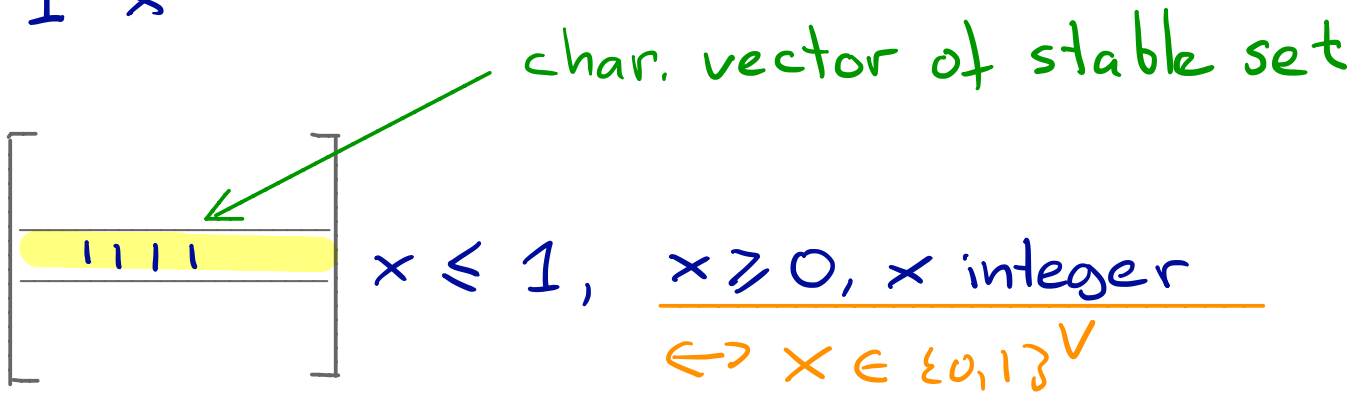

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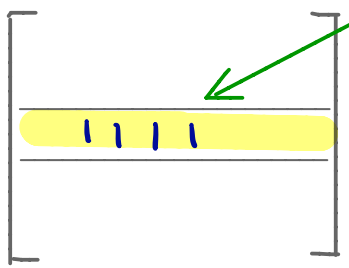

char. vector of stable set

$$x \leq \mathbb{1},\quad \underline{x \geq 0,\ x \text{ integer}}$$

$$\Leftrightarrow x \in \{0,1\}^V$$

$\Longrightarrow$ finds maximum size clique, i.e $\omega(G)$

Let

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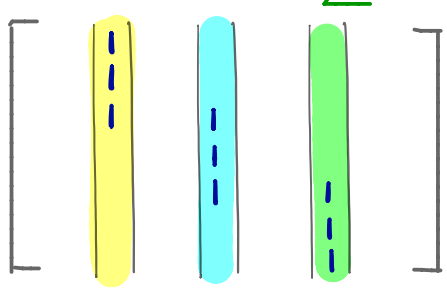

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$$y \geq \mathbb{1}, \quad \underline{y \geq 0,\ y \text{ integer}}$$

wma $y_s \in \{0,1\}$

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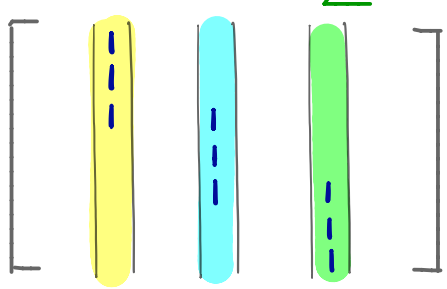

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wma $y_S \in \{0,1\}$

$\Longrightarrow$ finds minimum set of stable sets covering V, i.e $\chi(G)$.

Thus

$$\omega(G) = \max\{\mathbb{1}^T x : Mx \leq \mathbb{1},\ x \geq 0,\ x \text{ integer}\} \quad \text{(IP)}$$

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We showed:

To understand perfect matrices it suffices to understand perfect graphs !

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Is the converse true ?

From perfect graphs to perfect matrices

def: duplicating a vertex v is to add a new vertex $\bar{v}$, join it to all neighbors of v and to v.

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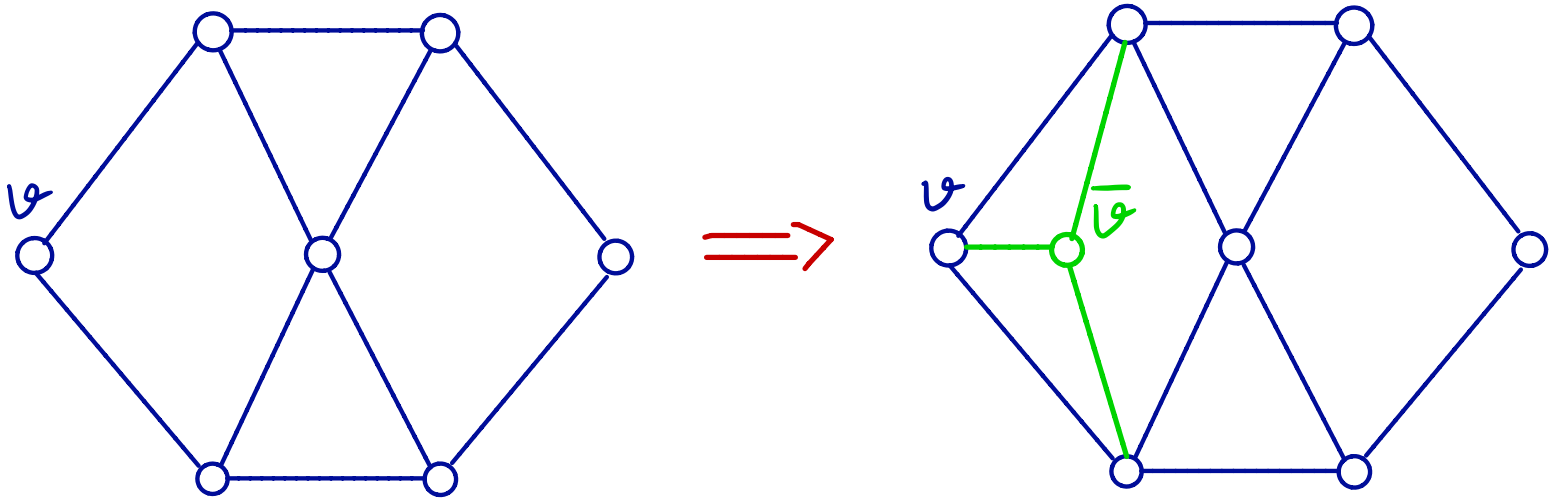

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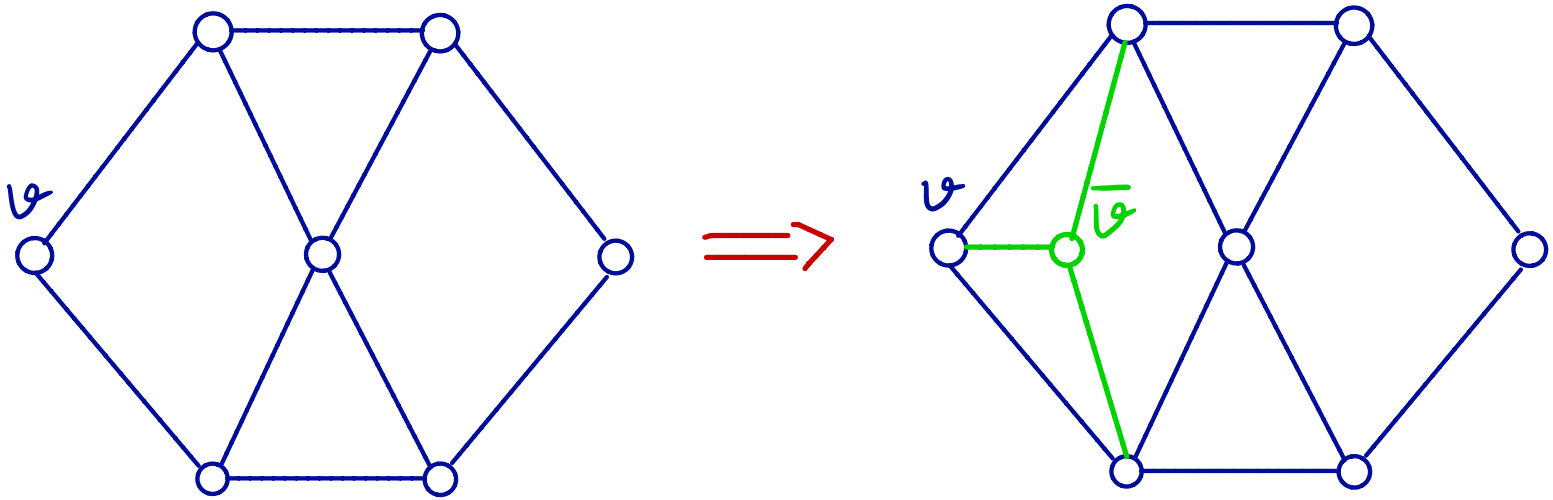

Duplication preserve perfection !

Pr[Lovász]

G perfect, H obtained from G by duplicating vertex v. Then H is perfect.

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$\exists$ colour classes $C_1, \ldots, C_{\omega(G)}$ of G.

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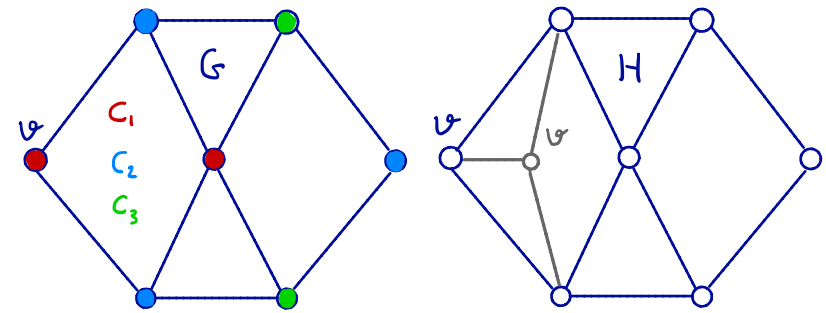
As $\{\bar{v}\}, C_1, \dots, C_{\omega(G)}$ is proper colouring of H

$\omega(H) = \chi(H)$ ✓

Case 2: v is in no maximum clique of G.



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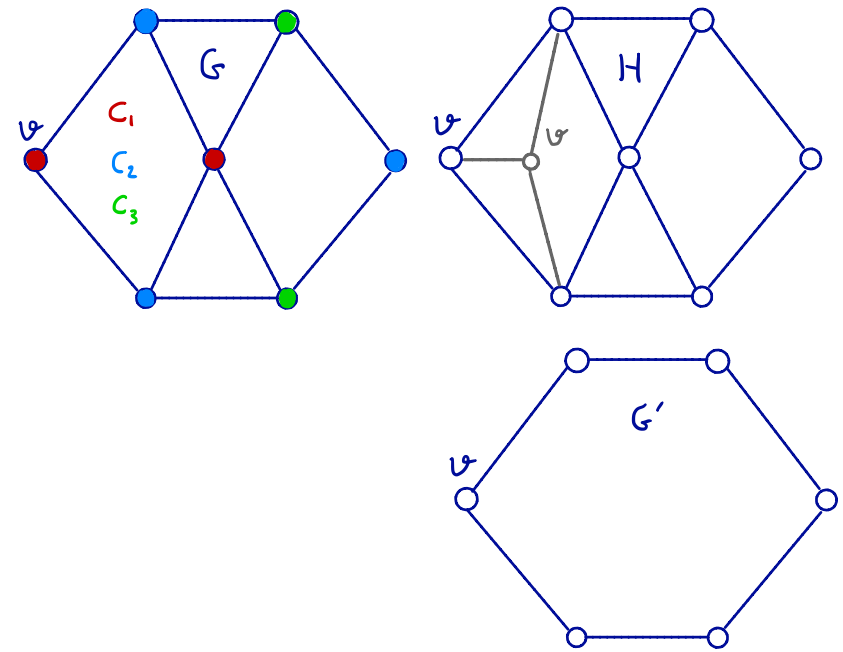
Let $G' = G \setminus (C_1 - v)$



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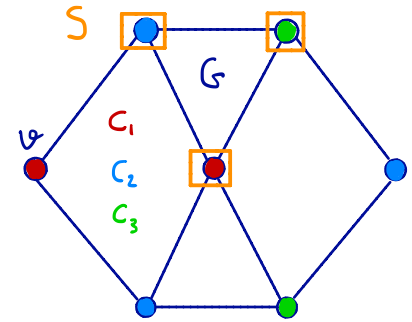

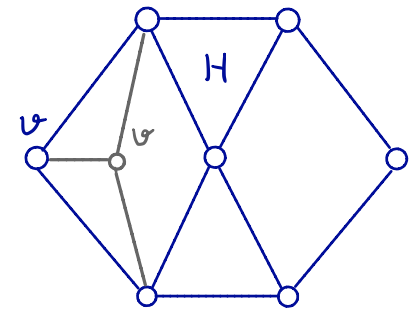

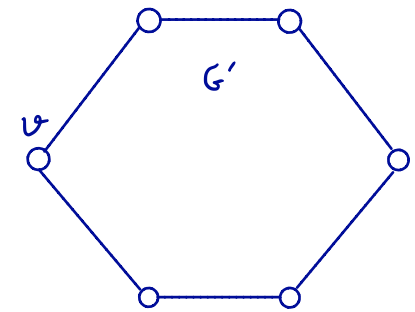

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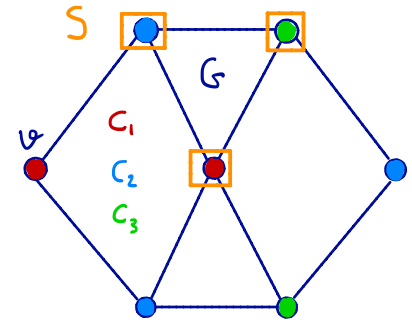

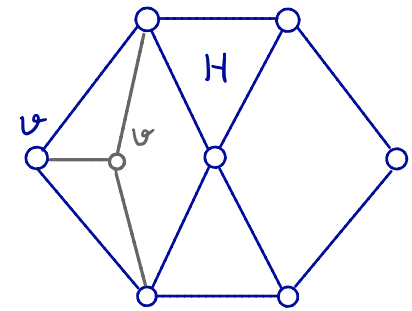

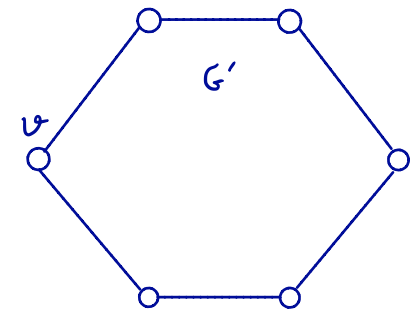

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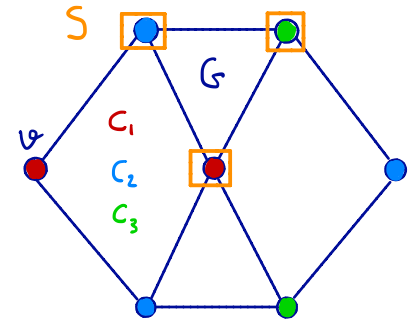

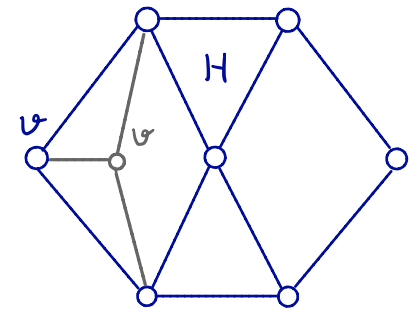

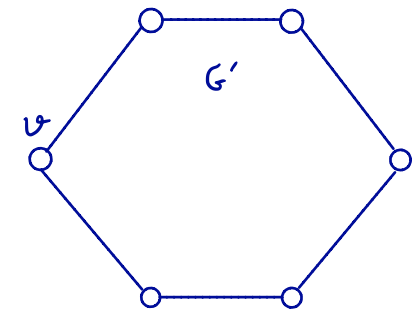

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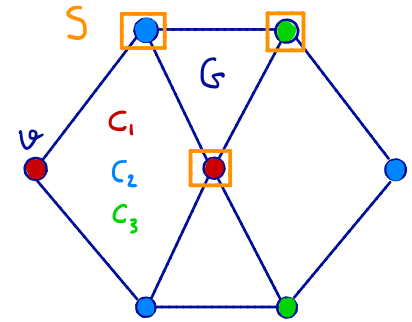

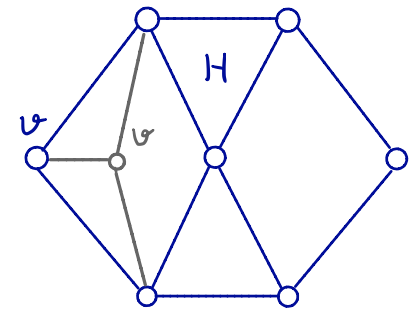

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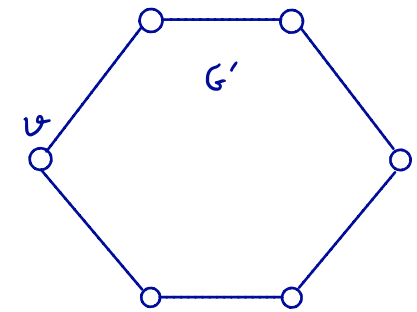

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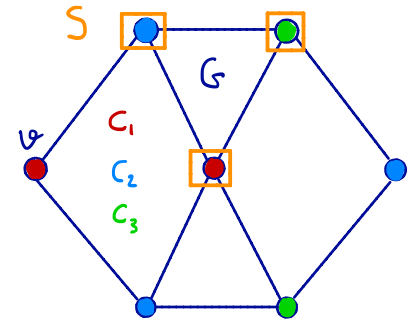

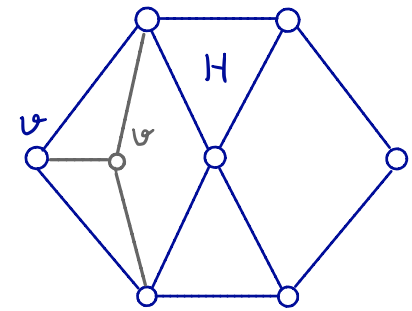

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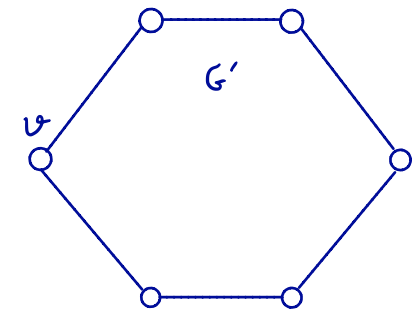

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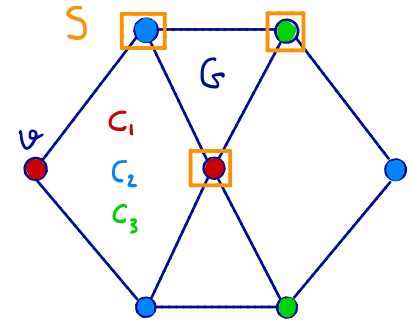

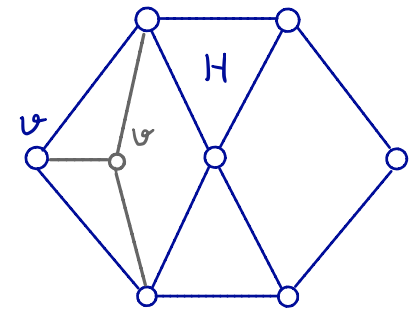

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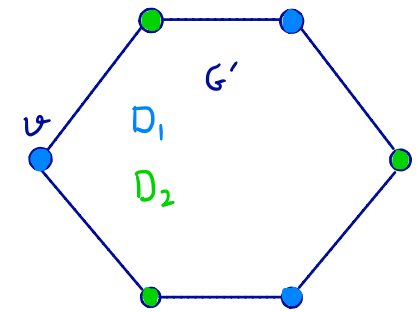

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$D_1, \dots, D_{\omega(G)-1}$ colouring of G'

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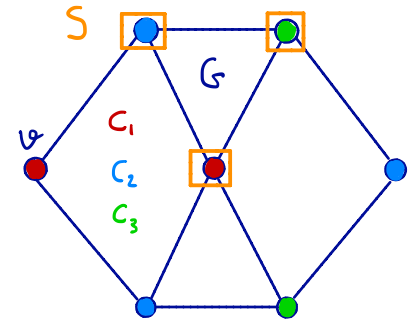

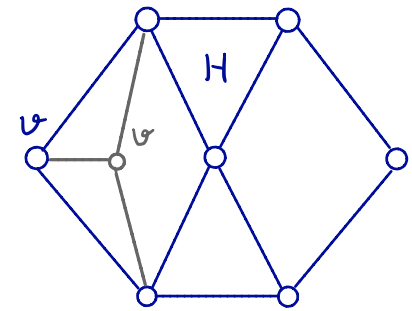

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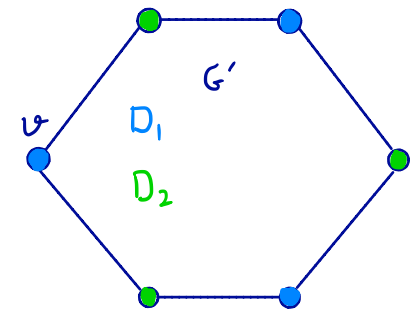

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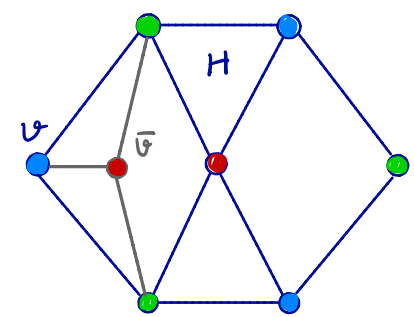

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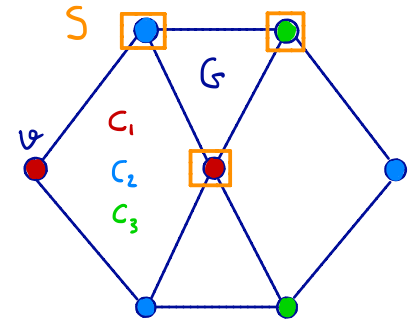
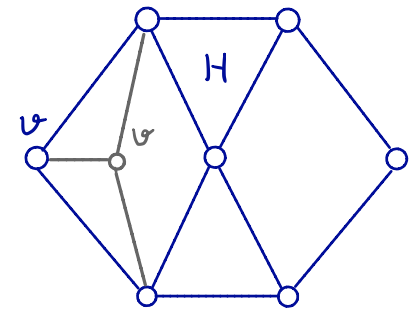
$\Longrightarrow D_1, \dots, D_{\omega(G)-1}, C_1 - v \cup \bar{v}$ colouring of H

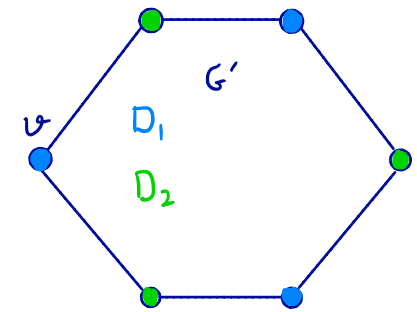
Case 2: v is in no maximum clique of G.





Let $G' = G \setminus (C_1 - v)$

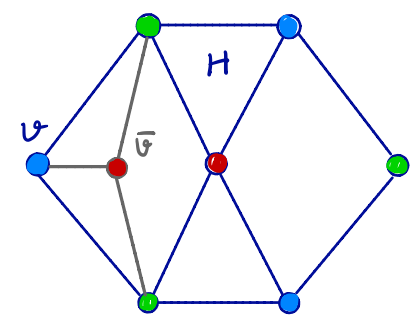
S maximum clique of G



Then $|S \cap C_i| \leq 1, \forall i = 1, \dots, \omega(G)$

$\Longrightarrow |S \cap C_i| = 1,$

$\Longrightarrow |S \cap (C_1 - v)| = 1$



$\Longrightarrow \omega(G) - 1 = \omega(G') = \chi(G').$

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$\Longrightarrow \omega(H) = \chi(H)$ $\blacksquare$

Pr[Chvátal]

G perfect graph, M stable matrix of G

$\Longrightarrow$ M perfect

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pf:

$$z_{IP} := \max\{c^T x : Mx \leq 1, x \geq 0, x \text{ integer}\} \quad (IP)$$

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To show: $\forall c \in \mathbb{Z}_+^n$: z_P integer.

Let G_c be obtained from G by $\forall v \in V$:

- if $c_v = 0$: delete v
- if $c_v \geq 2$: duplicate v, $c_v - 1$ times

① $z_{IP} = w(G_c)$

pf:

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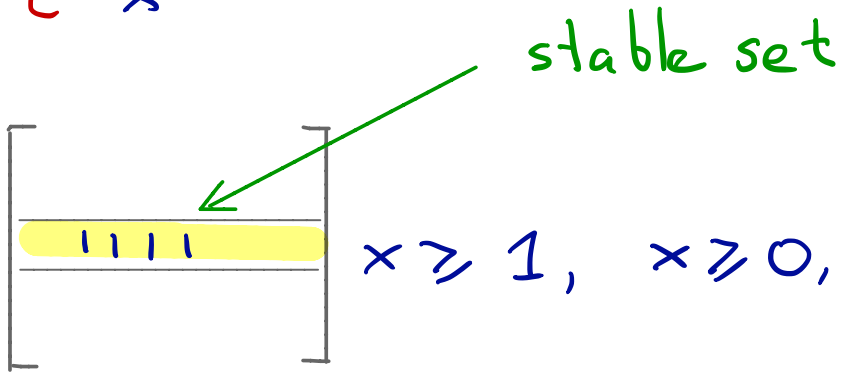
pf:

$\max \; c^T x$

st



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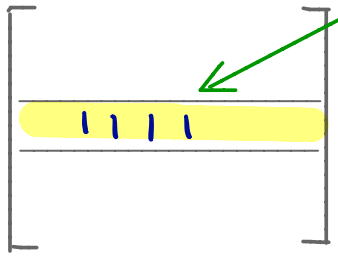

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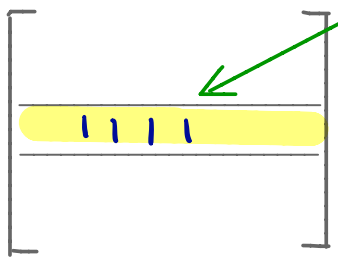

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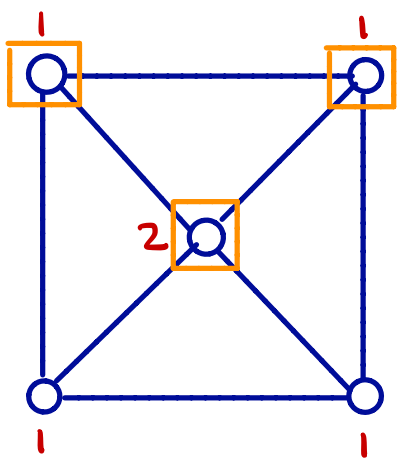
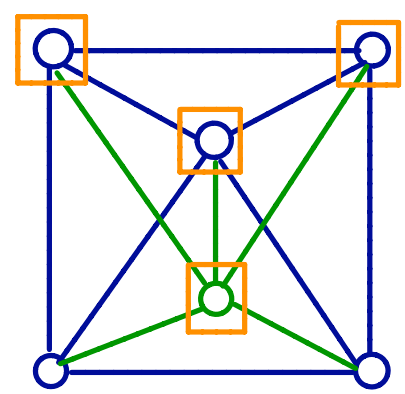
G





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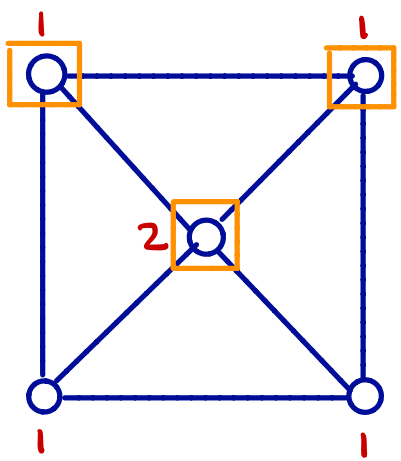
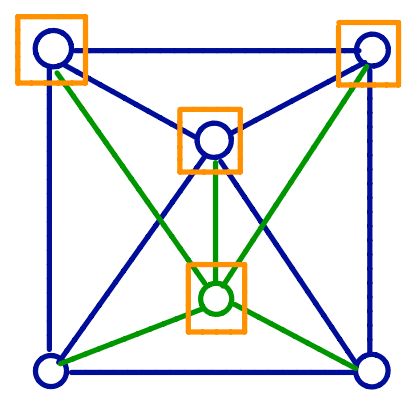
stable set

$$\begin{bmatrix} \\ 1\,1\,1\,1 \\ \\ \end{bmatrix} x \geq 1, \quad x \geq 0, \; x \text{ integer} \qquad (IP)$$

$\Longrightarrow$ find max weight clique of G $= w(G_c)$

G





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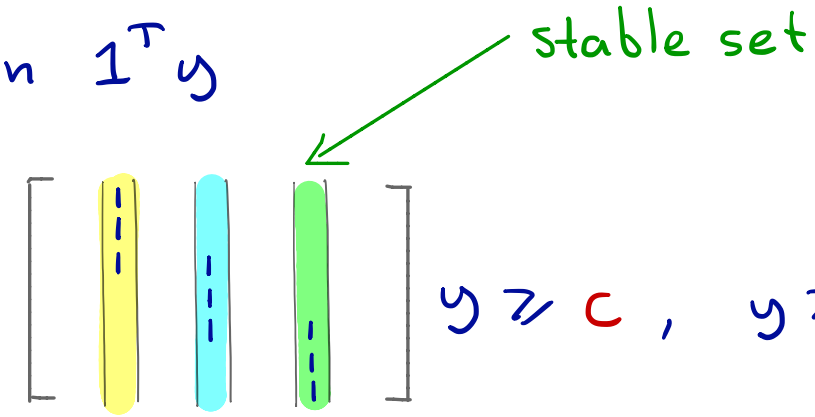

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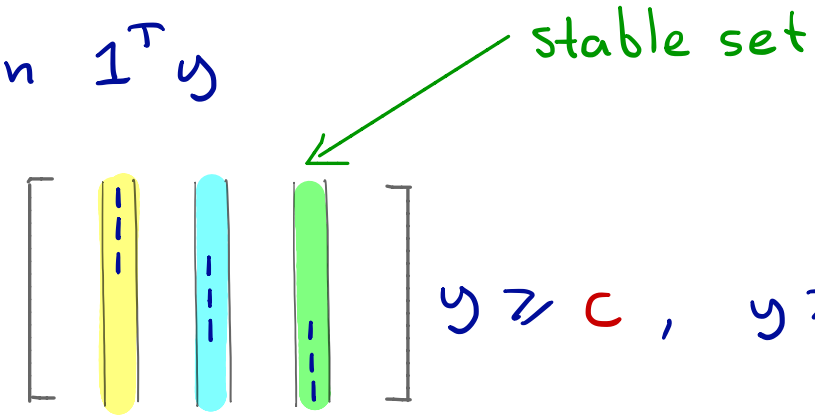

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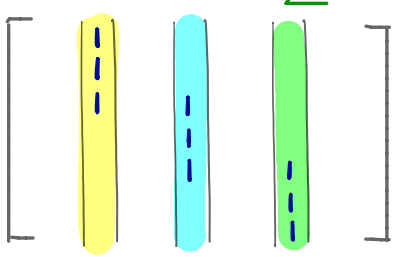

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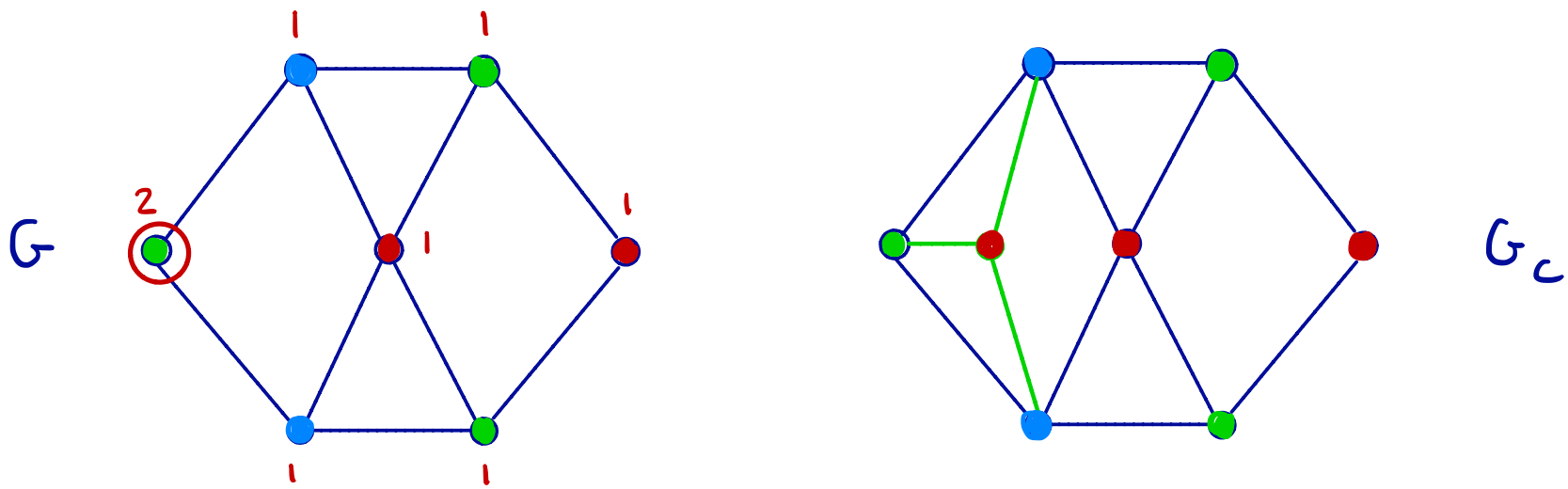

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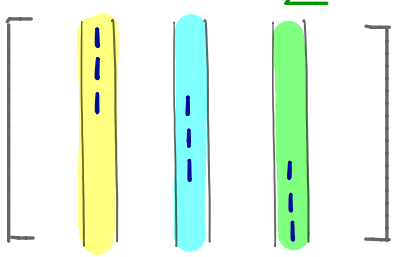

stable set

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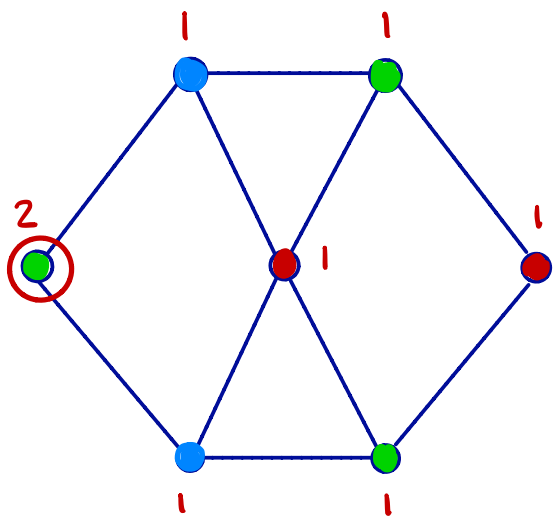
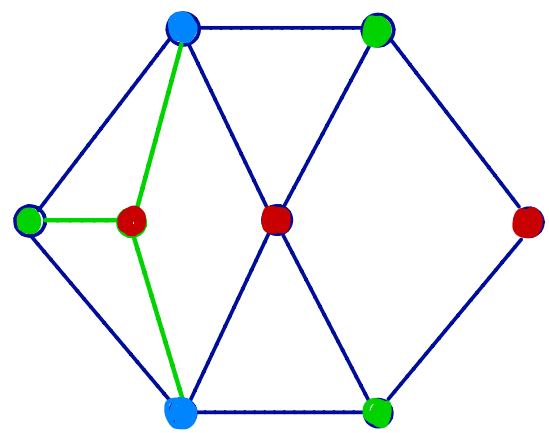
G





G_c

Then $\forall\, c \in \mathbb{Z}_+^n$

$$\omega(G_c) \overset{①}{=} z_{IP} \leq z_P = z_D \leq z_{ID} \overset{②}{=} \chi(G_c)$$

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We showed:

To understand perfect graphs it suffices to understand perfect matrices.

A characterization of perfect graphs/matrices

2 ways of characterizing a minor-closed class $\mathcal{C}$ of objects

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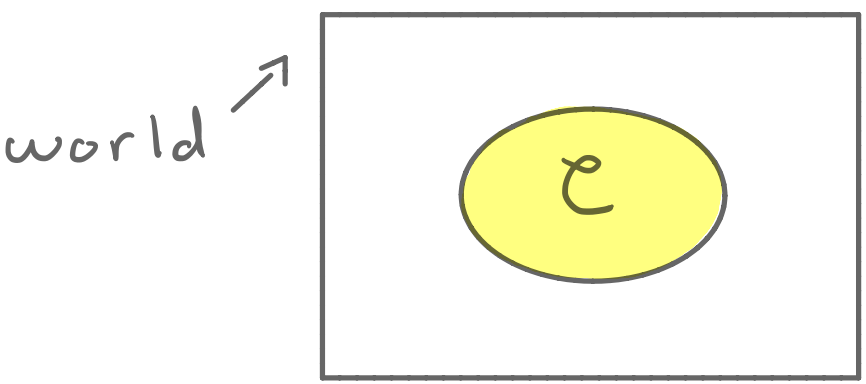
① describe what is inside

structure theorem



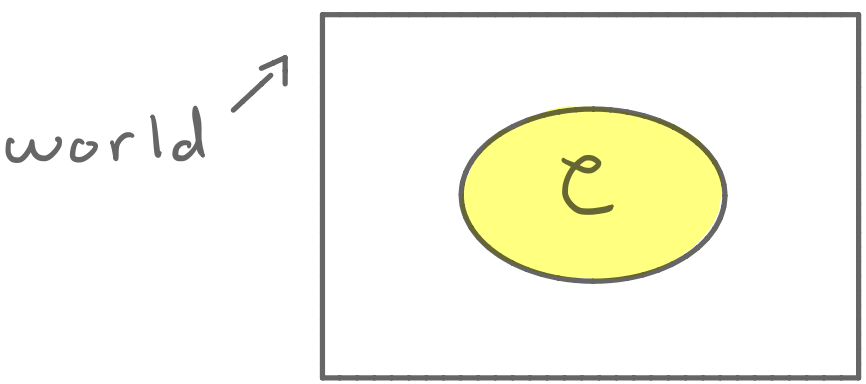
2 ways of characterizing a minor-closed class $\mathcal{C}$ of objects

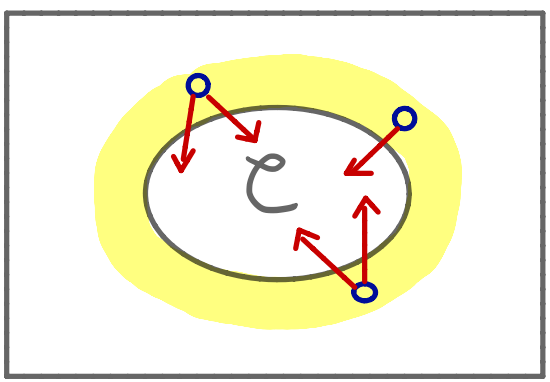
① describe what is inside

structure theorem

② describe what is just outside

excluded minor theorem





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Examples ?

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Examples ?

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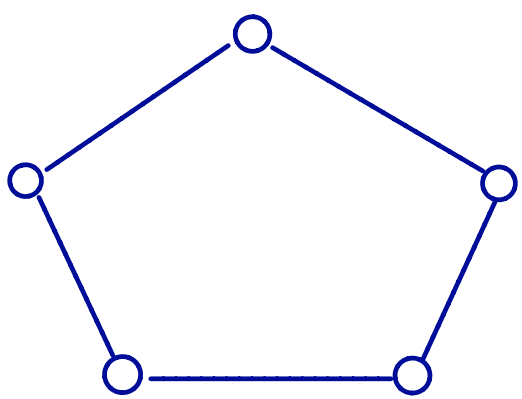

odd hole

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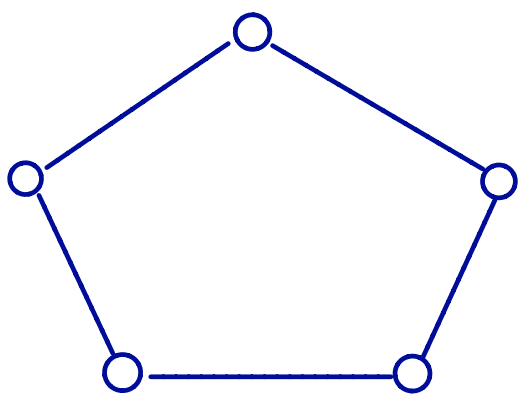
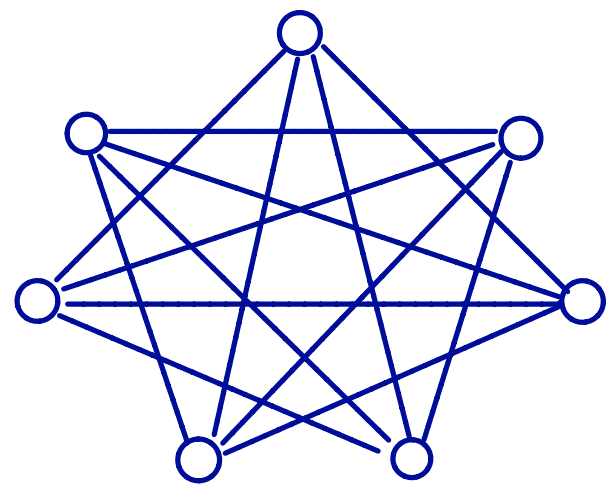
Examples ?

def: an odd hole is an odd length ≥ 5 cordless cycle. An odd antihole is the complement of an odd hole.



odd hole



odd antihole

Strong perfect graph theorem

[Chudnovsky, Robertson, Seymour, Thomas]

The only minimally imperfect graphs are odd holes and odd antiholes.

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Perfect graph theorem [Lovász]

If a graph is perfect so is its complement.

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What about perfect matrices?

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The only minimally imperfect matrices are

1

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3

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③

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & & 1 \\ 1 & & & \ddots & \\ & & & & 0 \\ \hline & & ? & & \end{array}\right] \quad \text{maximal rows}$$

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Th: ← restatement of Strong Perfect Graph th.

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