Mini-course

Packing & covering

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Part I: Introduction

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Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property

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Part IV : The Mengerian property

The Set packing problem

A max-min relation

Poset $(V, \leq)$

A max-min relation

Poset $(V, \leq)$, i.e $\forall a,b,c \in V$,

- $a \leq a$
- $a \leq b$ & $b \leq a \implies a = b$
- $a \leq b$ & $b \leq c \implies a \leq c$

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a, b incomparable otherwise

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Qu: maximum size of antichain ?

Th: [Dilworth]

max size of antichain =

min nb of chains needed to cover V.

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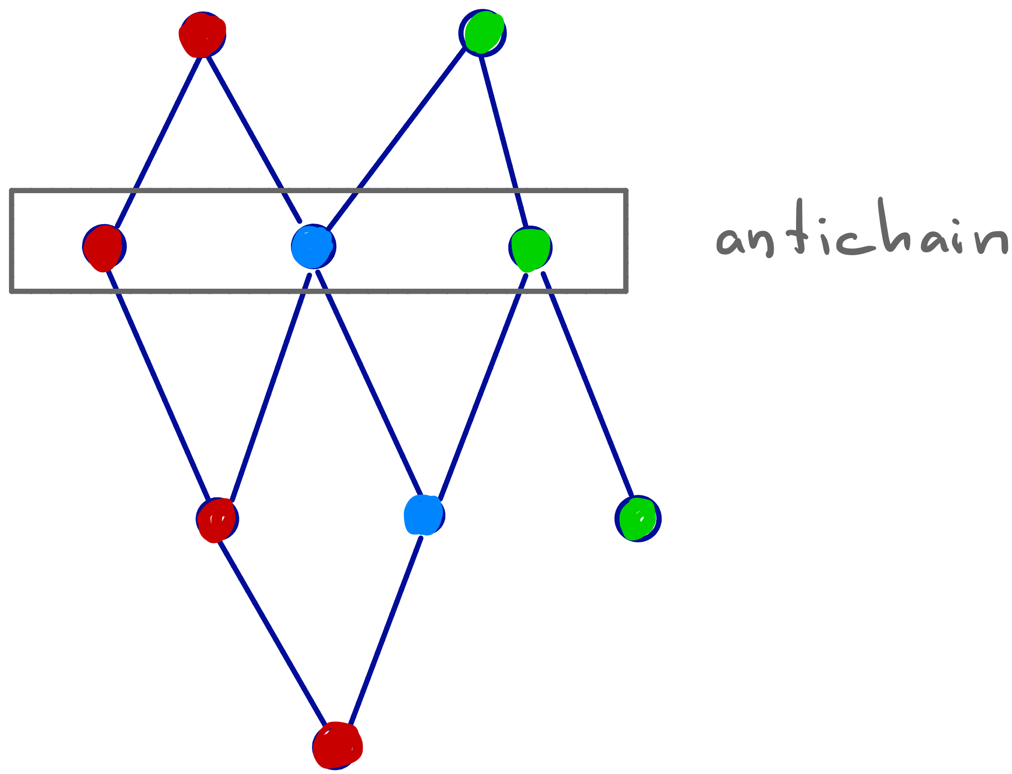

An Integer Programming framework

An Integer Programming framework

Let M be $m \times n$ 0,1 matrix, $w \in \mathbb{R}^n_+$

def: Set Packing IP

$$\max\{w^T x : Mx \leq 1,\ x \geq 0,\ x \text{ integer}\} \quad \text{(IP)}$$

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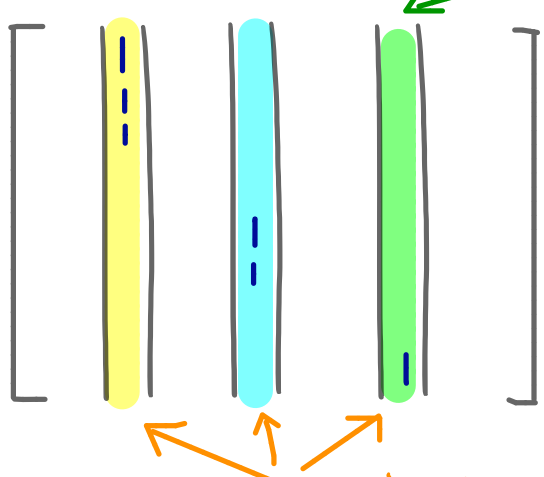
incidence vector of set

$M =$



$x_j = 1$

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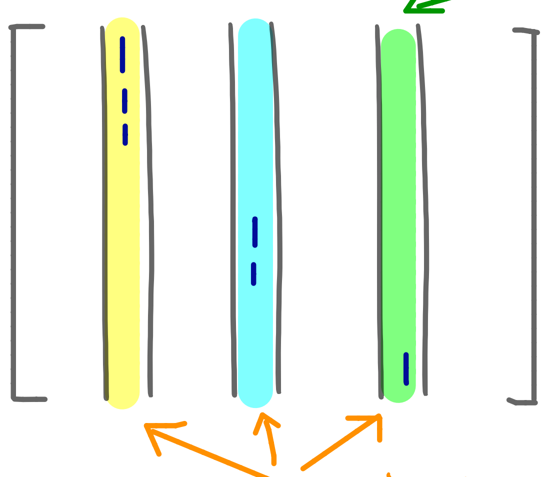

$x_j = 1$

Finds max weight family of disjoint sets.

$$\max \{ w^T x : Mx \leq 1, x \geq 0, x \text{ integer} \} \qquad \text{(IP)}$$

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Let z_{IP}, z_P, z_D, z_{ID} be optimal values for (IP), (P), (D), (ID) respectively,

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Let z_{IP}, z_P, z_D, z_{ID} be optimal values for (IP), (P), (D), (ID) respectively, then

$$\boxed{z_{IP} \le z_P = z_D \le z_{ID}}$$

strong duality

Restating Dilworth's theorem

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$(V, \leq)$ poset

M matrix where

- columns indexed by V
- rows = char. vectors of maximal chains.

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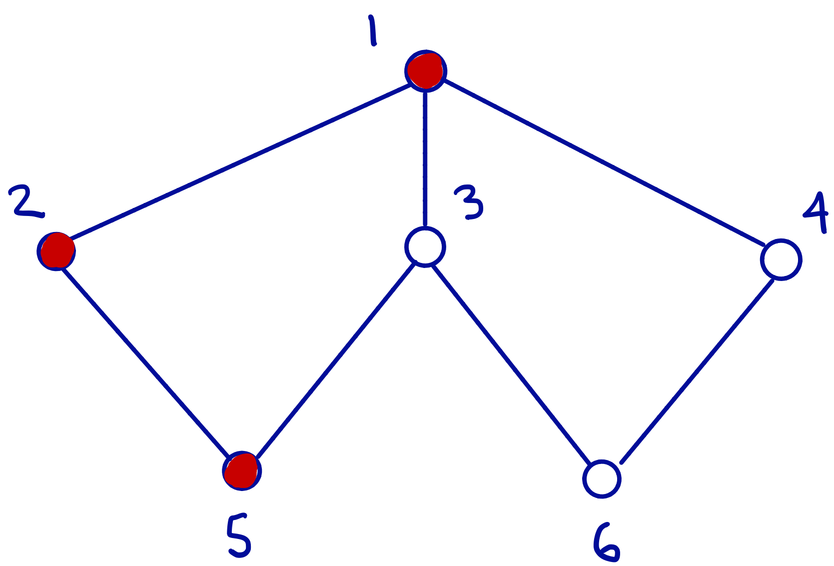

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & & & 1 & \\ 1 & & 1 & & 1 & \\ 1 & & 1 & & & 1 \\ 1 & & & 1 & & 1 \end{bmatrix} \end{array}$$

M matrix where

- columns indexed by V
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Question: What does (IP) finds for $w = 1$?

M matrix where

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Question: What does (IP) finds for $w = \mathbb{1}$?

max $\mathbb{1}^T x$

st

char. vector of chain

$$\begin{bmatrix} & \\ \hline 1\ 1\ 1\ 1 & \\ \hline & \end{bmatrix} x \leq \mathbb{1}, \quad \underline{x \geq 0,\ x \text{ integer}}$$

$\Leftrightarrow x \in \{0,1\}^V$

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$\Longrightarrow$ finds maximum size antichain

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min $\mathbb{1}^T y$

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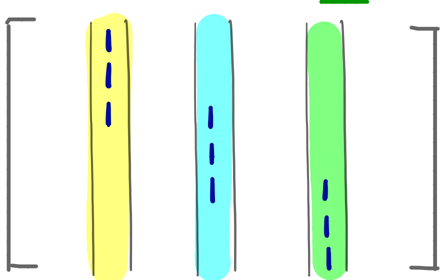

$y \geq \mathbb{1}$, $y \geq 0$, y integer

wma $y \in \{0,1\}^e$

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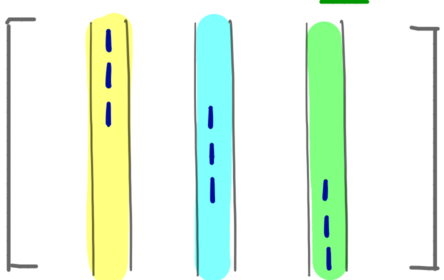

$y \geq \mathbb{1}$, $\quad \underline{y \geq 0,\ y \text{ integer}}$

wma $y \in \{0,1\}^e$

$\Longrightarrow$ finds minimum set of chains covering V

M matrix where

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Then Dilworth's theorem

$$\Longleftrightarrow \quad \boxed{z_{IP} = z_{ID} \quad \text{for} \quad w = 1}$$

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Exercise:

Show, $z_{IP} = z_{ID} \ \forall \ w \in \mathbb{Z}_+^V$.

What does this say in terms of posets?

Three questions

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To address these questions we need to review some polyhedral theory

Elements of polyhedral theory

Pr: Let $P = \{x \geq 0 : Ax \leq b\} \subseteq \mathbb{R}^n$.

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Consider primal / dual pair

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def: $Ax \leq b, x \geq 0$ Totally Dual Integral (TDI) if $\forall w \in \mathbb{Z}^n$ where (D) has optimal sol. it has an optimal <u>integer</u> sol.

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property of systems
not of polyhedra

Pr:

ⓐ $Ax \leq b,\ x \geq 0$ TDI & b integer $\Longrightarrow$

ⓑ $\{x \geq 0 : Ax \leq b\}$ integral

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reverse not true

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pf:

Pick $w \in \mathbb{Z}^n$ for which

$$z^* = \max\{w^T x : Ax \le b,\ x \ge 0\}$$
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By char. of integral polyhedra ✓ $\square$

Three questions — revisited

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$$z_{IP} \leq z_P = z_D \leq z_{ID}$$

Questions: For what class of matrices

1. $z_{IP} = z_{ID} \quad \forall w \in \mathbb{Z}_+^n$ — matrices satisfying 2 & 3
2. $z_{IP} = z_P \quad \forall w \in \mathbb{Z}_+^n$ — $\{x \geq 0 : Mx \leq 1\}$ integral
3. $z_{ID} = z_D \quad \forall w \in \mathbb{Z}_+^n$ — $Mx \leq 1, x \geq 0$ TDI

Questions: For what class of matrices

① $z_{IP} = z_{ID}$ $\forall w \in \mathbb{Z}^n_+$ matrices satisfying ② & ③

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③ $z_{ID} = z_D$ $\forall w \in \mathbb{Z}^n_+$ $Mx \leq 1, x \geq 0$ TDI

We will see:

$$\boxed{\begin{array}{l}\{x \geq 0 : Mx \leq 1\} \text{ integral } \Longrightarrow \\ Mx \leq 1, x \geq 0 \text{ TDI}\end{array}}$$

Questions: For what class of matrices

① $z_{IP} = z_{ID}$ $\forall w \in \mathbb{Z}^n_+$ matrices satisfying ② & ③

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We will see:

$$\boxed{\{x \geq 0 : Mx \leq 1\} \text{ integral} \Longrightarrow Mx \leq 1, x \geq 0 \text{ TDI}}$$

Thus ①, ②, ③ equivalent.

Perfection

def: A 0,1 matrix M is **perfect** if

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$\Longrightarrow$ study of perfect matrices = study of perfect graphs.

The set covering problem

A min-max relation

$G=(V,E)$ graph, $s,t \in V$ where $s \neq t$.

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edge

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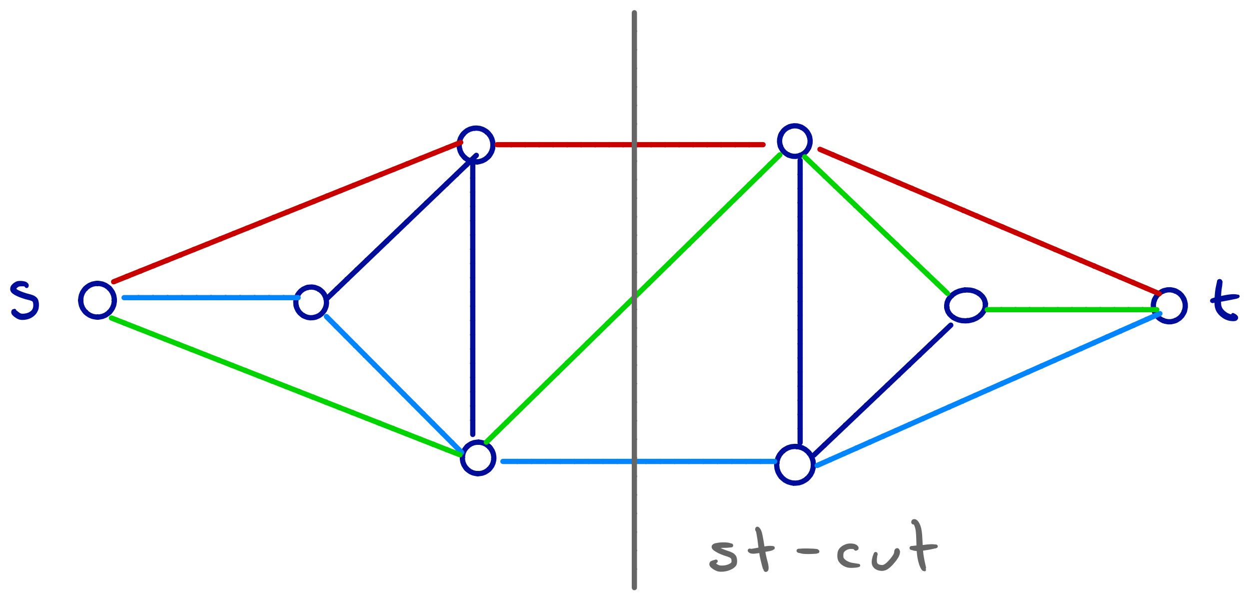

An Integer Programming framework

Let M be $m \times n$ 0,1 matrix, $w \in \mathbb{R}^n_+$

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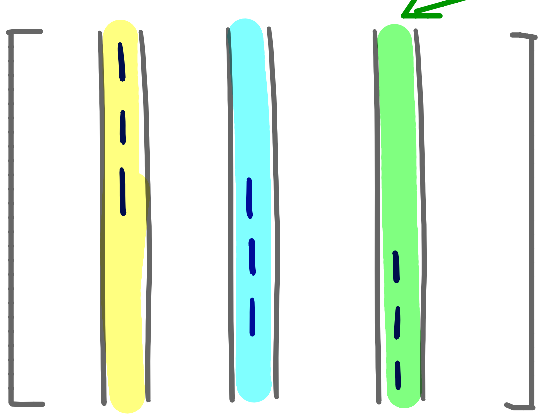
$M =$ 

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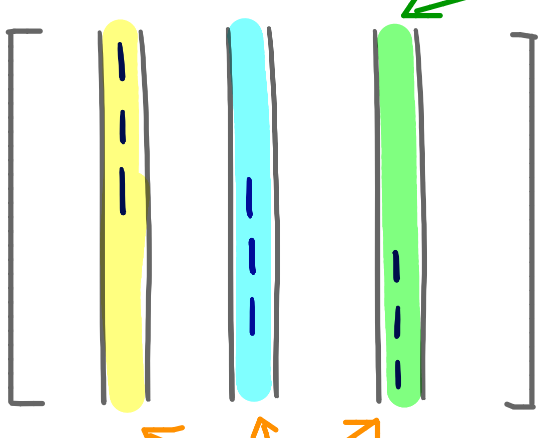

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Finds min weight family of sets covering groundset

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Restating Menger's th

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M matrix where

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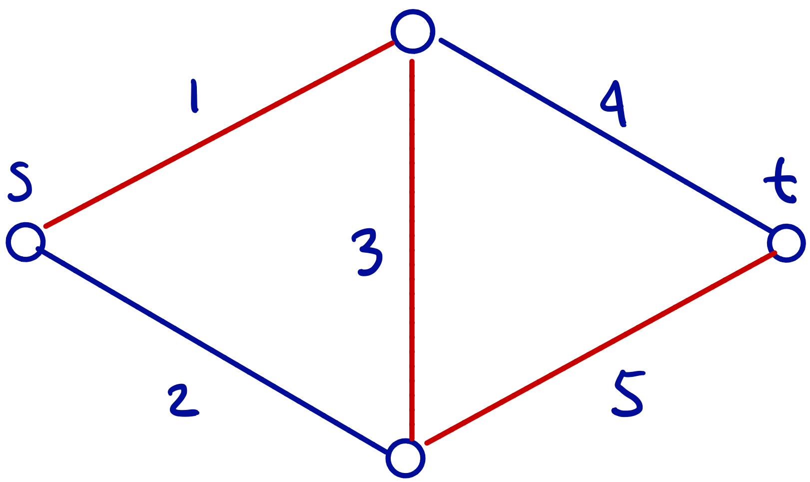

$$M=\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & & 1 & & 1 \\ 1 & & & 1 & \\ & 1 & 1 & 1 & \\ & 1 & & & 1 \end{bmatrix} \end{array}$$

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Question: What does (IP) finds for $w = 1$?

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Question: What does (IP) finds for $w = \mathbf{1}$?

$$\min \ \mathbf{1}^T x$$

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$$\begin{bmatrix} & & & & \\ 1 & 1 & 1 & 1 & \\ & & & & \end{bmatrix} x \geq \mathbf{1}, \quad x \geq 0, \ x \text{ integer}$$

char. vector of st-path

$\Leftrightarrow x \in \{0,1\}$ ✓

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$\Longrightarrow$ finds min size st-cut

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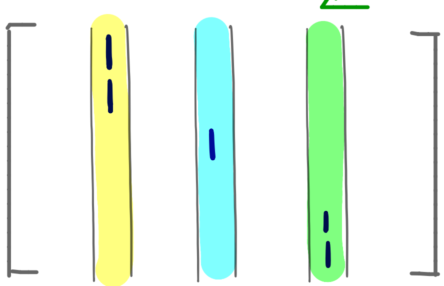

$y \le \mathbb{1}$, $\quad \underline{y \ge 0, \ y \text{ integer}}$

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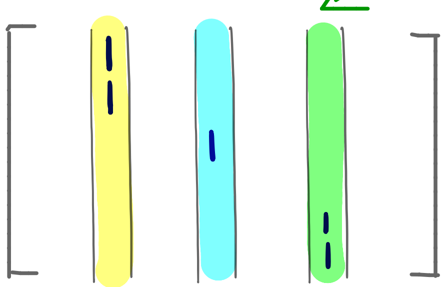

$y \le \mathbb{1}$, $y \ge 0$, y integer

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$\Longrightarrow$ finds maximum nb of disjoint st-paths

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Then Menger's theorem

$$\Longleftrightarrow \quad \boxed{z_{IP} = z_{ID} \quad \text{for} \quad w = 1}$$

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Exercise:

Show, $z_{IP} = z_{ID} \quad \forall\, w \in \mathbb{Z}^n_+$

What does this say in terms of graphs?

undirected flows

Three questions

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For $i = 1,2,3$ let $\mathcal{M}_i$ class of matrices satisfying i

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Question: does converse hold?

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- ideal, why?
- not Mengerian: for $w = 1$ we have

$$2 = z_{IP} > z_{ID} = 1$$

No single combinatorial object associated to ideal or Mengerian matrices.

contrast with perfection

Remainder of lectures

Part II : Perfection

Part III : Idealness

Part IV : The Mengerian property