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Performance-Based Resource Allocation – A Cautionary Tale[1]

Benito Müller,[2] Matthias Roesti,[3] Sam Fankhauser,[4] and Cameron Hepburn[5]

May 2, 2020

Abstract

Performance-based allocation (PBA) frameworks are popular among funding agencies because of their perceived objectivity. Measurable criteria are thought to ensure that funds are directed to the most deserving recipients. Multilateral development institutions now disburse over $ 20 billion annually mostly using PBA. Here we raise a methodological concern that casts doubt over the objectivity and robustness of PBA frameworks. The problem arises when frameworks fail to distinguish between two types of performance assessments: (i) *cardinal* performance measurement, where differences between performance levels possess a meaningful quantitative interpretation; and (ii) *ordinal* performance ranking, where they do not. We demonstrate the risks of committing a 'cardinal fallacy' where ordinal performance rankings are treated as if they were cardinal measurements. The real-world repercussions are substantial. Using a stylised variant of the World Bank IDA PBA framework we demonstrate that even a slight change in the arbitrarily chosen numerical scale for the ordinal performance rankings, say from the current 1-6 to 0-5, would in 2017, for example, have led to a reallocation of IDA funding of about $ 750 million. Countries like Afghanistan, Haiti and Yemen could see their IDA allocation change by more than a third.

Keywords Development funding, performance-based resource allocation
JEL codes D61, F63, O29

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I Introduction

In recent years, multilateral development institutions disbursed more than \$ 20 billion[1] annually mostly using *performance-based allocation* (PBA) systems (GEF 2014). PBA systems are popular because they are seen as objective and transparent. They are designed to facilitate the efficient use of available resources, while also taking into account the needs of recipients.

Country performance assessments typically involve a set of broad categories which are evaluated using ordinal criteria. The resulting ordinal performance rankings are often expressed using a numerical scale, with the numerical values serving as inputs to some function yielding the final resource allocation schedule. Crucially, this process treats ordinal performance levels as if they were cardinal in nature, thus committing what we call the 'cardinal fallacy'.

The essential fallacies linked to the use of non-cardinal data have already been well described in Stevens (1946). A large literature has since emerged with a main focus on how to aggregate such data, most notably in the area of social choice, and more recently in the methodological scrutiny of composite indices (see e.g. Freudenberg, 2003; Böhringer and Jochem, 2007; Munda, 2012 or Greco et al., 2019). While most PBA systems also rely on a composite performance index based on some type of aggregation, the 'cardinal fallacy' is more fundamental in the sense that its negative repercussions do not require any aggregation scheme to be present. The fallacy poses a considerable threat to the robustness of many existing PBA systems, yet we are not aware of any formal characterisation of the underlying problem in this context. We fill this gap by distinguishing between *cardinal performance measurements* and *ordinal performance rankings* and highlighting the consequences of scale transformations in both cases.

[1] As measured by average annual disbursements over a replenishment cycle.

Specifically, we suggest that a necessary condition for an allocation mechanism to be 'performance-based' is that the resulting allocation should not change if the scale or unit of measurement is altered in a way which preserves the original information on performance. This 'reality check' implies that the allocative outcome should not change if we alter the units of a cardinal performance measurement from, say, USD to GBP, or the levels of an ordinal performance ranking from $\{A, ..., F\}$ to $\{B, ..., G\}$, or from $\{1, ..., 6\}$ to $\{0, ..., 5\}$.

The practical implications of violating this condition are far from trivial. Using a stylised version of the World Bank PBA framework to allocate International Development Association (IDA) funds, we find that a seemingly innocuous rescaling of ordinal inputs has profound effects on the resulting IDA allocations. For instance, a simple transformation from the original $\{1, ..., 6\}$ to a $\{0, ..., 5\}$ scale would have reallocated about $ 750 million to different recipients in the fiscal year 2017. Choosing a scale of $\{2, ..., 7\}$ instead would have reallocated $ 400 million. The IDA allocation of countries like Afghanistan, Rwanda and Yemen would have changed by about one third under each transformation. The comparison between the two alternative scales - $\{0, ..., 5\}$ and $\{2, ..., 7\}$ - is even more staggering, resulting in IDA allocations that differ by a factor of two or more for countries like Afghanistan, Haiti and Yemen and by almost a factor of 10 for South Sudan.

These differences are troubling because by definition, ordinal inputs have no natural scale. Any choice of scale is arbitrary, which by extension implies a considerable degree of capriciousness in the allocations purely resulting from such decisions. It raises doubts about the claim that PBA systems are more robust, transparent and objective than alternative allocation methods.

The remainder of this paper is organised as follows: Section II contains a formal mathematical characterisation of the problem. Section III demonstrates its allocative implications empirically using the example of the World Bank PBA framework. Section IV discusses the ramifications of our findings for PBA frameworks more generally. Section V concludes.

II The Fundamental Problem

II.A Definitions

A.1 Performance Assessments

a) A ***performance assessment*** of (a class $\mathcal{C}$ of) subjects (e.g. countries) is an unambiguous assignment of a performance level to each of the relevant subjects, formally represented as a function $\pi : \mathcal{C} \to \mathrm{L} = \langle \mathcal{L} | \cong, \ldots \rangle$, where L is a structured set of performance levels $l \in \mathcal{L}$. Note that the minimal structure required for L to be a set is given by the identity relation '$\cong$' between the elements of the domain $\mathcal{L}$.

b) A performance assessment is ***numerical*** if it involves numerical performance levels: i.e. if $\mathcal{L} \subseteq \mathbb{R}$, with $\mathbb{R}$ the set of real numbers.

c) A ***performance measurement***, formally represented as a function $\mu : \mathcal{C} \to \mathrm{L}$, is a ('cardinal') performance assessment carried out by measuring a quantity $\mathrm{L} = \langle \mathcal{L} | \cong, \prec, \oplus, \odot \rangle$ associated with the subjects in question. The set $\mathcal{L}$ of cardinal levels of L is well-ordered ($\prec$), with magnitudes m that can be added: $m_1 \oplus m_2$, and multiplied with a scale factor: $r \odot m$ ($r \in \mathbb{R}^{>0}$, the set of positive real numbers).

d) A ***performance ranking*** $\omega : \mathcal{C} \to \mathrm{L}$ is an ('ordinal') performance assessment carried out in terms of a well-ordering $\mathrm{L} = \langle \mathcal{L} | \cong, \prec \rangle$, such as: $\langle \{A, B, C, D\} | \cong, \prec_{ABC} \rangle$, with $\{A, B, C, D\}$ a set of purely ordinal levels identified by the first 4 letters of the

Latin alphabet and $\prec_{ABC}$ the order of that alphabet.

A.2 Scales

a) A numerical ***scale*** for a structured set of (performance) levels $L = \langle \mathcal{L} | \cong, \ldots \rangle$(an 'L-scale') is a structure-preserving unambiguous assignment of numbers to the levels in $\mathcal{L}$, formally represented by a structure preserving function $\sigma : L \to \langle \sigma[\mathcal{L}] \subseteq \mathbb{R} | =, \ldots \rangle$, with $'\mathbb{R}'$ the set of real numbers, and $'\sigma[\mathcal{L}]'$ the image of $\mathcal{L}$ under σ. The scale is ***positive*** if it is restricted to positive numbers ($\mathbb{R}^{>0}$).

If σ_1 and σ_2 are L-scales (i.e. if $\sigma_1, \sigma_2 \in \mathfrak{S}^L$), then the function

$$\tau_{\sigma_1,\sigma_2} : \langle \sigma_1[\mathcal{L}] \subseteq \mathbb{R} | \ldots \rangle \to \langle \sigma_2[\mathcal{L}] \subseteq \mathbb{R} | \ldots \rangle, \tau_{\sigma_1,\sigma_2}(x) \overset{\text{def}}{=} \sigma_2 \left(\sigma_1^{-1}(x) \right) \qquad (1)$$

is called '(the L-) ***scale transformation*** (from σ_1 to σ_2)': $\tau_{\sigma_1,\sigma_2} \in \mathfrak{T}^L$.

b) ***Quantity (unit) scales***. Scalar quantities (e.g. length, mass) are measured with units (e.g. metre, gramme). The measurements are based on (physical) operations which can be interpreted arithmetically. In particular, there is a standard concatenation of units ($\oplus$) which not only gives rise to the cardinal structure of the quantity $L = \langle \mathcal{L} | \cong, \prec, \oplus, \odot \rangle$, but also to a unique numerical representation of the quantity for any chosen unit u, here referred to referred to as ***u-scales*** (e.g. metre scale, gramme scale): $\sigma_u : L \to \left\langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \right\rangle$, with

$$\sigma_{\mathrm{u}}(u) = \sigma_{\mathrm{u}}(1 \odot u) = 1 \text{ and } \sigma_{\mathrm{u}} \left[(r_1 \odot u) \oplus (r_2 \odot u) \right] = r_1 + r_2.$$

Any quantity level q can be used as a measurement unit, and for any two levels there is a unique transformation: $q_1 = r \odot q_2$ with $r \in \mathbb{R}^{>0}$ and $q_1, q_2 \in \mathcal{L}$. This means that if $\sigma_{\mathrm{u}_2}(u_1) = t \in \mathbb{R}^{\geq 0}$– say as in $\sigma_{\mathrm{g}}(1\mathrm{kg}) = 1000$ – then $\sigma_{\mathrm{u}_2}(q) = t \cdot \sigma_{\mathrm{u}_1}(q)$, i.e.

$\sigma_{\mathrm{g}}(q) = 1000 \cdot \sigma_{\mathrm{kg}}(q)$, with $q \in \mathcal{L}$.

c) An ***ordinal scale*** for a set $L = \langle \mathcal{L} | \cong, \prec \rangle$ of ordinal performance levels is an order-preserving unambiguous assignment of numbers to these levels, formally represented by a ***monotonic*** function $\sigma : L \to \langle \sigma[\mathcal{L}] | =, < \rangle$, with: $\sigma(l_1) < \sigma(l_2)$ if and only if $l_1 \prec l_2$.

A.3 Numerical Representations of Performance Assessments

A performance assessment π is numerically represented by applying an appropriate numerical scale $\sigma \in \mathfrak{S}^{L}$:

$$\varphi_{\sigma}^{\pi} \overset{\text{def}}{=} [\sigma \circ \pi] : \mathcal{C} \underset{\pi}{\to} \mathrm{L} = \langle \mathcal{L} | \cong, \ldots \rangle \underset{\sigma}{\to} \Big(\sigma[\mathcal{L}] \subseteq \mathbb{R} | =, \ldots \rangle ; \varphi_{\sigma}^{\pi}(c) \overset{\text{def}}{=} \sigma(\pi(c)) \; .$$

Applying an appropriate unit-scale σ_{u} to a performance measurement μ generates numerical representation of that measurement: $\varphi_{\mathrm{u}}^{\mu} : \mathcal{C} \to \langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \rangle$.

Applying an appropriate ordinal scale σ to a performance ranking ω generates a numerical representation of that ranking: $\varphi_{\sigma}^{\omega} : \mathcal{C} \to \langle \mathbb{R}^{\geq 0} | =, < \rangle$.

A.4 Assignments and Allocations

a) A ***reward allocation*** for a class $\mathcal{C}$ is an unambiguous assignment of rewards to the members of $\mathcal{C}$, formally represented as a function $\alpha : \mathcal{C} \to \mathrm{R}$, where R is the (structured) set of available rewards $\mathrm{R} = \langle \mathcal{R} | \cong, \ldots \rangle$. Rewards can be quantitative, meaning that R has the structure of a quantity $\langle \mathcal{R} | \cong, \prec, \oplus, \odot \rangle$ [A.1.c].

b) A reward allocation is ***performance-based*** if it involves the concatenation of a performance assessment $\pi : \boldsymbol{C} \to \mathrm{L} = \langle \mathcal{L}| \ldots \rangle$ with a ***reward assignment*** ρ^{π}:

$$\alpha^{\pi}_{\rho}(c) \overset{\text{def}}{\cong} \rho^{\pi}(\pi(c)), \text{ with } c \in \mathcal{C}.$$

Reward assignments ρ^{π} can be 'simple', i.e. independent of the performance assessment in the sense of assigning each performance level a reward defined without reference to the values of the underlying assessment π, or they can be 'complex', defined as a function of the the values of π.

c) ***Resource allocation***. A reward allocation α is said to allocate an amount a of a resource quantity $\mathrm{R} = \langle \mathcal{R}| \cong, \prec, \oplus, \odot \rangle$ to the members of $\mathcal{C}$ if the sum total of the values for all the members equals a, i.e. if:

$$\sum_{\mathcal{C}}^{\mathrm{R}} \alpha = a, \text{ where } \sum_{\mathcal{C}}^{\mathrm{R}} \alpha \overset{\text{def}}{=} \sum_{c \in \mathcal{C}}^{\mathrm{R}} \alpha(c)$$

d) A performance-based reward allocation $\alpha : \mathcal{C} \underset{\pi}{\to} L \underset{\rho}{\to} R$ is (performance) ***proportional***, if

$$\alpha\left(c_1\right) :^{R} \alpha\left(c_2\right) = \pi\left(c_1\right) :^{L} \pi\left(c_2\right), \text{ for all } c_1, c_2 \in \mathcal{C}.$$

A.5 Numerical Representations of Performance-based Reward Allocations

a) For any performance-based reward allocation $\alpha^{\pi}_{\rho} : C \underset{\pi}{\to} L \underset{\rho^{\pi}}{\to} R$, and any scale $\sigma \in \mathfrak{S}^{L}$, we can define a '***numerical representation***' ρ^{π}_{σ} of the involved reward assignment

$$\rho^{\pi}_{\sigma} : \left\langle \sigma[\mathcal{L}] \subseteq \mathbb{R}^{\geq 0} | =, <, +, \cdot \right\rangle \to \mathrm{R} = \langle \mathcal{R}| \cong, \ldots \rangle; \rho^{\pi}_{\sigma}(x) \overset{\text{def}}{\cong} \rho^{\pi}\left(\sigma^{-1}(x)\right).$$

This definition reflects the idea that the numerical representations $\sigma(l)$ of any

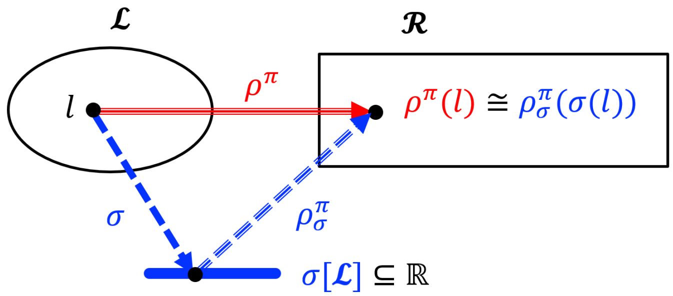

Figure 1

given performance level $l \in \mathcal{L}$ should all be assigned the same reward, namely $\rho^\pi(l)$, i.e.

$$\rho^\pi_\sigma(\sigma(l)) \cong \rho^\pi(l), \text{ for all } l \in \mathcal{L} \text{ and all } \sigma \in \mathfrak{S}^L. \quad (2)$$

b) Such a representation can, in turn, be used to form a 'scale expansion' $\alpha^\pi_{\rho\sigma}$ of α^π_ρ (i.e. the 'σ-expansion of α^π_ρ'):

$$\alpha^\pi_{\rho\sigma} : \boldsymbol{C} \underset{\pi}{\rightarrow} L \underset{\sigma}{\rightarrow} \left\langle \sigma[\mathcal{L}] \subseteq \mathbb{R}^{\geq 0} | \leq, +, \cdot \right\rangle \underset{\rho^\pi_\sigma}{\rightarrow} \mathrm{R}, \text{ which because}$$

$$\alpha^\pi_\rho \equiv \alpha^\pi_{\rho\sigma} \text{ for all } \sigma \in \mathfrak{S}^L,$$ [2]

can be interpreted as a ***numerical representation*** of α^π_ρ.

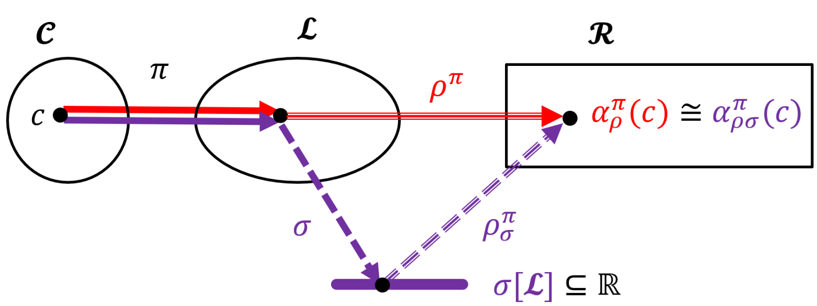

Figure 2

[2]**Proof**: Note that α^π differs from ρ^π only to the extent of the addition of the scale-independent mapping $\mathcal{C} \underset{\pi}{\rightarrow} L$. It is thus sufficient to show that $\rho^\pi(l) \cong \rho^\pi_\sigma(l) \forall \sigma \in \mathfrak{S}^L$: Let $l \overset{\text{def}}{=} \sigma^{-1}(x)$, then $\rho^\pi(l) = \rho^\pi\left(\sigma^{-1}(x)\right) = \rho^\pi_\sigma(x)$. q.e.d.

II.B A simple 'Reality Check'

The fact that all numerical representations $\sigma(l)$ of a given performance level $l \in \mathcal{L}$ are meant to be assigned the same reward, namely $\rho^\pi(l)$ (see equation (2)), gives rise to a very simple 'reality check' for being a numerical representation ρ^π_σ of a reward assignment:

$$[RC] \quad \rho^\pi_\sigma(x) \cong \rho^\pi_{\tau\circ\sigma}(\tau(x)), \text{ for all } x \in \sigma[\mathcal{L}] \text{ all } \sigma \in \mathfrak{S}^L, \text{ and all } \tau \in \mathfrak{T}^L.$$ [3]

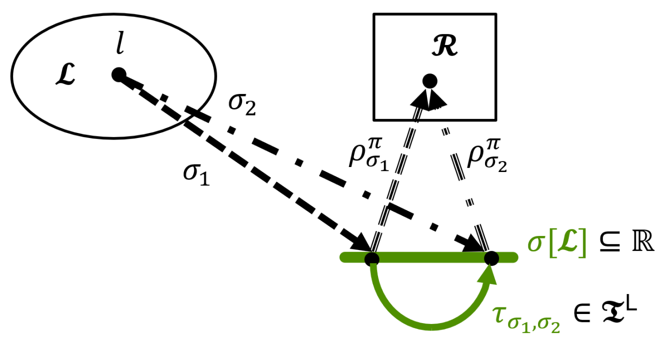

Figure 3

To illustrate the significance of this, consider the case where the underlying performance assessments are defined in terms weight measurements, using weight levels as performance levels.

Let $\sigma_{\mathrm{kg}}, \sigma_{\mathrm{t}} : \mathrm{L} = \langle \mathcal{W} | =, \prec, \oplus, \odot \rangle \to \langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \rangle$ be the (metric) kilogramme and tonne weight scales, and $\tau_{\mathrm{t,kg}} : \langle \mathbb{R}^{\geq 0} | \ldots \rangle \to \langle \mathbb{R}^{\geq 0} | \ldots \rangle$ the scale transformation from tonnes to kilograms, i.e. $\tau_{\mathrm{t,kg}}(x) \overset{\mathrm{def}}{=} \sigma_{\mathrm{kg}}\left(\sigma_{\mathrm{t}}^{-1}(x)\right) = 1000 \cdot x$. What does [RC] mean in this context?

Consider, for example, the case of $x = 5$. In this case [RC] amounts to the

[3] **Proof**: Let $l \overset{\mathrm{def}}{\cong} \sigma^{-1}(x)$, then
$\rho^\pi_{\tau\circ\sigma}(\tau(x)) \overset{\mathrm{def}}{\cong} \rho^\pi\left((\tau\circ\sigma)^{-1}[(\tau\circ\sigma)(l)]\right) \cong \rho^\pi(l) \cong \rho^\pi\left(\sigma^{-1}(x)\right) \cong \rho^\pi_\sigma(x)$. q.e.d.

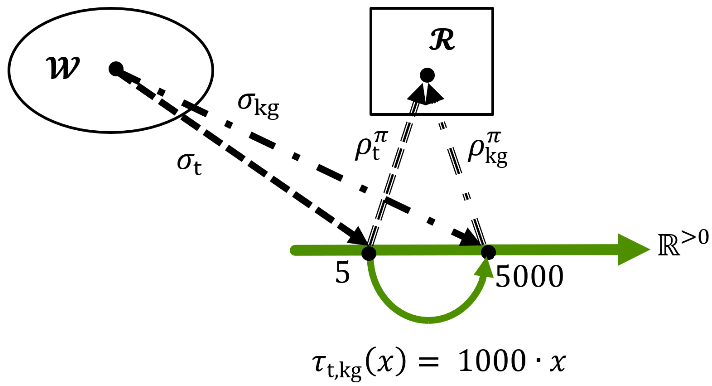

Figure 4

requirement that:

$$\rho^{\pi}_{\mathrm{t}}(5) \cong \rho^{\pi}_{\mathrm{kg}}(5000). \tag{3}$$

The 'meaning' of this equation is tied to the fact that 5 and 5000 are used to numerically represent a particular weight by way of the (metric) tonne and kilogram scales $\sigma_{\mathrm{t}}, \sigma_{\mathrm{kg}}$, respectively, i.e. the fact that:

$$5 \in \sigma_{\mathrm{t}}[\mathcal{L}], 5000 \in \sigma_{\mathrm{kg}}[\mathcal{L}], \text{ and } \sigma_{\mathrm{t}}^{-1}(5) \cong \sigma_{\mathrm{kg}}^{-1}(5000). \tag{4}$$

The standard convention to reflect this notationally is by adding the relevant unit symbols to the numerical representations to refer to the represented quantity levels, i.e.:

$$''5t'' \stackrel{\text{def}}{=} \sigma_{t}^{-1}(5),\ ''5000\text{kg}'' \stackrel{\text{def}}{=} \sigma_{\mathrm{kg}}^{-1}(5000), \text{ and generally } ''\text{xu}'' \stackrel{\text{def}}{=} \sigma_{\mathrm{u}}^{-1}(x). \tag{5}$$

Given that $\rho^{\pi}_{\sigma}(x) \stackrel{\text{def}}{\cong} \rho^{\pi}\left(\sigma^{-1}(x)\right) \cong \rho^{\pi}(x\ \mathrm{u})$, , (3) thus means that

$$\rho^{\pi}(5\mathrm{t}) \cong \rho^{\pi}(5000\mathrm{kg}), \tag{6}$$

which, given that $5\mathrm{t} = 5000\mathrm{kg}$ (see (4)), simply reflects the idea that reward assignments ρ^π based on a given performance assessment π are meant to assign a unique reward to each of the relevant performance levels. [RC] ensures that this is reflected in the numerical representations of these assignments, which in turn is tantamount to the idea that the choice of scale is arbitrary and should have no impact on reward assignments.

But what has this got to do with a 'reality check'? As mentioned in [A.2.b], quantity scale transformations are similarity transformations, i.e.: $\tau(x) = r \cdot x$ with $r \in \mathbb{R}^{>0}$. According to [RC], a function $\varphi : \langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \rangle \to \mathrm{R}$ can only be a scale *in*dependent[4] numerical representation of an award allocation based on performance measurements $\mu : \mathcal{C} \to \mathrm{L}$ if:

$$\varphi(x) \cong \varphi(r \cdot x) \text{ for all } r \in \mathbb{R}^{>0}. \tag{7}$$

As mentioned above, this reflects the intuition that the choice of a measurement unit should not affect the resulting measurement-based allocations, in the same way in which, say, it is not meant to have an effect on the numerical formulation of laws of physics. Take Einstein's famous equation: $E = m \cdot c^2$, linking energy E and mass m with the speed of light c. The standard SI units of mass, speed and energy are *kilogram* [kg], *metres per second* [m/s], and *Joule*, or *Newton metre*:$[\mathrm{J}] = \left[\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2\right]$. But Einstein's equation is equally true if, say, mass is measured in grams [g] and consequently energy in $[\mathrm{mJ}] \stackrel{\text{def}}{=} \left[\mathrm{g} \cdot \mathrm{m}^2/\mathrm{s}^2\right]$ because it satisfies (7), and $\mu_{\mathrm{g}}(x) = 1000 \cdot \mu_{\mathrm{kg}}(x)$.[5] Indeed, (7) has a physical correlate: mathematical

[4] $\varphi(x) \cong \rho^\pi_\sigma(x)$ for all $\sigma \in \mathfrak{S}^L$.

[5] **Proof**: Given that 1kg = 1000g, so 1J = 1000mJ, we have that $E[\mathrm{mJ}] = 1000 \cdot E[\mathrm{J}]$ and also that $m[\mathrm{kg}] \cdot c[\mathrm{m/s}]^2 = 1000 \cdot m[\mathrm{g}] \cdot c[\mathrm{m/s}]^2$, which means: $E[\mathrm{J}] = m[\mathrm{kg}] \cdot c[\mathrm{m/s}]^2$ if and only if $E[\mathrm{mJ}] = m[\mathrm{g}] \cdot c[\mathrm{m/s}]^2$.

formula[6] $\Phi(x_1, \ldots, x_n)$ can only be a representation of a physical law if

$$\Phi(x_1 \ldots, x_k, \ldots, x_n) \text{ iff } \Phi(x_1, \ldots, r \cdot x_k, \ldots, x_n) \text{ for all } r \in \mathbb{R}^{>0} \text{ and } 1 \leq k \leq n. \tag{8}$$

A mathematical equation cannot reflect a 'real' equation between physical magnitudes if it is not invariant under the relevant scale transformations associated with the switching measurement units. Mathematical functions as representations of performance-based reward allocations are equally unable to do justice to their intended purpose if the outcome depends on the (arbitrary) choice of a numerical scale.

II.C Normalized Performance-based Resource Allocations

Performance-based resource allocations are most commonly defined by 'normalizing' numerical representations of the involved performance assessments.

C.1 Definitions

The ***normalization of a numerical representation*** $\varphi_\sigma^\pi : \mathcal{C} \to \left\langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \right\rangle$ ***of a performance assessment*** $\pi : \mathcal{C} \to \mathrm{L} = \langle \mathcal{L} | \ldots \rangle$ with an L-scale σ, is defined as:

$$\bar{\varphi}_\sigma^\pi : \left\langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \right\rangle \to \langle [0,1] | =, <, +, \cdot \rangle ; \bar{\varphi}_\sigma^\pi(x) \stackrel{\text{def}}{=} x / \sum_{\mathcal{C}} \varphi_\sigma^\pi .^7 \tag{9}$$

Given an amount a of a resource quantity $\mathrm{R} = \langle \mathcal{R} | \cong, \prec, \oplus, \odot \rangle$ this normalized numerical representation of the underlying performance assessment π can be used

[6] Say an n-variable equation.

[7] $\Sigma_{\mathcal{C}} \varphi_\sigma^\pi \stackrel{\text{def}}{=} \sum_{c \in \mathcal{C}} \varphi_\sigma^\pi(c)$.

to define a ***normalized allocation*** of a as follows:

$$\begin{aligned} &\bar{\alpha}^{\pi}_{\sigma}[a] : \mathcal{C} \underset{\varphi^{\pi}_{\sigma}}{\rightarrow} \left\langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \right\rangle \underset{\bar{\varphi}^{\pi}_{\sigma}}{\rightarrow} \langle [0,1] | =, <, +, \cdot \rangle \underset{\odot a}{\rightarrow} \mathrm{R};^{8} \\ &\bar{\alpha}^{\pi}_{\sigma}[a](c) \cong a \odot [\bar{\varphi}^{\pi}_{\sigma} \circ \varphi^{\pi}_{\sigma}](c). \end{aligned} \tag{10}$$

But is this resource allocation actually 'based on the performance assessment π' in the sense discussed in [A.5]? In other words, is

$$\bar{\varphi}^{\pi}_{\sigma}[a] : \left\langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \right\rangle \rightarrow \mathrm{R}; \bar{\varphi}^{\pi}_{\sigma}[a](x) \cong a \odot \bar{\varphi}^{\pi}_{\sigma}(x)$$

a (σ-based) numerical representation $\bar{\rho}^{\pi}_{\sigma}$[A.5.a] of an underlying (π-based) reward assignment $\bar{\rho}^{\pi} : \mathrm{L} \rightarrow \mathrm{R}$?

In discussing this, we shall focus on the two fundamental types of performance assessments, that is performance measurements [A.1.c], and performance rankings [A.1.d]. In either case, we begin by applying our check [RC], which in this context is tantamount to:

$$[RCN] \quad \bar{\varphi}^{\pi}_{\sigma}(x) = \bar{\varphi}^{\pi}_{\tau\circ\sigma}(\tau(x)) \text{ for all } x \in \sigma[\mathcal{L}], \text{ all } \sigma \in \mathfrak{S}^{L}, \text{ and all } \tau \in \mathfrak{T}^{L}.$$

C.2 Quantitative Normalized Resource Allocations

Consider, in a first instance, the case of a quantitative performance assessment, that is a performance measurement $\mu : \mathcal{C} \rightarrow \mathrm{L} = \langle \mathcal{L} | =, \prec, \oplus, \odot \rangle$, using the magnitudes of a quantity as performance levels. Given that scale transformations for a quantity are simply similarity transformations of the form $\tau(x) = r \cdot x$ (for some $r \in \mathbb{R}^{>0}$)

[8] $\bar{\alpha}^{\pi}_{\sigma}[a] : \mathcal{C} \underset{\pi}{\rightarrow} \mathrm{L} = \langle \boldsymbol{L} | \ldots \rangle \underset{\sigma}{\rightarrow} \langle \mathbb{R}^{\geq 0} | =, <, +, \cdot \rangle \underset{\bar{\varphi}^{\pi}_{\sigma}}{\rightarrow} \langle [0,1] | =, <, +, \cdot \rangle \underset{\odot a}{\rightarrow} \mathrm{R}.$

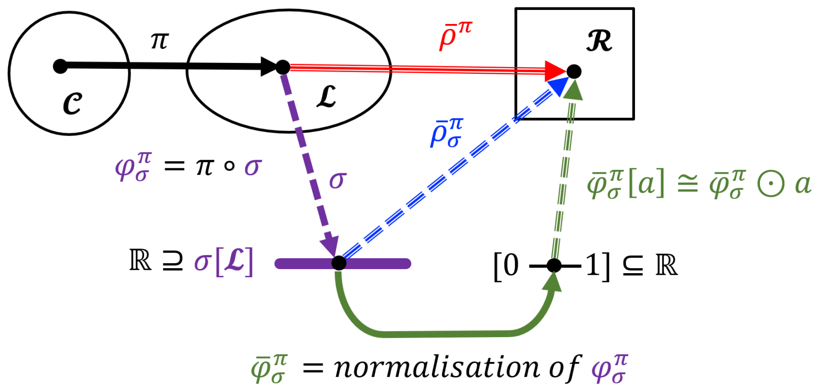

Figure 5

[A.2.a], it is easy to see that $\bar{\varphi}^{\mu}_{\sigma}$ does satisfy [RCN].[9] However, given that our reality check only amounts to a necessary condition, conformity with [RCN] does not by itself entail the existence of a reward assignment numerically represented by $\bar{\rho}^{\mu}_{\sigma}[a]$.

But it is easy to define such an assignment by normalizing the underlying performance assessment itself, that is by normalizing the performance measurement μ:

$$\bar{\mu} : \mathrm{L} \to \langle [0,1] | =, <, +, \cdot \rangle ; \bar{\mu}(l) \overset{\text{def}}{=} l :^{L} \sum_{\mathcal{C}}^{\mathrm{L}} \mu \tag{11}$$

Given that this function can be interpreted as being numerically represented by $\bar{\varphi}^{\mu}_{\sigma}$ in the sense that:

$$\bar{\varphi}^{\mu}_{\sigma}(x) = \bar{\mu}\left(\sigma^{-1}(x)\right), ^{10} \tag{12}$$

it follows that $\bar{\rho}^{\mu}_{\sigma}[a]$ is indeed a numerical representation of $\rho^{\bar{\mu}}[a](l) \overset{\text{def}}{=} a \odot \bar{\mu}(l)$ in the sense that $\bar{\rho}^{\mu}_{\sigma}[a](x) = \rho^{\bar{\mu}}[a]\left(\sigma^{-1}(x)\right)$,[11] and consequently that:

$$\alpha^{\bar{\mu}}[a] : \mathcal{C} \underset{\mu}{\to} \mathrm{L} \underset{\rho^{\bar{\mu}}[a]}{\to} \mathrm{R} \tag{13}$$

[9]**Proof**: $\bar{\varphi}^{\mu}_{\tau\circ\sigma}(\tau(x)) \overset{\text{def}}{=} \tau(x) / \sum_{c \in \mathcal{C}} \varphi^{\mu}_{\tau\circ\sigma}(c) = r \cdot x / \sum_{c \in \mathcal{C}} r \cdot \varphi^{\mu}_{\sigma}(c) \overset{\text{def}}{=} \bar{\varphi}^{\pi}_{\sigma}(x)$. q.e.d.

[10]**Proof**: Assume μ is measured with unit u, i.e. $\sigma = \sigma_{\mathrm{u}}$. Then $\mu(\mathrm{c}) = \mathrm{u} \odot \varphi^{\mu}_{\sigma_{\mathrm{u}}}(c)$, and hence $\Sigma^{\mathrm{L}}_{\mathcal{C}} \mu = \mathrm{u} \odot \sum_{\mathcal{C}} \varphi^{\mu}_{\sigma_{\mathrm{u}}}$. Therefore (see Fig. 6), $\bar{\mu}(\sigma_{\mathrm{u}}^{-1}(x)) = (l / \sum^{\mathrm{L}}_{\mathcal{C}} \mu) = \mathrm{u} \odot \sigma_{\mathrm{u}}(l) / \mathrm{u} \odot \sum_{\mathcal{C}} \varphi^{\mu}_{\sigma_{\mathrm{u}}} = x / \sum_{\mathcal{C}} \varphi^{\mu}_{\sigma_{\mathrm{u}}} = \bar{\varphi}^{\mu}_{\sigma_{\mathrm{u}}}$. q.e.d.

[11]**Proof**: $\bar{\rho}^{\pi}_{\sigma}[a](x) \overset{\text{def}}{\cong} a \odot \bar{\varphi}^{\pi}_{\sigma}(x) \cong a \odot \bar{\mu}\left(\sigma^{-1}(x)\right) \overset{\text{def}}{\cong} \rho^{\bar{\mu}}[a]\left(\sigma^{-1}(x)\right)$. q.e.d.

is the 'real' normalized resource allocation[12] represented by $\bar{\alpha}^{\mu}_{\sigma}[a]$ (see (10)) for any L-scale σ, in the sense that:

$$\alpha^{\bar{\mu}}[a](c) = \bar{\alpha}^{\pi}_{\sigma}[a](c), \text{ for all } c \in \mathcal{C} \text{ and all } \sigma \in \mathfrak{S}^{L}. \tag{14}$$

An important point here is that the values of $\bar{\mu}$ are not just numbers between 0 and 1, they represent genuine ***shares of a total amount***, namely the sum of all performance magnitudes assigned to the elements of $\mathcal{C}$ (i.e. $\Sigma^{L}_{\mathcal{C}}\mu$) for which it makes sense to say, for example, that "the amount l makes up $100\bar{\mu}(l)\%$ of $\sum^{L}_{c}\mu$" or "the share of l in $\sum^{L}_{c}\mu$ is $\bar{\mu}(l)$".

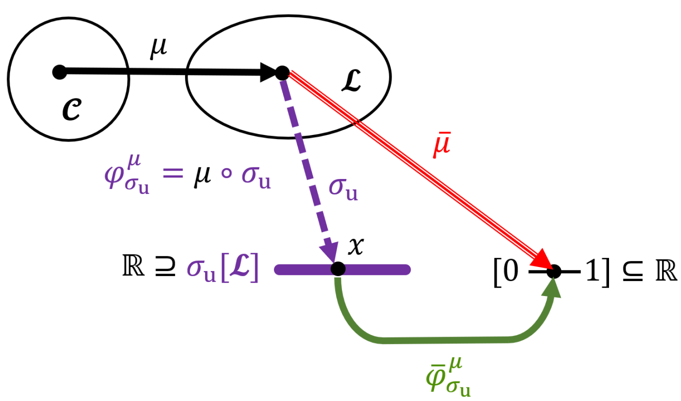

Figure 6

Moreover, normalised quantitative resource allocations are proportional, in the sense of [A.4.d]:

$$\alpha^{\bar{\mu}}[a]\,(c_1) :^{\mathrm{R}} \alpha^{\bar{\mu}}[a]\,(c_2) \text{ iff } \mu\,(c_1) :^{\mathrm{L}} \mu\,(c_2) \text{ for all } c_1, c_2 \in \mathcal{C}. \tag{15}$$

Given Aristotle's dictum that "what is just is what is proportional, and what is unjust

[12]**Lemma**: $\bar{\alpha}^{\mu}[q]$ is an allocation of q, i.e. $q = \sum^{Q}_{c\in\mathcal{C}} \bar{\alpha}^{\mu}[q]$,
Proof: Let $\sum\mu \overset{\text{def}}{=} \sum^{L}_{c\in\mathcal{C}} \mu(c)$. Then $\sum^{Q}_{c\in\mathcal{C}} \bar{\alpha}^{\mu}[q] \overset{\text{def}}{=} \sum^{Q}_{c\in\mathcal{C}} q \odot [\mu(c) :^{L} \sum\mu] = q \odot \left[\sum^{Q}_{c\in\mathcal{C}} \mu(c)\right] :^{L} \sum\mu = q$. q.e.d.

is what violates the proportion"[13] this means that if the performance measure μ (say economic size measured as GDP or population size) is accepted as fair, then the normalised resource allocation based on this measure will, according to Aristotle, be a fair allocation.

C.3 Ordinal Normalized Resource Allocations

Turning now to the case of an ordinal performance assessment, that is a performance ranking (see [A.1.d]), the situation is markedly different. Take the following example, drawing on J.R. Tolkien's fictional realm of Middle Earth. More precisely, consider a four-level performance ranking of three of the Middle Earth kingdoms: $\mathcal{ME} =$ {Gondor, Mordor, Rohan}

$$\omega_{AD} : \mathcal{ME} \to \mathrm{AD} = \left\langle \{A, B, C, D\} \right| \cong, \prec_{\alpha\beta} \rangle \tag{16}$$

as given in Table 1.b. Given the purely ordinal structure of the chosen performance levels, it follows that the relevant class of scale transformations $\mathfrak{T}^{AD}$ is the class of monotonic functions on the positive real numbers $\mathbb{R}^{\geq 0}$. This includes similarity transformations, and given our discussion in the preceding section, it will not surprise that the normalization of a numerical representation $\bar{\varphi}_{\sigma}^{\omega_{AD}}$ of a performance ranking such as ω_{AD}, i.e. $\bar{\varphi}_{\sigma}^{\omega_{AD}}(x) = x / \sum_{\mathcal{ME}} \varphi_{\sigma}^{\omega_{AD}}$, will satisfy [RCN] for similarity transformations, such as $\tau_{\sigma_0,\sigma_1}(x) = 2x$ (see Table 1.a) with

$$\bar{\varphi}_{\sigma_0}^{\omega_{AD}}(x) = \bar{\varphi}_{\sigma_1}^{\omega_{AD}}(2x), \text{ for } x = 0, 1, 2, 3. \tag{17}$$

However, the situation changes fundamentally in the case of monotonic transformations known as '***translations***': $\tau(x) = x + t$, such as $\tau_{\sigma_0,\sigma_2}(x) = x + 1$, where $\bar{\varphi}_{\sigma}^{\omega_{AD}}$

[13] Aristotle, *Nicomachean Ethics*: Bk V: Ch.3.

clearly fails our reality check [RCN],[14] because the reward of, say, Rohan would differ under the two normalized scale representations $\bar{\varphi}_{\sigma_0}^{\omega_{AD}}$ and $\bar{\varphi}_{\sigma_2}^{\omega_{AD}}$ (see Table 1.c).

The Middle Earth Performance Assessment							
a. Numerical Scales							
	Level	σ_0	σ_1	σ_2			
	A	0	0	1			
	B	1	2	2			
	C	2	4	3			
	D	3	6	4			
b. Performance Ranking					**c. Normalized**		
Φ:	ω_{AD}	$\varphi_{\sigma_0}^{\omega_{AD}}$	$\varphi_{\sigma_1}^{\omega_{AD}}$	$\varphi_{\sigma_2}^{\omega_{AD}}$	$\bar{\varphi}_{\sigma_0}^{\omega_{AD}}$	$\bar{\varphi}_{\sigma_1}^{\omega_{AD}}$	$\bar{\varphi}_{\sigma_2}^{\omega_{AD}}$
Gondor	B	1	2	2	0.2	0.2	0.25
Mordor	B	1	2	2	0.2	0.2	0.25
Rohan	D	3	6	4	0.6	0.6	0.5
$\Sigma_{\mathcal{ME}}\Phi$	n/a	5	10	8	1	1	1

Table 1

C.4 The Cardinal Fallacy

The fact that normalized numerical representations $(\bar{\varphi}_{\sigma}^{\omega})$ of performance rankings (ω) fail the relevant reality check [RCN] implies that ordinal normalised resource allocations:

$$\bar{\rho}_{\sigma}^{\omega}[q](x) \overset{\text{def}}{\cong} q \odot \bar{\varphi}_{\sigma}^{\omega}(x), \tag{18}$$

do ***not*** represent performance ranking based reward assignments. In particular, unlike in the cardinal case (15), there is no underlying normalised performance

[14] $\bar{\varphi}_{\sigma_0}^{\widehat{\omega}}(x) \neq \bar{\varphi}_{\sigma_2}^{\widehat{\omega}}(x+1)$, for $x = 0, 1, 2, 3$.

ranking:

$$\bar{\omega}_{AD}(l) \stackrel{\text{def}}{\cong} l :^{\text{AD}} \sum_{\mathcal{ME}}^{\text{AD}} \omega_{AD}, \tag{19}$$

simply because the alphabetically ordered set of four performance levels (AD) does not have the presupposed operations. In particular "$\Sigma_{\mathcal{ME}}\omega_{AD}$", that is to say "B plus *B* plus *D*" (see Table 1.b), is simply meaningless. Unlike in the quantitative case (12) the values of $\bar{\varphi}_{\sigma_0}^{\omega_{AD}}$ are just numbers between 0 and 1, they do ***not*** represent shares of a quantitative amount. To pretend they do, in using them to define normalized allocations [C.1] is to commit what we call the '***cardinal fallacy***' of treating purely ordinal structures as if they were cardinal in nature. The results, as we shall demonstrate in the next section, can be significant.

III Allocative Impacts of the Cardinal Fallacy - A Real World Example

The World Bank's *Country Performance and Institutional Assessment* (CPIA) provides the basis of one of the most prominent and most sophisticated PBA frameworks currently in use. Specifically, CPIA is used to guide the allocation of concessional finance via the International Development Association (IDA), which is one of the largest, if not *the* largest PBA scheme of its kind in monetary terms.[15] For this reason, we demonstrate the allocative impacts of committing a cardinal fallacy using a simplified World Bank PBA formula, abstracting from a number of complexities such as performance independent base allocations and special funds for high vulnerability countries/regions immaterial for the present purposes.

The CPIA consists of a set of 16 criteria grouped into four clusters: economic

[15]In the IDA18 3-year replenishment cycle, total resources amount to $ 75 billion – although not all of these funds are allocated based on performance – see World Bank (2017b).

management (CPIA_A), structural policies (CPIA_B), social inclusion (CPIA_C) and public sector management (CPIA_D).[16] Every country c receives a score ranging from 1 (worst) to 6 (best) for each criterion. More precisely, in our terminology, the CPIA involves 16 ordinal rankings ω of countries in terms of 11 ordinal performance levels $\mathcal{L} = \{l_1, \ldots, l_{11}\}$, numerically represented by a scale $\sigma[\mathcal{L}] = \{1, 1.5, 2, 2.5, \ldots, 6\}$.

These numerically scaled rankings are used to produce what is known as the country performance rating, or the performance-based component φ_σ^ω of the IDA PBA formula:

$$\varphi_\sigma^\omega(c) = \left[0.24 \cdot \overline{CPIA}_{AC}(c) + 0.68 \cdot \overline{CPIA}_D(c) + 0.08 \cdot PPI(c)\right], \tag{20}$$

where PPI is a portfolio performance index and $\overline{CPIA}_{AC}(c)$ denotes the arithmetic average over all the CPIA criteria in clusters A to C for country c, with $\overline{CPIA}_D(c)$ being defined analogously.

There is also a needs based component in the IDA PBA formula, defined in terms of (measurements of) population size (Pop) and Gross National Income (GNI) figures:

$$\varphi^\mu(c) = \text{Pop}(c) \cdot [GNI(c)/Pop(c)]^{-0.125}. \tag{21}$$

Combined, the two factors produce the numerical performance assessment formula for the IDA resource allocation framework:

$$\varphi_\sigma^\pi(c) = \varphi_\sigma^\omega(c)^4 \cdot \varphi^\mu(c).^{17} \tag{22}$$

In this setting, country c's share (= normalised allocation) of the funding envelope

[16]See Annex 2 in World Bank (2017b).

[17]The functional form of the numerical performance assessment presented here is based on the PBA framework for the IDA17 replenishment period (see World Bank (2010, 2017b)). For the IDA18 replenishment cycle, the exponent of $\varphi_\sigma^\omega(c)$ is reduced from 4 to 3. However, this change has no

of size a is given by (see [C.1]):

$$\bar{\alpha}_{\sigma}^{\pi}[a] = \varphi_{\sigma}^{\pi} \circ \bar{\varphi}_{\sigma}^{\pi}[a], \tag{23}$$

where φ_{σ}^{π} is the IDA performance assessment formula (based on scale σ), and $\bar{\varphi}_{\sigma}^{\pi}[a]$ is the allocation of the resource envelope a in proportion to its normalised 'performance shares'

$$\bar{\varphi}_{\sigma}^{\pi}[a] = \frac{\varphi_{\sigma}^{\pi}(c)}{\Sigma c \varphi_{\sigma}^{\pi}} \odot a, \tag{24}$$

implying that the reward function allocates funds in proportion to the numerical 'value share' of the performance assessment of a country.

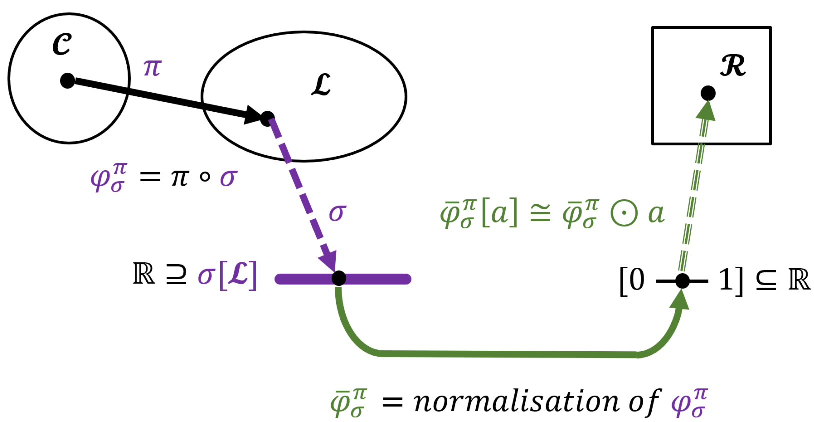

Figure 7

But because the CPIA criteria are ordinal in nature, the magnitude of these numerical values has no specific meaning. Committing the cardinal fallacy of ignoring the nature of the data has undesirable effects, which are most apparent if we consider two simple changes in scale from $\sigma[\mathcal{L}] = \{1, 1.5, 2, 2.5, \ldots, 6\}$ to $\sigma'[\mathcal{L}] = \{0, 0.5, 1, 1.5, \ldots, 5\}$ and $\sigma''[\mathcal{L}] = \{2, 2.5, 3, 3.5 \ldots, 7\}$.

The consequences of using these alternative scales are striking. Using data for 73 recipient countries from 2015 (the basis for 2017 allocations), Table 2 shows that

qualitative impact on our findings.

these simple scale transformations change the IDA allocations of countries like Afghanistan, Rwanda and Yemen by around one third compared to the current formula. Comparing the σ' and σ'' scales – $\{0,...,5\}$ and $\{2,...,7\}$, respectively – reveals differences of a factor of two or more for countries like Afghanistan, Haiti and Yemen. For South Sudan, the difference between the two scales is almost a factor of 10.[18]

The changes are equally significant in absolute monetary terms. Moving to a $\{0,...,5\}$ scale would have reallocated roughly $ 750 million to different recipients in FY 2017 according to our simplified model of the World Bank framework.[19] Moving in the other direction, a $\{2,...,7\}$ scale reallocates about $ 400 million relative to the baseline.

Accordingly, we find $ 1.15 billion in IDA flows changing destination when comparing the $\{0,...,5\}$ and $\{2,...,7\}$ scales.[20] This figure could be further increased for other, more dissimilar scales.

While the exact value of these numbers should not be taken at face value given

[18]All the CPIA data is sourced from World Bank (2017a). For 14 countries, missing GNI/capita data has been added to maximise sample size (source: https://data.worldbank.org/ - GNI/capita (Atlas method) series). Due to revisions, there are some minor differences between this series and the data used in World Bank (2017a), but they are immaterial for the purpose of our exposition. Eritrea, Sudan, Zimbabwe and Somalia have been dropped from the sample because they did not receive any funding in the sample period (due to 'inactive' status, or 'credits in non-accrual' in case of Eritrea, see p. 7 in World Bank (2017a)).

[19]These calculations are based on equation (23) and a total funding pot of SDR 9.176 billion converted at 1.424 USD/SDR (end of 2017 exchange rate as published on https://www.imf.org/external/np/fin/data/rms_sdrv.aspx) minus the base allocation of SDR 4 million per country, which amounts to roughly $ 12.6 billion (see World Bank, 2017a). In our simplified model, we do not consider any front- or backloading, regional or intra-regional reallocations, special assistance funding, disaster or all the additional funding distributed through the regional program (RP) or turn-around regime set-asides (TAR). We also disregard the discounting of grant allocations or adjustments due to debt cancellation under the Multilateral Debt Relief Initiative (MDRI) as well as more specific exceptions laid out in Annex II of World Bank (2014). The monetary difference between our model and the actual allocations can be found in Table 5 of Appendix B.

[20]To compare, in absence of the needs-based adjustment, i.e. if allocations were based on equation (20) alone (but adding the exponent of 4), this figure would amount to about $ 1.45 billion. For individual countries, the relative changes in the allocated share is still very similar to the case with the needs-based adjustment – see Table 3 in Appendix B.

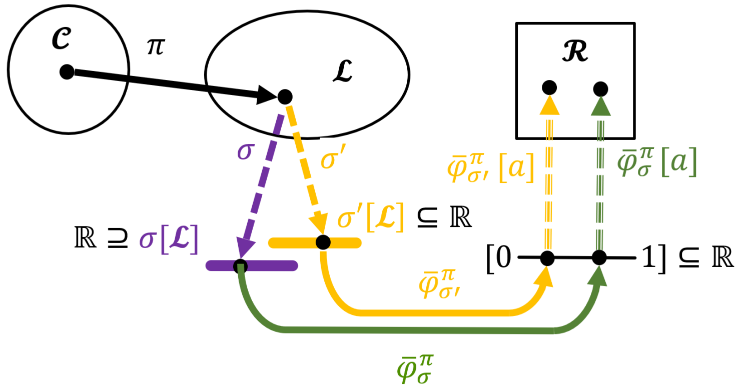

Figure 8

our simplifications, they are certainly in the correct order of magnitude. They tend to be a conservative estimate of the inherent ambiguity given that our scale transformations have been chosen for mere simplicity of exposition rather than to maximise differences in outcomes.

Figure 9 provides another perspective on the vulnerability of the current system to changes in scale once we allow for a broad set of monotonic scale transformations. It contains the distribution of allocative impacts for four countries located at different parts of the performance distribution under a simulation based on 100,000 random scales. Every scale was generated by drawing a series of random numbers from a uniform distribution on the interval (0,10) and taking the cumulative sums of these numbers as numerical scale levels to ensure these values are monotonically increasing as we move up the scale.[21] While the simulated average change in allocation per country is about 14.8%,[22] this figure masks the true heterogeneity

[21]Note that the allocations are invariant to scaling all the inputs by a (non-zero) constant (see also discussion on similarity transformations in section II.C.3), such that whether the numbers are drawn from (0,10) or $(0, 500000)$ does not affect the outcome. By taking the cumulative sum, we mean that a draw of, say, $\{4, 2.5, 1.1\}$ would result in a scale of $\{4, 4+2.5, 4+2.5+1.1\} = \{4, 6.5, 7.6\}$. Because the original $\{1, \ldots, 6\}$ CPIA-scale also contains half steps (i.e. 1.5, 2.5 and so on), we generate 11 distinct values in each draw.

[22]Our calculations include both the performance and the needs-based component of the CPIA formula. Without the needs-based component, the average change in allocation would be 15.05% instead.

Country	**Baseline share of total funding**	**Share of funding under 0-5 scale**	**Rel. Diff. to baseline (%)**	**Share of funding under 2-7 scale**	**Rel. Diff. to baseline (%)**
South Sudan	0.0010	0.0002	**-78%**	0.0019	**84%**
Central African Republic	0.0012	0.0006	**-50%**	0.0016	**37%**
Guinea-Bissau	0.0004	0.0002	**-50%**	0.0005	**37%**
Yemen, Republic of	0.0058	0.0031	**-47%**	0.0078	**34%**
Samoa	0.0003	0.0004	**41%**	0.0002	**-18%**
Afghanistan	0.0099	0.0062	**-37%**	0.0123	**24%**
Haiti	0.0032	0.0020	**-37%**	0.0040	**24%**
Cape Verde	0.0006	0.0008	**34%**	0.0005	**-16%**
Timor-Leste	0.0004	0.0002	**-33%**	0.0004	**21%**
Rwanda	0.0164	0.0215	**31%**	0.0141	**-14%**

Table 2: Allocative consequences of rescaling to a 0-5 or 2-7 scale under the World Bank PBA framework (17th replenishment cycle), including the needs-based adjustment factor. Only the 10 countries with the largest response are listed here - see Appendix B for the full table.

in observable patterns. We can see that the most drastic changes are produced for countries either at the bottom (South Sudan) or the top (Samoa) of the World Bank's country performance rating, covering a range spanning a multiple of the respective baseline allocation. But countries very close to the average rating like Nepal are still exposed to a wide range of potential allocative outcomes caused entirely by the particular choice of scale.

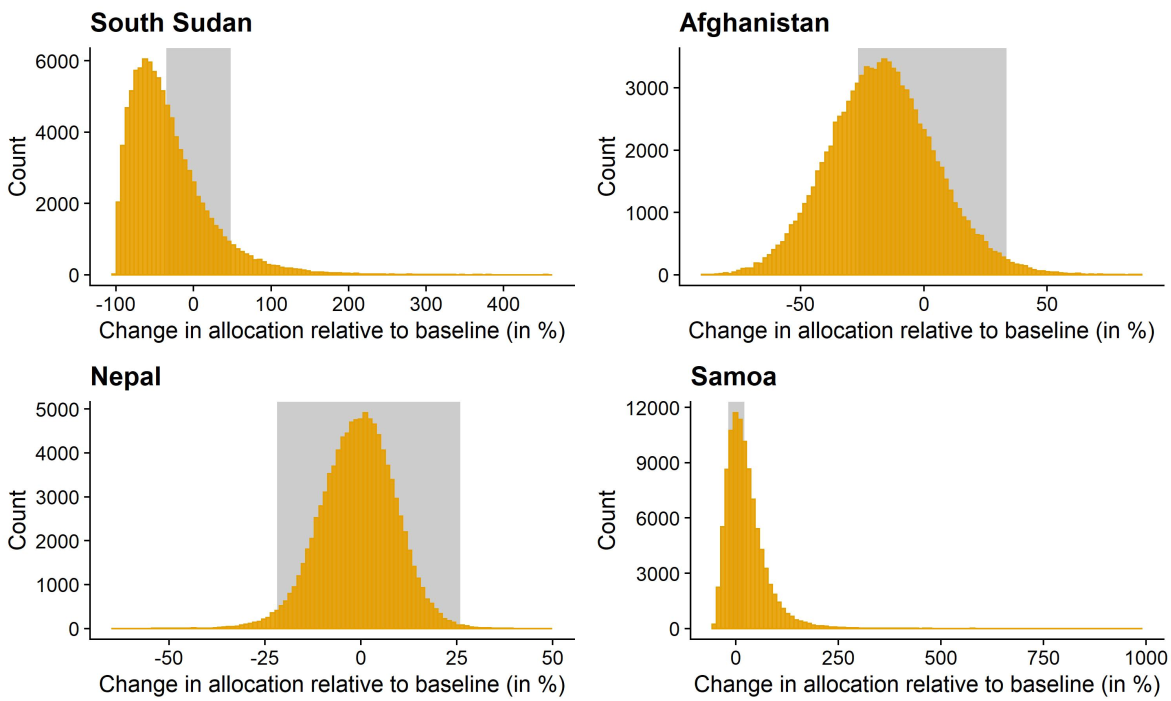

Figure 9: Distribution of allocative impacts of employing 100,000 random scales. Shaded areas mark the interval of the resulting allocations following an increase/decrease in performance rating φ_σ^ω by 0.5 standard deviations for that country in the baseline scale.

To put this in context, Figure 9 also contains information on the range of allocations under the baseline if the country had been assigned a higher or lower performance rating (± 0.5 standard deviations of φ_σ^ω). The length of this interval covers a one standard deviation performance difference, which is sizeable given that 30 out of 73 countries in our sample are contained in an interval of the same size around the performance mean. But even at this order of magnitude, our simulations show considerable mass outside the grey areas. In other words: For many countries – particularly those towards either end of the performance distribution – a change in scale can matter far more than a change in actual performance.

IV Discussion

There are ways in which rankings can be 'structurally augmented' so as not to fall foul of such arbitrary changes. However, it is beyond the scope of this paper to discuss the assumptions best suited to bridge the gap between ordinal performance rankings and cardinal rewards, especially as it would involve both taking a position on the assumed true nature of performance as well as a normative judgement on how to reward a particular performance. The simplest way of complying with our reality check is by way of a simple performance-based reward allocation $\alpha_{\rho}^{\pi} : \mathcal{C} \underset{\pi}{\rightarrow} \mathrm{L} \underset{\rho^{\pi}}{\rightarrow} \mathrm{R}$ (see [A.4.b]) which does not depend on any particular scale σ, but directly on the underlying set of performance levels L and thus satisfies [RC] and [RCN] by definition.[23] Returning to the Middle Earth example, we can assign rewards of, say, $\{1, 2, 3, 4\}$ to the ordinal performance levels $\{A, B, C, D\}$ – or whichever way they may be labelled – without reference to a specific scale. Consequently, the scale-invariance property of such a reward scheme allows these rewards to be chosen in a transparent way, be it based on theoretical considerations, or be it based on an explicit policy decision on how different performance levels should be rewarded relative to each other.

One might argue that some choice of scale is necessary, and that the resulting

[23] By definition $\alpha_{\rho}^{\pi} \cong \alpha_{\rho\sigma}^{\pi} \forall \sigma \in \mathfrak{S}^{L}$ in the present example. And by the above definition, we also have $\rho_{\sigma}^{\pi}(x) \cong \rho^{\pi}(x) \forall \sigma \in \mathfrak{S}^{L}$, which implies $\rho_{\sigma}^{\pi}(x) \cong \rho_{\tau\circ\sigma}^{\pi}(\tau(x))$ for all $x \in \sigma[\mathcal{L}]$ all $\sigma \in \mathfrak{S}^{L}$, and all $\tau \in \mathfrak{T}^{L}$.

allocation can be 'tweaked' until deemed 'satisfactory' by adjusting weighting parameters (e.g. the exponent of equation (23) or the weights in (20)). This argument is inherently flawed: There is no requirement to assign numbers to ordinal performance levels and to directly convert performance assessments based on such numbers into monetary allocations relying on one single formula. Doing so leads to (avoidable) problems: Setting aside the question of how to determine what a 'satisfactory' allocation means in this context, the chosen scale and functional form of the reward function imposes clear limits to any parameter 'tweaking'.

This is most apparent if we again consider the unidimensional Middle Earth performance ranking. Assume the relevant Middle Earth policy makers initially agree that a performance level of D, C, and B should imply allocation bonuses of 60%, 40% and 20% relative to A. For a numerical scale $\sigma_0 = \{0, 1, 2, 3\}$, this would imply a reward assignment of the form $\rho^{\pi}_{\sigma_0}(x) = 1 + 0.2x$, for $x = \{0, 1, 2, 3\}$.[24]

Now suppose new research highlights that top performers should get even more resources, and hence the corresponding bonus for D should be increased from 60% to 80%. Under the scale σ_0, the parameters of the above reward functions are unable to produce the suggested allocation. 'Tweaking' the existing parameters might produce something like $\rho^{\pi}_{\sigma_0}(x) = 1 + \mathbf{0.25}x$, which only approximates the allocation deemed desirable. Alternatively, the functional form could be overhauled entirely to $\rho^{\pi}_{\sigma_0}(x) = 1 + \mathbb{1}\{x > 0\}(0.1 * 2^x)$, which is neither particularly intuitive, nor would it be flexible enough to accommodate future revisions which do not fit into this very specific functional form. Again, assigning rewards of $\{1, 1.2, 1.4, 1.8\}$ directly to $\{A, B, C, D\}$ avoids both complex changes to a functional form aimed at a specific numerical scale as well as the necessity of any haphazard 'tweaking' of parameters.

In any case, the World Bank PBA system provides an interesting point of depar-

[24]Note that these rewards can be normalized to represent shares of a total funding envelope (see [II.C])

ture for future work in this area. It contains the added complication of not only having a simple ordinal performance indicator, but a whole set of ordinal performance rankings for different categories belonging to the same subject. The degree to which such rankings can be aggregated into a 'meaningful' one-dimensional performance is still actively debated (see Greco et al. (2019) for a recent overview), and shall be determined outside of the present contribution.

V Conclusion

Committing the 'cardinal fallacy' of treating ordinal structures as if they were cardinal in nature can have profound real-world implications in performance-based resource allocation frameworks. We provide a formal characterisation of the problem, from which we are able to derive a simple 'reality check': the choice of numerical scale associated with such performance levels should not affect the resulting allocation.

And yet it does in practice: Relying on data from the World Bank International Development Agency's resource allocation framework, we demonstrate that a simple rescaling of the ordinal input components strongly affects the resulting allocations, even though the available information on the underlying performance remains exactly the same. We show that simply changing the scale of the ordinal performance ranking from $\{1, \ldots, 6\}$ to $\{0, \ldots, 5\}$ would have reallocated some \$ 750 million to different recipients in the fiscal year 2017, according to our stylised version of the World Bank framework. Some countries would have seen their allocated shares fall by more than a third (e.g. -37% for Afghanistan or -50% for the Central African Republic), with other shares increasing by a similar margin (e.g. +31% for Rwanda and +41% for Samoa). Changing to a $\{2, \ldots, 7\}$ scale would have seen funding changes worth \$ 400 million in the opposite direction.

These impacts call for an alternative allocation mechanism not prone to such behaviour. While it is beyond the scope of this paper to make specific recommendations on a given resource allocation system, we hope it provides the necessary basis from which alternative allocation mechanisms can be devised, such that committing the cardinal fallacy will no longer play an integral role in the allocation of funds dedicated to help the poorest people in the world.

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A Glossary of Notation

A.1. Performance Assessments

$S = \langle \mathcal{S} | \ldots \rangle$: a ***structured set*** with domain $\mathcal{S}$ and the structure given by the listed relations and operations ("...") , such as the structured set of real numbers: $\langle \mathbb{R} | =, <, +, \cdot, / \rangle$, with the numerical identity $=$ and order $<$, and the operations of numerical addition ($+$), multiplication ($\cdot$), and division ($/$). $\mathbb{R}^{\geq 0}$: the set of non-negative real numbers (i.e. greater or equal to zero). [A.1]

$\mathcal{C}$: class of subjects to be assessed. [A.1.a]

$\mathrm{L} = \langle \mathcal{L} | \cong, \ldots \rangle$: structured set of performance levels. [A.1.a]

$\pi : \mathcal{C} \rightarrow \mathrm{L}$: ***performance assessment***. [A.1.a]

$\mu : \mathcal{C} \rightarrow L$: cardinal performance assessment ('***performance measurement***'), with $L = \langle \mathcal{L} | \cong, \prec, \oplus, \odot \rangle$ a set of ***cardinal*** (i.e. quantitative) ***performance levels***. [A.1.c]

$\omega : \mathcal{C} \rightarrow \mathrm{L}$: ordinal performance assessment ('***performance ranking***') with $\mathrm{L} = \langle \mathcal{L} | \cong, \prec \rangle$ a set of (purely) ***ordinal performance levels***. [A.1.d]

A.2. Scales

$\sigma : L = \langle \mathcal{L} | \ldots \rangle \rightarrow \langle \sigma[\mathcal{L}] \subseteq \mathbb{R} | \leq, +, \cdot \rangle$: numerical ***scale*** for L ('L-scale'). [A.2.a]

$\mathfrak{S}^L$: the set of L-***scales*** σ. [A.2.a]

σ_{u} : ***unit quantity scale*** based on unit 'u'. [A.2.b]

$\tau_{\sigma_1,\sigma_2} : \langle \sigma_1[\mathcal{L}] \subseteq \mathbb{R} | =, <, +, \cdot \rangle \rightarrow \langle \sigma_2[\mathcal{L}] \subseteq \mathbb{R} | =, <, +, \cdot \rangle, \tau_{\sigma_1,\sigma_2}(x) \stackrel{\mathrm{def}}{=} \sigma_2(\sigma_1^{-1}(x))$: scale transformation from σ_1 to σ_2. [A.2.a]

$\mathfrak{T}^L$: the set of 'L-***scale transformations***'. [A.2.a]

A.3. Numerical Representations of Performance Assessments

$\varphi_\sigma^\pi : \mathcal{C} \underset{\pi}{\rightarrow} \mathrm{L} \underset{\sigma}{\rightarrow} \langle \sigma[\mathcal{L}] \subseteq \mathbb{R} | =, \ldots \rangle ; \varphi_\sigma^\pi(c) \overset{\text{def}}{=} \sigma(\pi(c))$, with $\sigma \in \mathfrak{S}^L$. [A.3]

A.4. Assignments and Allocations

$\mathrm{R} = \langle \mathcal{R} |\ \cong, \ldots \rangle$ (or $\langle \mathcal{R} |\ \cong, \prec, \oplus, \odot \rangle$) : structured set of (quantitative) rewards. [A.4.a]

$\alpha : \mathcal{C} \rightarrow \mathrm{R}$: ***reward allocation***. [A.4.a]

$\alpha_\rho^\pi : \mathcal{C} \underset{\pi}{\rightarrow} \mathrm{L} \underset{\rho^\pi}{\rightarrow} \mathrm{R}; \alpha_\rho^\pi(c) = \rho^\pi(\pi(c))$: ***performance-based*** reward allocation. [A.4.b]

$\rho^\pi : L \rightarrow \mathrm{R}$: performance ***reward assignment***. [A.4.b]

$\Sigma_{\mathcal{C}}^{\mathrm{R}} \alpha = a$ where $\Sigma_{\mathcal{C}}^{\mathrm{R}} \alpha \overset{\text{def}}{=} \sum_{c \in \mathcal{C}}^{\mathrm{R}} \alpha(c)$: ***Resource allocation*** of amount a of a resource quantity $\mathrm{R} = \langle \mathcal{R} |\ \cong, \prec, \oplus, \odot \rangle$ to the members of $\mathcal{C}$. [A.4.c]

A.5. Numerical Representations of Performance-based Reward Allocations

$\rho_\sigma^\pi : \langle \sigma[\mathcal{L}] \subseteq \mathbb{R}^{\geq 0} | =, <, +, \cdot \rangle \rightarrow \mathrm{R} = \langle \mathcal{R} |\ \cong, \ldots \rangle ; \rho_\sigma^\pi(x) \overset{\text{def}}{\cong} \rho^\pi \left(\sigma^{-1}(x) \right)$: σ-based numerical representation of ρ^π.

$\alpha_{\rho\sigma}^\pi : \mathcal{C} \underset{\pi}{\rightarrow} \mathrm{L} \underset{\sigma}{\rightarrow} \langle \sigma[\mathcal{L}] \subseteq \mathbb{R}^{\geq 0} | \leq, +, \cdot \rangle \underset{\rho_\sigma^\pi}{\rightarrow} \mathrm{R}$: σ-based numerical representation of α_ρ^π.

C.1. Definitions

$\bar{\varphi}_\sigma^\pi(x) \overset{\text{def}}{=} x / \sum_{\mathcal{C}} \varphi_\sigma^\pi$: the ***normalization*** of φ_σ^π. [A.3]

$\bar{\alpha}_\sigma^\pi[a](c) \overset{\text{def}}{\cong} a \odot \left[\bar{\varphi}_\sigma^\pi \circ \varphi_\sigma^\pi \right](c)$: ***normalized allocation*** of a.

$\bar{\rho}_\sigma^\pi[a](x) \overset{\text{def}}{\cong} a \odot \bar{\varphi}_\sigma^\pi(x)$.

C.2. Quantitative Normalized Resource Allocations

$\bar{\mu} : \mathrm{L} \to \langle [0,1] | =, <, +, \cdot \rangle; \bar{\mu}(l) \stackrel{\mathrm{def}}{=} l :^{\mathrm{L}} \Sigma_{\mathcal{C}}^{\mathrm{L}} \mu$: normalisation of μ.

C.3. Ordinal Normalized Resource Allocations

ω_{AD} : The Middle Earth performance ranking (see Table 1).

B Allocative Impacts in World Bank Example

Table 3: Allocative consequences of rescaling to a 0-5 or 2-7 scale under our (stylized) World Bank PBA framework (17th replenishment cycle), without the needs-based adjustment factor.

Country	**Baseline share of total funding**	**Share of funding under 0-5 scale**	**Rel. Diff. to baseline (%)**	**Share of funding under 2-7 scale**	**Rel. Diff. to baseline (%)**
South Sudan	0.0016	0.0003	**-79%**	0.0031	**87%**
Central African Republic	0.0041	0.0020	**-52%**	0.0058	**39%**
Guinea-Bissau	0.0041	0.0020	**-52%**	0.0058	**39%**
Yemen, Republic of	0.0044	0.0022	**-50%**	0.0060	**37%**
Afghanistan	0.0058	0.0035	**-40%**	0.0073	**27%**
Haiti	0.0059	0.0036	**-39%**	0.0074	**26%**
Timor-Leste	0.0064	0.0041	**-36%**	0.0079	**23%**
Samoa	0.0341	0.0460	**35%**	0.0284	**-17%**
Comoros	0.0068	0.0045	**-33%**	0.0083	**21%**
Togo	0.0068	0.0045	**-33%**	0.0083	**21%**
Burundi	0.0069	0.0047	**-33%**	0.0084	**21%**
Congo, Democratic Republic of	0.0071	0.0049	**-32%**	0.0085	**20%**
Congo, Republic of	0.0071	0.0049	**-32%**	0.0085	**20%**
Chad	0.0072	0.0050	**-31%**	0.0086	**19%**
Djibouti	0.0075	0.0052	**-30%**	0.0088	**18%**
Marshall Islands	0.0077	0.0055	**-29%**	0.0090	**17%**
Cape Verde	0.0293	0.0376	**28%**	0.0251	**-14%**
Solomon Islands	0.0084	0.0062	**-25%**	0.0096	**15%**
Rwanda	0.0272	0.0341	**25%**	0.0237	**-13%**
Bhutan	0.0258	0.0318	**23%**	0.0228	**-12%**
Madagascar	0.0091	0.0071	**-22%**	0.0102	**12%**
Micronesia	0.0091	0.0071	**-22%**	0.0102	**12%**
Dominica	0.0248	0.0300	**21%**	0.0220	**-11%**
Papua New Guinea	0.0095	0.0075	**-20%**	0.0105	**11%**
Guinea	0.0096	0.0077	**-20%**	0.0106	**11%**
Gambia, The	0.0097	0.0078	**-19%**	0.0107	**10%**
Tonga	0.0237	0.0283	**19%**	0.0213	**-10%**
Cameroon	0.0101	0.0083	**-18%**	0.0111	**9%**
Senegal	0.0227	0.0266	**17%**	0.0206	**-9%**
St. Vincent and the Grenadines	0.0227	0.0266	**17%**	0.0206	**-9%**
Liberia	0.0104	0.0087	**-16%**	0.0113	**9%**
Nigeria	0.0104	0.0087	**-16%**	0.0113	**9%**
St. Lucia	0.0222	0.0258	**16%**	0.0202	**-9%**
Bangladesh	0.0108	0.0092	**-15%**	0.0116	**7%**

Ghana	0.0213	0.0243	**14%**	0.0195	**-8%**
Myanmar	0.0111	0.0096	**-14%**	0.0119	**7%**
Tajikistan	0.0111	0.0096	**-14%**	0.0119	**7%**
Cambodia	0.0112	0.0098	**-13%**	0.0120	**6%**
Malawi	0.0118	0.0105	**-11%**	0.0124	**5%**
Sierra Leone	0.0118	0.0105	**-11%**	0.0124	**5%**
Tuvalu	0.0118	0.0105	**-11%**	0.0124	**5%**
Burkina Faso	0.0197	0.0218	**11%**	0.0184	**-7%**
Sao Tome and Principe	0.0120	0.0107	**-10%**	0.0126	**5%**
Vietnam	0.0194	0.0214	**10%**	0.0182	**-6%**
Maldives	0.0123	0.0112	**-9%**	0.0128	**4%**
Ethiopia	0.0190	0.0208	**9%**	0.0179	**-6%**
Kenya	0.0190	0.0208	**9%**	0.0179	**-6%**
Lao People's Democratic Republic	0.0125	0.0114	**-9%**	0.0129	**4%**
Nicaragua	0.0188	0.0204	**9%**	0.0177	**-6%**
Bolivia	0.0126	0.0116	**-8%**	0.0131	**3%**
Guyana	0.0126	0.0116	**-8%**	0.0131	**3%**
Mali	0.0126	0.0116	**-8%**	0.0131	**3%**
Pakistan	0.0128	0.0118	**-8%**	0.0132	**3%**
Cote d'Ivoire	0.0131	0.0122	**-7%**	0.0134	**3%**
Sri Lanka	0.0177	0.0188	**6%**	0.0169	**-4%**
Kosovo	0.0175	0.0185	**6%**	0.0168	**-4%**
Moldova	0.0173	0.0182	**5%**	0.0166	**-4%**
Honduras	0.0136	0.0129	**-5%**	0.0138	**2%**
Kiribati	0.0136	0.0129	**-5%**	0.0138	**2%**
Uzbekistan	0.0136	0.0129	**-5%**	0.0138	**2%**
Mongolia	0.0171	0.0179	**5%**	0.0165	**-4%**
Nepal	0.0138	0.0131	**-5%**	0.0140	**1%**
Lesotho	0.0140	0.0134	**-4%**	0.0141	**1%**
Grenada	0.0167	0.0173	**4%**	0.0162	**-3%**
Tanzania	0.0167	0.0173	**4%**	0.0162	**-3%**
Zambia	0.0141	0.0136	**-4%**	0.0142	**1%**
Benin	0.0165	0.0171	**3%**	0.0161	**-3%**
Uganda	0.0143	0.0139	**-3%**	0.0144	**0%**
Kyrgyz Republic	0.0163	0.0168	**3%**	0.0159	**-3%**
Vanuatu	0.0158	0.0159	**1%**	0.0155	**-2%**
Mauritania	0.0150	0.0149	**-1%**	0.0149	**-1%**
Mozambique	0.0152	0.0151	**0%**	0.0151	**-1%**
Niger	0.0152	0.0151	**0%**	0.0151	**-1%**

Table 4: Allocative consequences of rescaling to a 0-5 or 2-7 scale under our (stylized) World Bank PBA framework (17th replenishment cycle), including the needs-based adjustment factor.

Country	**Baseline share of total funding**	**Share of funding under 0-5 scale**	**Rel. Diff. to baseline (%)**	**Share of funding under 2-7 scale**	**Rel. Diff. to baseline (%)**
South Sudan	0.0010	0.0002	**-78%**	0.0019	**84%**
Central African Republic	0.0012	0.0006	**-50%**	0.0016	**37%**
Guinea-Bissau	0.0004	0.0002	**-50%**	0.0005	**37%**
Yemen, Republic of	0.0058	0.0031	**-47%**	0.0078	**34%**
Samoa	0.0003	0.0004	**41%**	0.0002	**-18%**
Afghanistan	0.0099	0.0062	**-37%**	0.0123	**24%**
Haiti	0.0032	0.0020	**-37%**	0.0040	**24%**
Cape Verde	0.0006	0.0008	**34%**	0.0005	**-16%**
Timor-Leste	0.0004	0.0002	**-33%**	0.0004	**21%**
Rwanda	0.0164	0.0215	**31%**	0.0141	**-14%**
Comoros	0.0003	0.0002	**-31%**	0.0003	**19%**
Togo	0.0027	0.0019	**-31%**	0.0032	**19%**
Burundi	0.0046	0.0032	**-30%**	0.0054	**18%**
Congo, Republic of	0.0015	0.0010	**-29%**	0.0017	**17%**
Congo, Democratic Republic of	0.0307	0.0219	**-29%**	0.0360	**17%**
Bhutan	0.0009	0.0012	**28%**	0.0008	**-13%**
Chad	0.0051	0.0037	**-28%**	0.0060	**17%**
Djibouti	0.0003	0.0002	**-27%**	0.0004	**16%**
Dominica	0.0001	0.0001	**26%**	0.0001	**-13%**
Marshall Islands	0.0000	0.0000	**-26%**	0.0000	**15%**
Tonga	0.0001	0.0001	**24%**	0.0001	**-12%**
Senegal	0.0171	0.0209	**22%**	0.0152	**-11%**
St. Vincent and the Grenadines	0.0001	0.0001	**22%**	0.0001	**-11%**
Solomon Islands	0.0002	0.0002	**-22%**	0.0003	**13%**
St. Lucia	0.0002	0.0002	**21%**	0.0002	**-11%**
Ghana	0.0276	0.0329	**19%**	0.0249	**-10%**
Madagascar	0.0122	0.0099	**-19%**	0.0134	**10%**
Micronesia	0.0000	0.0000	**-19%**	0.0000	**10%**
Papua New Guinea	0.0031	0.0026	**-17%**	0.0034	**9%**
Guinea	0.0066	0.0055	**-16%**	0.0072	**9%**
Gambia, The	0.0011	0.0009	**-16%**	0.0012	**8%**
Burkina Faso	0.0186	0.0216	**16%**	0.0171	**-8%**
Vietnam	0.0814	0.0937	**15%**	0.0749	**-8%**
Ethiopia	0.1004	0.1145	**14%**	0.0928	**-8%**
Kenya	0.0420	0.0479	**14%**	0.0388	**-8%**
Cameroon	0.0113	0.0097	**-14%**	0.0121	**7%**

Nicaragua	0.0052	0.0060	**14%**	0.0049	**-7%**
Liberia	0.0026	0.0023	**-13%**	0.0028	**7%**
Nigeria	0.0827	0.0721	**-13%**	0.0882	**7%**
Bangladesh	0.0847	0.0753	**-11%**	0.0894	**6%**
Sri Lanka	0.0157	0.0174	**11%**	0.0147	**-6%**
Kosovo	0.0013	0.0015	**10%**	0.0012	**-6%**
Myanmar	0.0291	0.0262	**-10%**	0.0305	**5%**
Tajikistan	0.0046	0.0041	**-10%**	0.0048	**5%**
Moldova	0.0028	0.0031	**10%**	0.0026	**-6%**
Cambodia	0.0087	0.0078	**-9%**	0.0090	**5%**
Mongolia	0.0022	0.0024	**9%**	0.0020	**-5%**
Tanzania	0.0437	0.0473	**8%**	0.0416	**-5%**
Grenada	0.0001	0.0001	**8%**	0.0001	**-5%**
Benin	0.0091	0.0098	**8%**	0.0087	**-5%**
Kyrgyz Republic	0.0048	0.0051	**7%**	0.0046	**-4%**
Sierra Leone	0.0041	0.0038	**-7%**	0.0042	**3%**
Tuvalu	0.0000	0.0000	**-7%**	0.0000	**3%**
Malawi	0.0116	0.0107	**-7%**	0.0119	**3%**
Sao Tome and Principe	0.0001	0.0001	**-7%**	0.0001	**3%**
Vanuatu	0.0002	0.0002	**6%**	0.0002	**-4%**
Maldives	0.0002	0.0002	**-5%**	0.0002	**2%**
Lao People's Democratic Republic	0.0039	0.0037	**-5%**	0.0040	**2%**
Guyana	0.0004	0.0004	**-4%**	0.0004	**2%**
Mali	0.0114	0.0109	**-4%**	0.0116	**2%**
Bolivia	0.0058	0.0056	**-4%**	0.0059	**2%**
Niger	0.0169	0.0176	**4%**	0.0165	**-3%**
Mozambique	0.0227	0.0236	**4%**	0.0220	**-3%**
Pakistan	0.1148	0.1104	**-4%**	0.1163	**1%**
Mauritania	0.0030	0.0031	**3%**	0.0029	**-2%**
Cote d'Ivoire	0.0142	0.0138	**-3%**	0.0143	**1%**
Uganda	0.0292	0.0295	**1%**	0.0288	**-1%**
Honduras	0.0050	0.0049	**-1%**	0.0049	**0%**
Uzbekistan	0.0193	0.0191	**-1%**	0.0192	**0%**
Kiribati	0.0001	0.0001	**-1%**	0.0001	**0%**
Zambia	0.0108	0.0109	**1%**	0.0107	**-1%**
Nepal	0.0203	0.0202	**0%**	0.0202	**0%**
Lesotho	0.0014	0.0014	**0%**	0.0014	**-1%**

Table 5: Comparison of Allocations - comparing our stylised models with and without scale transformation to the actual WB-figures for FY 2017. 'Actual WB Allocation' refers to the original PBA allocations prior to adjustment for front- and backloading as well as regional/intra-regional reallocation. SDR figures have been converted to US$ at the end-of-2017 exchange rate. Every model allocation also includes a base allocation of 4 million SDR consistent with stated World Bank practice for 2017 (World Bank, 2017b). Note that the relative changes in allocations displayed in the previous tables do not include this base allocation - they refer exclusively to the performance-based part of the allocation. Therefore, there is no exact correspondence between the changes displayed here and the relative changes provided above.

Country	**Actual WB Allocation ($ Mio.)**	**Model Baseline**	**Model Rescaled 0-5**	**Model Rescaled 2-7**
Afghanistan	167.48	130.82	84.32	161.28
Bangladesh	1234.01	1075.63	956.35	1134.67
Benin	109.52	120.98	129.87	115.67
Bhutan	17.09	17.36	20.64	15.78
Bolivia	59.81	79.41	76.21	80.59
Burkina Faso	235.84	241.10	277.87	221.67
Burundi	77.33	63.39	46.12	73.98
Cambodia	130.88	114.96	104.66	119.88
Cameroon	134.30	148.60	128.56	159.00
Cape Verde	14.10	13.62	16.31	12.38
Central African Republic	14.38	20.36	13.05	25.77
Chad	50.27	70.39	52.18	81.37
Comoros	7.55	9.23	8.15	9.90
Congo, Democratic Republic of	500.72	393.01	281.69	460.52
Congo, Republic of	20.37	24.05	18.77	27.25
Cote d'Ivoire	166.62	184.76	179.99	186.00
Djibouti	9.68	9.63	8.57	10.26
Dominica	6.55	6.92	7.24	6.77
Ethiopia	1310.48	1272.71	1450.84	1176.97
Gambia, The	13.96	19.22	17.09	20.36
Ghana	282.83	354.16	421.11	319.95
Grenada	6.55	6.50	6.57	6.46
Guinea	70.92	89.09	75.46	96.38
Guinea-Bissau	20.37	10.69	8.20	12.53
Guyana	7.55	11.02	10.78	11.10
Haiti	40.16	46.17	31.38	55.81
Honduras	47.71	68.20	67.57	68.07
Kenya	615.94	536.05	610.61	495.97

Kiribati	5.41	6.43	6.43	6.43
Kosovo	22.50	22.38	24.11	21.40
Kyrgyz Republic	67.65	66.04	70.37	63.42
Lao People's Democratic Republic	55.97	55.41	52.97	56.36
Lesotho	24.07	23.47	23.49	23.33
Liberia	48.85	38.85	34.58	41.02
Madagascar	233.98	159.50	130.77	175.28
Malawi	142.98	151.59	141.14	156.24
Maldives	6.98	8.14	8.00	8.19
Mali	114.22	149.42	143.18	151.73
Marshall Islands	4.70	6.09	5.99	6.15
Mauritania	29.05	43.37	44.65	42.44
Micronesia	4.98	6.18	6.09	6.23
Moldova	42.87	41.14	44.64	39.15
Mongolia	34.18	32.97	35.51	31.50
Mozambique	231.28	291.80	303.07	283.93
Myanmar	506.99	373.41	336.63	391.20
Nepal	299.21	262.27	261.13	261.01
Nicaragua	57.96	71.94	80.91	67.09
Niger	216.89	219.38	227.79	213.50
Nigeria	1104.41	1050.45	915.97	1119.02
Pakistan	978.38	1454.98	1400.13	1473.80
Papua New Guinea	48.71	45.43	38.71	49.05
Rwanda	241.82	213.23	276.77	183.27
Samoa	8.83	9.30	10.78	8.65
Sao Tome and Principe	5.41	7.11	7.01	7.15
Senegal	185.56	221.16	269.10	197.25
Sierra Leone	51.41	56.93	53.26	58.56
Solomon Islands	8.12	8.59	7.95	8.96
South Sudan	68.50	18.82	8.59	29.80
Sri Lanka	213.48	203.60	225.19	191.50
St. Lucia	7.69	7.87	8.33	7.64
St. Vincent and the Grenadines	6.98	6.82	7.07	6.70
Tajikistan	65.37	63.39	57.62	66.18
Tanzania	572.79	557.22	602.73	530.41
Timor-Leste	10.82	10.17	8.69	11.09
Togo	33.61	39.51	29.19	45.88
Tonga	6.55	6.94	7.24	6.79
Tuvalu	4.56	5.76	5.76	5.76
Uganda	354.04	373.96	378.40	368.96
Uzbekistan	260.90	248.86	246.42	248.37

Vanuatu	7.55	8.30	8.44	8.21
Vietnam	1087.32	1033.56	1188.76	951.10
Yemen, Republic of	74.62	79.13	44.28	104.30
Zambia	110.23	142.43	143.33	140.97