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Modeling Coupled Climate, Ecosystems, and Economic Systems

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Abstract

Human economies and ecosystems form a coupled system coevolving in time and space, since human economies use ecosystems services and at the same time affect ecosystems through their production and consumption activities. The study of the interactions between human economies and ecosystems is fundamental for the efficient use of natural resources and the protection of the environment. This necessitates the development and use of models capable of tracing the main interactions, links and feedbacks. In developing this chapter, our objective was to focus on a segment of rapidly developing literature on coupled ecological/economic models with an emphasis on climate change. The advantage of this approach is that it introduces the reader to a very important current research topic, but it also allows, by using climate as the reference ecosystem, the exploration of new modeling approaches which are relevant and useful for the modeling of other types of coupled ecological/economic systems. These include modeling of deep structural uncertainty by using robust control methods, exploring modeling through cumulative carbon budgeting, studying spatial transport phenomena and spatial aspects in economic/ecological modelling.

Keywords: Coupled ecological/economic models, climate change, deep uncertainty, robust control,cumulative carbon budgeting, energy balance climate models, spatial aspects in ecological/economic modeling.

JEL Classification: Q20, Q40, Q54, Q57

1 Introduction

Human economies and ecosystems form a coupled system coevolving in time and space, since human economies use ecosystems services[1] and at the same time affect ecosystems through their production and consumption activities. The study of the interactions between human economies and ecosystems is fundamental for the efficient use of natural resources and the protection of the environment through the design of policy and management rules. This necessitates the development and use of models capable of tracing the main interactions, links and feedbacks. Models are necessary in order to understand the issues involved and to derive efficient policies. It is clear that, in order to attain these objectives, these models should be coupled models of ecosystems and economics systems.

The modeling of coupled ecological and economic systems can be traced back to models dealing with management of natural resources. The natural link between ecosystems and human economies has been manifested in the traditional development of resource management or bio-economic models (for example, Clark, 1990), in which the main focus has been on fishery or forestry management where the impact of humans on ecosystems is realized through harvesting and biomass depletion. Closer links have been developed, however, as both disciplines evolve.

Thus the classical phenomenological-descriptive approach to species competition based on Lotka-Volterra systems has been complemented by mechanistic resource-based models of species competition for limiting resources (Tilman, 1982, 1988). This approach has obvious links to competition among economic agents for limited resources. Furthermore, new insights into the fundamental issues of the valuation of ecosystems or the valuation of biodiversity have been derived (e.g., Weitzman, 1992, 1998a; Brock and Xepapadeas, 2003) by linking the functioning of natural ecosystems with the provision of useful services to humans; or by using concepts such as ecosystems productivity or insurance from the genetic diversity of ecological systems against catastrophic events; or by developing new products using genetic resources existing in natural ecosystems (Heal, 2000).

The size and the strength of the impact of human economies in ecosystems depend on the way in which certain actions, such as harvesting, extraction of resources, emissions of pollutants, or investment in harvesting or pollution abatement capacity, which can be chosen by humans and which influence the

[1]Examples of useful services to humans include provisioning services, such as food, water, fuel, or genetic material; regulation services, such as climate regulation or disease regulation; and cultural services and supporting services, such as soil formation or nutrient cycling (see Millennium Ecosystem Assessment, 2005).

evolution of ecosystems, are actually chosen. These actions can be regarded as control variables, and the way in which they are chosen affects the evolution of quantities describing the state of the coevolving coupled ecosystem and economic system. The state of the coupled system depends on the evolution of ecological variables, such as species biomasses or stock of pollutants or greenhouse gases, which determine the flow of ecosystem services, along with traditional economic variables such as consumption, investment, and stock of produced or human capital.

The typical approach in economics is to associate the choice of the control variables with forward-looking optimizing behavior. Thus, the control variables are chosen so that a criterion function is optimized, and the economic problem of ecosystem management – where management means choice of control variables – is defined as a formal optimal control problem. In this problem the objective is the optimization of the criterion function subject to the constraints imposed by the structure of the ecosystem and the structure of the economy. These constraints provide the transition equations as well as other possible exhaustibility constraints associated with the optimal control problem. For example in models of resource harvesting with generalized resource competition, the ecosystem dynamics describe both biomass and limited resource evolution (e.g., Brock and Xepapadeas, 2002; Tilman et al., 2005) in biodiversity valuation problems. Brock and Xepapadeas (2003) show that genetic constraints associated with development of resistance should be part of the optimal control of the coupled system.

The solution of coupled ecosystem-economic system models, provided it exists, will determine the paths of the state and the control variables and the steady state of the system. These paths will determine the long-run equilibrium values of the ecological and economic variables as well as the approach dynamics to the steady state.[2]

In principle, two types of solution can be characterized: (i) a socially optimal solution where all known constraints associated with the problem and externalities associated with action of individual agents are taken into account, and (ii) a privately optimal solution which corresponds to an unregulated market equilibrium where forward-looking agents maximize private profits and externalities are not internalized. The deviations between the private solution and the social optimum justify regulation. Thus, policy design in this context implies that instruments, such as taxes or quotas, are determined so that the social optimum is implemented in a competitive equi-

[2]Managed ecological systems which are predominantly nonlinear could exhibit dynamic behavior characterized by multiple, locally stable and unstable steady states, limit cycles, or the emergence of hysteresis, bifurcations or irreversibilities.

librium.

In developing this chapter, our objective was not to review the large body of literature on modeling of coupled ecosystem and economic systems, but rather to focus on a segment of rapidly developing literature on coupled ecological/economic models with an emphasis on climate change. The advantage of this approach is that it introduces the reader to a very important current research topic, but it also allows, by using climate as the reference ecosystem, the exploration of new modeling approaches which are relevant and useful for the modeling of other types of coupled ecological/economic systems. These include modeling of deep structural uncertainty by using robust control methods, exploring modeling through cumulative carbon budgeting, studying spatial transport phenomena and spatial aspects in economic/ecological modelling.

2 Coupled Ecological/Economic Modeling for Robustness

Consider the following social optimization model of an economy dependent upon a biosphere stock x given by

$$\begin{array}{l} \max_c \int_{t=0}^{\infty} e^{-\rho t} u(c\,(t)\,, x\,(t)\,)dt \\ s.t. \\ \dot{x}\,(t) = F(x\,(t)\,, c\,(t)\,),\ x(0) = x_0. \end{array} \tag{1}$$

For example in the Steele-Henderson (1984) model of a fishery below,

$$\begin{array}{l} F(x,c) = rx(1-k/x) - c - aR(x) \\ R(x) = x^2/(b^2+x^2). \end{array} \tag{2}$$

$x\,(t)$ is a valuable stock, e.g. biomass of fish. [3]The term $R\,(x)$ introduces nonlinear feedbacks, which are physical processes that further impact on initial change of the system under study. Feedbacks could be positive if the impact is such that the initial perturbation is enhanced, or negative if the initial perturbation is reduced. In the context of renewable resources feedbacks can be related to nonlinear predation terms. In the analysis of eutrophication of lakes, positive feedbacks are related to the release of phosphorus that has been slowly accumulated in sediments and submerged vegetation, while in climate change issues they can be related for example with the permafrost

[3]To ease notation in many case we will omit the explicit dependence of a variable on time t and write x instead of $x\,(t)$ and so on.

carbon pool (Brock, Engström, and Xepapadeas, 2014a). Nonlinear feedbacks in the resource dynamics introduce nonconvexities which are related to the existence of multiple steady states, hysteresis, or irreversibilities and cause the emergence of Sciba points (see the collection edited by Dasgupta and Mäler, 2004).

In the case of an ecosystem that is stressed by consumptive activities (e.g., Dasgupta and Mäler, 2004),

$$\begin{array}{l} \dot{x} = F(x,c) = \lambda c - \delta x + aR(x),\ x(0) = x_0 \\ R(x) = x^2/(b^2 + x^2), \end{array} \tag{3}$$

where x is a stock of something bad, e.g. the stock of phosphorous sequestered in algae in a lake ecosystem (Carpenter et al., 1999; Dasgupta and Mäler, 2004; Mäler et al., 2003) and c is consumptive activities that yield utility but damage the services that enjoyers obtain, $u(c,x)$, where the utility function $u(c,x)$ increases in c but decreases in x, and the nonlinear feedback term $R(x)$ introduces nonconvexities. An example in which optimal control c can be obtained in a closed form solution analytically is the case

$$a = 0,\ u(c,x) = \ln(c^{\alpha} e^{-Dx}),\ D > 0,\ 0 < \alpha < 1. \tag{4}$$

Thus the current value Hamiltonian for problem (1)-(3) is

$$H(x,c,p) = u(c(t), x(t)) + p[\lambda c - \delta x + aR(x)] \tag{5}$$

and the first order necessary conditions (FONCs) resulting from the maximum principle, using (4), imply:

$$\frac{1}{c} = -\lambda p \tag{6}$$

$$\dot{c} = (\rho + \delta) c + (1 - aR'(x)) \lambda c^2 \tag{7}$$

$$\dot{x} = \lambda c - \delta x + aR(x). \tag{8}$$

All cases may be analyzed using phase diagram techniques in co-state and state space since the FONCs of the optimal control problem result in two ordinary differential equations (ODEs) for the stock $x(t)$ and its shadow value $p(t)$ which are autonomous. We refer to the collection edited by Dasgupta and Mäler (2004) for analysis of selected cases and Crépin et al. (2012) and Levin et al. (2013) for many examples. Mäler et al. (2003) and Kossioris et al. (2008), by using a utility function which is logarithmic in benefits and quadratic in damages,

$$u(c,x) = \ln c - \beta x^2, \beta > 0, \tag{9}$$

study a situation in which enjoyers interact strategically. The lake ecosystem problem is analyzed as a differential game with open loop and nonlinear feedback Nash equilibrium with the feedback Nash equilibrium strategies obtained numerically. For the same problem, Kossioris et al. (2011) study the structure of optimal state-dependent taxes that steer the combined economic-ecological system towards the trajectory of optimal management, and provide an algorithm for calculating such taxes. More examples, together with discussion of early warning signals of impending regime changes and tipping points, are given in the collection published in *Theoretical Ecology,* edited by Dakos and Hastings (2013).

Before continuing with our analysis, we insert a word of caution. Brook et al. (2013) caution that one-dimensional models like the above that focus on nonlinear responses to anthropogenic forcing must be restricted to the appropriate time and spatial scales in order to be relevant. For example, they identify settings in which such models might be relevant: (i) there must be enough spatial homogeneity in drivers and responses; (ii) there must be enough interconnectivity at the spatial and temporal scales under scrutiny. At the global level they argue that the "usual suspects" - climate change, land use change, habitat fragmentation, and species richness - are not likely to satisfy the conditions needed for a strong enough nonlinearity at the global scale to induce a global scale tipping point. They do not dispute, however, that tipping points may occur at smaller regional scales.

We turn now to introducing robustness into the analysis of management models of human-dominated ecosystems.

2.1 Robust control methods in coupled ecological/economic systems

2.1.1 An introduction to robust control methods

Robustness is related to the major and interrelated uncertainties associated with coupled ecological/economic systems. These uncertainties are primarily associated with two basic factors: (a) the high structural uncertainty over the physical processes of environmental phenomena and (b) the high sensitivity of model outputs to modeling assumptions. As a result, separate models may arrive at dramatically different policy recommendations, generating significant uncertainty over the magnitude and timing of desirable policies. These uncertainties may impede adequate scientific understanding of the underlying ecosystem mechanisms and the impacts of policies applied to ecosystems.

A central feature of the above structure of uncertainty is that it might be difficult or even impossible to associate probabilities with uncertain prospects affecting the ecosystem evolution. This is close to the concept of uncertainty as introduced by Knight (1921) to represent a situation in which probabilities cannot be assigned to events because there is ignorance insufficient information. Knight argued that uncertainty in this sense of unmeasurable uncertainty is more common in economic decision making. Knightian uncertainty should be contrasted to risk (measurable or probabilistic uncertainty) where probabilities can be assigned to events and are summarized by a subjective probability measure or a single Bayesian prior.

Inspired by the work of Knight and subsequently by Ellsberg (1961), economic theorists have questioned the classical expected utility framework and attempted to formally model preferences when probabilistic beliefs are not of sufficiently high quality to generate prior distributions. Gilboa and Schmeidler (1989) developed the axiomatic foundations of maxmin expected utility, an alternative to classical expected utility for economic environments featuring unknown risk. They argued that when the underlying uncertainty of an economic system is not well understood, it is sensible - and axiomatically compelling - to optimize over the worst-case outcome (i.e., the worst-case prior) that could conceivably come to pass.

Motivated by concerns about model misspecification in macroeconomics, Hansen and Sargent (2001a, 2001b, 2008) and Hansen et al. (2006) extended Gilboa and Schmeidler's insights to dynamic optimization problems, thus introducing the concept of robust control to economic environments. A decision maker characterized by robust preferences takes into account the possibility that the model used to design regulation, call it benchmark or approximating model $\mathbb{P}$, may not be the correct one but only an approximation of the correct one. Other possible models, say $\mathbb{Q}_1, \ldots, \mathbb{Q}_J$, which surround $\mathbb{P}$, should also be taken into account with the relative differences among these models measured by an entropy measure, or an entropy ball containing the approximate model $\mathbb{P}$. Hansen and Sargent (2003) characterize robust control as a theory "... [that] instructs decision makers to investigate the fragility of decision rules by conducting worst-case analyses," and suggest that this type of model uncertainty can be related to ambiguity or deep uncertainty so that robust control can be interpreted as a recursive version of maxmin expected utility theory. The models inside the entropy ball are close enough to the benchmark model that they are difficult to distinguish with finite data sets. Then robust decisions rules are obtained by introducing a fictitious 'adversarial agent' which we will refer to as Nature. Nature promotes robust decision rules by forcing the regulator, who seeks to maximize (minimize) an objective, to explore the fragility of decision rules with regard to departures from the benchmark model. A robust decision rule means that lower bounds to the rule's performance are determined by Nature – the adversarial agent – which acts as a minimizing (maximizing) agent when constructing these lower bounds.

In terms of applications, climate change is an area where ambiguity and concerns about model misspecification are present and significant. As Weitzman (2009) points out, the high structural uncertainty over the physics of environmental phenomena makes the assignment of precise probabilistic model structure untenable, while there is high sensitivity of model outputs to alternative modeling assumptions such as the functional form of the chosen damage function and the value of the social discount rate (e.g., Stern, 2006; Weitzman, 2010). Thus robust control approaches fit very well with climate change problems, as well as with more general environmental and resource economics problems, given the deep uncertainties associated with these issues.[4] For example a specific density function for climate sensitivity from the set of densities reported by Meinshausen et al. (2009) can be regarded as the benchmark model, but other possible densities should be taken into account when designing regulation. One of these densities that corresponds to the least favorable outcome regarding climate change impacts can be associated with the concept of the worst case.

To provide a more formal presentation, let the set of states of the world be Ω, and consider an individual observing some realization $\omega_t \in \Omega$. The basic idea underlying the multiple priors approach is that beliefs about the evolution of the process $\{\omega_t\}$ cannot be represented by a probability measure. Instead, beliefs conditional on ω_t are too vague to be represented by such a single probability measure and are represented by a *set* of probability measures (Epstein and Wang, 1994). Thus for each $\omega \in \Omega$, we consider $\mathcal{P}(\omega)$ as a set of probability measures about the next period's state.[5]

The individual ranks uncertain prospects or acts α. Let u be a standard utility function. The utility of any act α in an atemporal model is defined as (Gilboa and Schmeidler, 1989; Chen and Epstein, 2002)

$$U(c) = \min_{Q \in \mathcal{P}} \int u(\alpha)\, dQ, \tag{10}$$

[4]Issues of regulation under ambiguity have been studied using two main approaches: smooth ambiguity and robust control. Smooth ambiguity (Klibanoff et al., 2005) parameterizes uncertainty or ambiguity aversion in terms of preferences and nests the worst-case, corresponding to robust control, as a limit of absolute ambiguity aversion. The approach has been used in climate change issues (e.g., Millner et al., 2010), but questions regarding the calibration of the regulator's ambiguity aversion remain open. Robust control methods have been applied to climate change by Athanassoglou and Xepapadeas (2012).

[5]Formally $\mathcal{P}$ is a correspondence $\mathcal{P} : \Omega \to \mathcal{M}(\Omega)$ assumed to be continuous, compact-valued and convex-valued and $\mathcal{M}(\Omega)$ is the space of all Borel probability measures.

while in continuous time framework, recursive multiple prior utility is defined as:

$$V_t = \min_{Q \in \mathcal{P}} E_Q \left[\int_t^T e^{-\rho(s-t)} u(\alpha)\, ds \right]. \tag{11}$$

These definitions of utility in the context of multiple-priors correspond to an intuitive idea of the 'worst case'. Utility is associated with the utility corresponding to the least favorable prior. With utility defined in this way, decision making by using the maxmin rule follows naturally, since maximizing utility in the multiple-priors case implies the maxmin criterion

Given the set of probability measures $\mathcal{P}$, the decision maker considers the reference probability measure $\mathbb{P}$ and another measure $\mathbb{Q} \in \mathcal{M}(\Omega)$. The discrepancy between the two measures is determined by the discounted relative entropy

$$R(\mathbb{Q}//\mathbb{P}) = \int_0^{+\infty} e^{-\delta t} \mathbb{E}_Q[\frac{1}{2} h_t^2] dt, \tag{12}$$

where h is a measurable function associated with the distortion of the probability measure $\mathbb{P}$ to the probability measure $\mathbb{Q}$. To allow for the notion that even when the model is misspecified the benchmark model remains a "good" approximation, the misspecification error is constrained. Thus we only consider distorted probability measures $\mathbb{Q}$ such that

$$R(\mathbb{Q}//\mathbb{P}) = \int_0^{+\infty} e^{-\delta t} \mathbb{E}_Q[\frac{1}{2} h_t^2] dt \leq \eta < \infty. \tag{13}$$

Using (13) as the entropy constraint, Hansen and Sargent (2008) define two robust control problems, a constraint robust control problem and a multiplier robust control problem. Using problem (1) as reference the constraint robust control problem is written as:

$$\max_{c(t)} \min_{h(t)} \mathbb{E}_0 \int_{t=0}^{\infty} \mathrm{e}^{-\rho t} u(c(t), x(t)) dt \tag{14}$$

$$\text{subject to} \tag{15}$$

$$d(t) = [F(x(t), c(t)) + \sigma(x(t) h(t))] dt + \sigma(x(t)) dZ(t), \; x(0) = x_0 \tag{16}$$

$$\text{and (13),} \tag{17}$$

where $\{Z(t), t \geq 0\}$ is a Brownian motion in the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $h(t)$ is a measurable drift distortion which reflects the fact that the probability measure $\mathbb{P}$ is replaced by another measure $\mathbb{Q}$. The drift distortion incorporates omitted or misspecified dynamic effects on the dynamics of the state variable.

The multiplier robust control problem is defined as:

$$\mathbb{E}_0 \int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \left[u(c(t), x(t)) + \frac{1}{2}\theta h^2(t) \right] dt \tag{18}$$

$$\text{subject to (16) and } x(0) = x_0. \tag{19}$$

In both extremization problems, the distorting process $h(t)$ is such that allowable measures $\mathbb{Q}$ have finite entropy. In the constraint problem (14), the parameter η is the maximum expected misspecification error that the decision maker is willing to consider. In the multiplier problem (18), the parameter θ, which is called the robustness parameter, can be interpreted as a Lagrangean multiplier associated with entropy constraint (13). Our choice of θ lies in an interval $(\theta_{\min}, +\infty)$, where the lower bound $\theta_{\min}$ is a breakdown point beyond which it is fruitless to seek more robustness. This is because the minimizing agent is sufficiently unconstrained that he can push the criterion function to $-\infty$ despite the best response of the maximizing agent. Thus when $\theta \leq \theta_{\min}$, robust control rules cannot be attained. On the other hand, when $\theta \to +\infty$, or equivalently $\eta = 0$, there are no concerns about model misspecification and the decision-maker may safely consider just the benchmark model.

The multiplier robust control problem, which is the more analytically tractable of the two, is solved by using the Hamilton-Jacobi-Bellman-Isaacs (HJBI) condition (Fleming and Souganidis, 1989)

$$\begin{aligned} \rho V(x) = \max_{c(t)} \min_{h(t)} \Big\{ & u(c(t), x(t)) + \frac{1}{2}\theta h^2(t) + \\ & V'(x)\left[F(x(t), c(t)) + \sigma(x(t) h(t))\right] + \frac{1}{2}\sigma^2(x(t)) V''(x) \Big\}, \end{aligned} \tag{20}$$

where $V(x)$ is the value function for the problem. As shown in Hansen et al. (2006, Appendix D), if $\sigma(\cdot)$ is independent of the control then the HJBI condition is satisfied and the orders of maximization and minimization can be exchanged in (20). Thus $\rho V(x) = \max_{c(t)} \min_{h(t)} \{\cdot\} = \min_{h(t)} \max_{c(t)} \{\cdot\}$. In the rest of the chapter we assume that this independence assumption is satisfied.

Solution of problem (20) will determine the optimal robust paths $(c_\theta^*(t), x_\theta^*(t))$ for a given level of robustness θ which express the regulator's concerns about model misspecification. Solution of the same problem for $\theta \to \infty$ will provide paths $(c_\infty^*(t), x_\infty^*(t))$ when the regulator is not concerned about model misspecification and regards the benchmark model as adequate.

2.1.2 A deterministic approximation to robust control methods in ecosystem management

The stochastic differential game (20) can be simplified to a deterministic game, which simplifies considerably the solution without altering its structure. Consider the following robust control version of the last model, adapting the framework of Hansen et al. (2006), Hansen and Sargent (2008) and Anderson et al. (2014):

$$\begin{array}{l} \max_{\{c_i\}} \min_{\{h_i\}} E_0\{\int_{t=0}^{\infty} e^{-\rho t}(u(c,x) + (1/2)(\sum_i \theta_i(\varepsilon)h_i^2)dt\} \\ s.t. \\ dx = [(\lambda + \varepsilon^{1/2}\sigma_1 h_1)c - (\delta + \varepsilon^{1/2}\sigma_2 h_2)x + (a + \varepsilon^{1/2}\sigma_3 h_3)R(x)]dt \\ +\varepsilon^{1/2}\sigma_1 c dZ_1 - \varepsilon^{1/2}\sigma_2 dZ_2 + \varepsilon^{1/2}\sigma_3 R(x) dZ_3, \\ x(0) = x_0 \\ R(x) = x^2/(b^2 + x^2),\ x(0) = x_0. \end{array} \tag{21}$$

We will exploit the scaling methods introduced by Anderson et al. (2012) and used by Anderson et al. (2014) to scale the θs with ε in such a way that as $\varepsilon \to 0$ we obtain a deterministic robust control problem that can be analyzed quite easily. We believe that a very important agenda for future research is to extend the methods of Anderson et al. (2012), Hansen et al. (2006), and others in the recent robust control literature to management modeling of human-dominated ecosystems. We content ourselves with analysis of limit problems here. Assuming independence of the shocks $\{dZ_i, i = 1, 2, 3\}$, the HJBI equation for (21) is:

$$\begin{array}{l} \rho W(x) = \max_{\{c_i\}} \min_{\{h_i\}} \{u(c,x) + (1/2)(\sum_i \theta_i(\varepsilon)h_i^2) \\ +W_x[(\lambda + \varepsilon^{1/2}\sigma_1 h_1)c - (\delta + \varepsilon^{1/2}\sigma_2 h_2)x + (a + \varepsilon^{1/2}\sigma_3 h_3)R(x)] \\ +(1/2)[(\varepsilon^{1/2}\sigma_1 c)^2 + (\varepsilon^{1/2}\sigma_2 x)^2 + (\varepsilon^{1/2}\sigma_3 R(x))^2]W_{xx}\}. \end{array} \tag{22}$$

The FONCs for the minimizing agent are

$$\begin{array}{l} h_1 = -(1/\theta_1(\varepsilon))W_x\varepsilon^{1/2}\sigma_1 c \\ h_2 = (1/\theta_2(\varepsilon))W_x\varepsilon^{1/2}\sigma_2 x \\ h_3 = -(1/\theta_3(\varepsilon))W_x\varepsilon^{1/2}\sigma_3 R(x). \end{array} \tag{23}$$

The important thing to note is the following. Consider one of the terms containing an h_i in the HJBI equation, e.g.

$$\begin{array}{l} (1/2)\theta_2(\varepsilon)h_2^2 - W_x\varepsilon^{1/2}\sigma_2 h_2 x = -(1/(2\theta_2(\varepsilon)))(W_x\varepsilon^{1/2}\sigma_2 x)^2 \\ = -(1/(2\theta_2))(W_x\sigma_2 x)^2, \end{array} \tag{24}$$

and assume that $\theta_2(\varepsilon) = \theta_2\varepsilon$, and that the same linear scaling applies to all θs. Now take the limit of the HJBI equation as $\varepsilon \to 0$. Since the terms involving W_{xx} vanish as $\varepsilon \to 0$, this suggests (under appropriate regularity) that the limit HJBI equation is[6]

$$\begin{array}{l}\rho W(x) = \max_{\{h_i\}} \min_{\{c\}}\{u(c,x) + (1/2)(\sum_i \theta_i h_i^2) \\ +W_x[(\lambda + \sigma_1 h_1)c - (\delta + \sigma_2 h_2)x + (a + \sigma_3 h_3)R(x)]\}.\end{array} \tag{25}$$

From now on we confine our attention to robust deterministic problems under the assumption that scaling of the thetas and the sigmas as above has taken place. We find it more convenient to use optimal control theory rather than the HJBI equation to analyze (25). The Hamiltonian is

$$\begin{array}{l}H = \max_{\{c_i\}} \min_{\{h\}}\{u(c,x) + (1/2)(\sum_i \theta_i h_i^2) \\ +\mu[(\lambda + \sigma_1 h_1)c - (\delta + \sigma_2 h_2)x + (a + \sigma_3 h_3)R(x)]\}.\end{array} \tag{26}$$

The FONCs for the minimizing agent and the maximizing agent in (25) are

$$\begin{array}{l}h_1 = -(1/\theta_1)\mu\sigma_1 c \\ h_2 = (1/\theta_2)\mu\sigma_2 x \\ h_3 = -(1/\theta_3)\mu\sigma_3 R(x)\end{array} \tag{27}$$

$$\begin{array}{l}u_c(c,x) + \mu(\lambda + \sigma_1 h_1) = 0 \\ \dot{\mu} = \rho\mu - H_x = \rho\mu - \{u_x(c,x) + \mu[-(\delta + \sigma_2 h_2) + (a + \sigma_3 h_3)R'(x)]\}.\end{array} \tag{28}$$

The state equation is

$$\begin{array}{l}\dot{x} = (\lambda + \sigma_1 h_1)c - (\delta + \sigma_2 h_2)x + (a + \sigma_3 h_3)R(x) \\ x(0) = x_0.\end{array} \tag{29}$$

After solving for the hs and the optimal control c, the co-state and state equations are two autonomous of time ODEs which may be phase diagrammed and analyzed by standard qualitative methods to locate candidates for optimal solutions.

Before we turn to phase diagram analysis, we exhibit a special case where there is a closed form solution. Assume that

$$\begin{array}{l}u(c,x) = \ln(c^{\alpha}e^{-Dx}), 0 < \alpha < 1,\ D > 0 \\ 0 = a = \sigma_2 = \sigma_3.\end{array} \tag{30}$$

In this case, the co-state equation is

$$\dot{\mu} = (\rho + \delta)\mu + D, \quad or \tag{31}$$

$$\mu = -D/(\rho + \delta) \tag{32}$$

[6]For details regarding the limit of the HJBI equation as $\varepsilon \to 0$, see Campi and James (1996).

at the steady state. Insert the solution $h_1 = -(1/\theta_1)\mu\sigma_1 c$ into the FONCs for c,

$$\alpha/c = -\mu(\lambda + \sigma_1 h_1) = -\mu(\lambda + \sigma_1(-(1/\theta_1)\mu\sigma_1 c)) \tag{33}$$
$$c = \{\mu\lambda + [(\mu\lambda)^2 + 4\alpha\gamma\mu^2]^{1/2}\}/(2\mu^2\gamma) \tag{34}$$
$$\gamma \equiv \sigma_1^2/\theta_1. \tag{35}$$

We used L'Hospital's Rule to guide us to the correct root of the quadratic in c which is the positive root of equation (34). Call this root, c_+. More precisely, take $\gamma \to 0$ and obtain

$$c_+ \to \alpha(\rho + \delta)/(D\lambda). \tag{36}$$

This is the solution that would be obtained if $\sigma_1 = 0$ or $1/\theta_1 = 0$, i.e. "$\theta_1 = \infty$", i.e., no robustness. It is easy to check that

$$c_+ < c^* = \alpha(\rho + \delta)/(D\lambda), \tag{37}$$

where c^* denotes the steady state solution when the manager has no doubts about its specification of the dynamics, i.e. $\theta_1 = \infty$. A picture of the forces affecting the steady-state optimal control c, the doubts about the specification of the dynamics, θ_1 and the standard deviation of the shocks buffeting the dynamics, σ_1, can be obtained by running a numerical simulation of the optimal control given in (34). Assuming $\lambda = 1, \delta = 0.06, \rho = 0.02, D = 0.03$ and $\alpha = 0.5$, which imply, using (32), a steady-state $\mu = -0.375$, the steady-state optimal control $c(\sigma_1, \theta_1)$ is shown in figure 1.

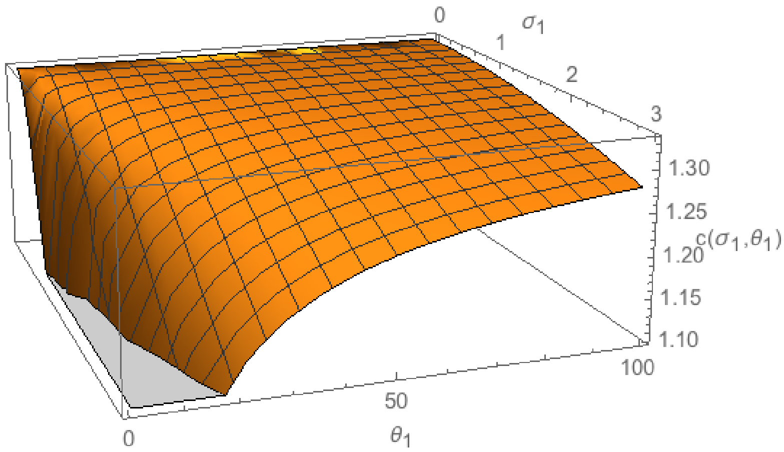


Figure 1: The steady-state optimal control $c(\sigma_1, \theta_1)$

The shape of the surface suggests that optimal consumptive activities at the steady state are reduced as concerns about model misspecification increase, i.e., θ_1 is reduced. As $\theta_1 \to \infty$, which implies that there are no concerns about model misspecification, consumptive activities increase towards the $c^* = 1.33$. Another interesting property of this model, stemming from the specification of the logarithmic utility function (30), is that the Hamiltonian system for the state and the costate variables is independent of (σ_1, θ_1) and can be written as

$$\dot{x} = -\frac{\alpha}{\mu} - \delta x \quad (38)$$

$$\dot{\mu} = (\rho + \delta)\mu + D, \quad (39)$$

with steady states

$$x^* = \frac{\alpha(\rho + \delta)}{\delta D}\ ,\ \mu^* = -\frac{D}{(\rho + \delta)}. \quad (40)$$

This implies that the paths for the state and the costate variables and the corresponding steady state, are independent of (σ_1, θ_1). This property will not extend to more general utility functions.

3 Some Recent Work in Climate Economics with Emphasis on New Modeling of Carbon Budgeting, Robustness, and Spatial Transport

It is standard in a handbook chapter to review the relevant literature in each section of the chapter. However, in the case of integrated assessment modeling in climate-economics, the literature is huge and there are already good sources that review this massive area of research. A few of the most recent sources that also give critiques are Nordhaus (2008, 2013), Pindyck (2013a, 2013c), and Stern (2013). Brock, Xepapadeas, and Yannacopoulos (2014b) review recent literature on inter-temporal spatial dynamic environmental economic modeling. Rather than going over terrain that is competently covered elsewhere, this section discusses some very recent work that emphasizes spatial transport phenomena in climate-economics models and also reviews a cumulative carbon budgeting approach that abstracts from the difficult issues surrounding the parametric specification of a damage function. We also include some discussion of robustness and stochastic forcing where the robustness parameter is scaled relative to the standard deviation of the stochastic shocks in such a way as to yield an approximate deterministic problem (Anderson et al., 2014).[7]

In addition we provide a brief discussion of policies needed to deal with greenhouse gases (GHGs) that have different lifetimes. Pierrehumbert (2014) argues that appropriate policies should focus on putting more emphasis on long-lived GHGs in contrast to basing policies on global warming potentials (GWPs) independent of lifetime of the GHGs. Even narrowing the focus to this much smaller slice of the area requires that we concentrate on a relatively narrow spectrum of the hierarchy of climate-economic models. Models of the climate component range from the complex general models, which are computer models with spatial resolution as fine as current computer technology can handle, to simple analytical energy balance models and "box" models. Models of the economic component also have a similar complexity hierarchy. We use the simplest possible models of both the climate component and the economic component here.

3.1 Cumulative Carbon Budgeting to Implement Temperature Limits

We begin this section by offering a potentially radical approach to mitigating some controversies in the literature IAMs. Pindyck has written a recent series of papers (see the references, especially Pindyck (2013a, 2013c) that argue that too many assumptions, especially regarding damage functions, are made that do not have strong support in reality. That is, he argues that the exact specifications of "damage functions" seen in a lot of the literature on IAMs are weakly supported by hard evidence. Roe and Baker (2007) explain why it is difficult to make progress on reducing the uncertainty about a key parameter, the climate sensitivity. Roe and Bauman (2013) critique the use of the uncertainty distribution of climate sensitivity in the existing literature on IAMs because much of the uncertainty is only relevant in the very distant future. Finally Roe (2013) argues that the whole IAM enterprise by arguing that it is just a "numbers game". To put it another way, these objections to the usual approach in climate economics based upon cost benefit analysis (CBA) are similar to the list of problems with CBA discussed by Held (2013). However, we do not want to overstate criticism of CBA. For

[7] While this type of scaling in small noise expansions in robustness analysis is a useful device for simplifying a complex stochastic problem into a simple deterministic problem for analytical work, it has to be handled with care because the scaling needed may be inconsistent with detection probabilities that are consistent with available data sets (Anderson et al., 2012).

example, recent research on damages at the regional level (e.g., Barreca et al., forthcoming) could be aggregated appropriately to produce a global scale damage function with stronger foundations than current specifications of global damage functions. Nevertheless it is worthwhile, for conceptual clarity, to investigate a stark threshold-based option for comparison with existing work.

Meanwhile Matthews et al. (2009) and Matthews et al. (2012) have advanced a very interesting argument that the increase in mean global yearly temperature, which we refer to as just "temperature" from now on, is approximately proportional to cumulated carbon emissions in each of the respected big climate models that they simulate. They call this constant of proportionality the cumulated carbon response parameter (CCR). Matthews et al. (2012) argue that this finding of approximate constancy of the CCR parameter allows a cumulated carbon budget to be set that should not be exceeded for a given threshold temperature. Matthews et al. (2009) and Matthews et al. (2012) argue that there is evidence to support the proportionality relationship in reality and, hence, this relationship could be used for policy purposes.

Here is the potentially radical idea that we develop in this section. Instead of struggling with the problem of specifying an exactly parameterized damage function, we will let the climate science community set a threshold temperature that they agree should not be exceeded in order to avoid catastrophic climate change. We then use the Matthews et al. (2009) CCR parameter to set a cumulated carbon budget which should not be exceeded. Of course this implies an "implicit" damage welfare cost function which is essentially plus infinity when the threshold is exceeded. To put it another way, output that is left over for consumption is full output until the threshold is reached, and then it becomes zero. Our effort could be also viewed as a very crude simplification of Weitzman's (2012) much more sophisticated treatment in which he replaces the "standard" quadratic damage function with a damage function that increases much more sharply as temperature increases and replaces thin-tailed distributions of climate sensitivity with fat-tailed distributions like the Pareto distribution. It can be argued that this is taking too extreme a stand on a particular threshold temperature. In any event we explore the conclusions that taking this stand implies. This stand implies that our job as economists is to design a set of efficient institutions, i.e., policy instruments to implement the optimal path of emissions of the economy that do not exceed this cumulated carbon budget. In some sense we are taking the position that the climate science community has the expertise to set the limiting global average temperature increase that the climate system can tolerate and the economic science community's job is to design the best set of policies to maximize the welfare of the world economy, subject to this cumulated carbon emissions budget constraint. This idea is not unique to us. It is similar to the cost effectiveness analysis (CEA) advocated by Held (2013). As far as we know, using the Matthews et al. (2009) and Matthews et al. (2012) climate module to implement Held's CEA and the discussion of implementation of CEA by decentralized market-based institutions is new ,although it is closely related to some of the analysis in Anderson et al. (2014).

We realize that our approach has uncertainty problems that may be as great or greater than the received approach with detailed specification of objects like damage functions and approaches that deal with the "fat-tailed" distribution of possible values of the usual climate sensitivity parameter (e. g., Weitzman, 2011; Roe and Baumann, 2013). For example climate science has not settled on what the value of the critical threshold temperature is and readers of Matthews et al. (2009) and Matthews et al. (2012) will notice right away that the CCR parameter varies across respected big climate models. In any event, for readers who feel that our approach is too radical, it is fairly straightforward to extend it to cases where the damage function increases sharply over a domain of temperatures for which the climate science community has a strong consensus that going beyond temperatures in this domain would be truly catastrophic. We discuss these problems later. It is easiest to explain our approach with formal modeling. We will do the simplest deterministic case first because that will be enough to explain the basic ideas before turning to more complicated and realistic cases.

We assume that the global average temperature evolves much as in the Anderson et al. (2014) working paper which used the temperature dynamics climate model from the Matthews et al. (2009) and Matthews et al. (2012) papers. To our knowledge the Anderson et al. paper is the first paper in the climate economics literature to use the Matthews et al. approach for coupled climate-economic models. Anderson et al. still use damage functions as in Golosov et al. (2014), Nordhaus (2008) and others. The approach we use here is new and is not in Anderson et al. although it is closely related to some of the analysis in section 2 of that paper.

3.1.1 Deterministic Case: The Simplest Possible Model

We start with a specification where a closed form solution is available. Suppose T_c is chosen by the climate science community as the temperature which should not be exceeded to avoid catastrophic climate change. Since it is standard to work with the increment to temperature since pre-industrial times, "temperature" here is always short for "incremental temperature". For example, a standard choice for T_c is 2°C, that is two degrees Centigrade. However, this choice has recently become controversial (e.g., Victor and Kennel 2014). Nevertheless, it stands to reason that climate scientists and policy makers in general might want to try to keep the temperature from getting much bigger than, say, 3°C even if they felt that 2°C was too cautious. Indeed some climate scientists and economists (e.g., Hansen et al., 2013) argue that 2°C is too high for the planet to tolerate without serious harm and that 1°C should be the limit. Held (2013) argues for the 2°C limit as a sensible limit to set based upon current knowledge in climate science.

Stern (2013) discusses many extremely unpleasant effects that might occur if the Earth's temperature reaches 4°C. While the approach based upon cumulative carbon budgeting that we use here is extremely stark it does have the advantage of separating the model component specification tasks according to relative expertise. That is, specification of the global average temperature target not to be exceeded is left to the climate science community, and design of incentive structures to implement a cumulative carbon budget not to exceed that target is left to the economic science community. Since specification of a target temperature not to be exceeded is specification of an "extreme" damage function of zero until the target is reached, then minus infinity for temperatures larger than target, what our approach is basically saying is this: The climate science community has expertise in specifying a "penalty function" on temperature increases and the economics community has expertise in incentive mechanism design that should be exploited. This separation of specification tasks in the modeling exercise has elements of transparency and specification task separation across science communities according to relative expertise that might appeal to writers like Stern (2013) and Pindyck (2013b).

Repetto (2014) in his review of Nordhaus (2013) takes issue with the cost benefit approach to climate economics. Our approach here could be viewed as a start in developing an alternative approach that avoids some of the problems with the CBA approach at the expense of introducing other problems. Our view is that it is useful to place our approach on the table for discussion. We will not take a stand on its value relative to received approaches such as Nordhaus (2013) and many others who use a CBA type of approach to climate economics. In any event, the Matthews et al. (2009) and Matthews et al. (2012) framework allows specification of a target cumulative carbon budget not to be exceeded once a target temperature not to be exceeded is specified. It is easiest to explain the approach proposed here by working out some simple examples.

The proportionality of temperature to cumulated carbon emissions can be specified as

$$T(t) - T(0) = \lambda \int_0^t E(u)\, du, \tag{41}$$

where $T(t)$ and $T(0)$ denote current temperature and initial (e.g. preindustrial) temperature respectively, λ is the CCR parameter and $E(u)$ is emissions of GHGs emitted at time $u \in [0, t]$. Thus the rate of change of temperature is proportional to current emissions or

$$\dot{T}(t) = \lambda E(t), T(0) = T_0. \tag{42}$$

Probably the simplest "precautionary" approach to cumulative carbon budgeting that might appeal to some climate scientists and that might be illustrated with a toy model like this one is to take a value of the CCR parameter λ from the high end of the distribution of values across models displayed in, for example Matthews et al. (2009), call it $\lambda_{\max}$, and simply solve the problem,

$$\max_{E(t)} \left\{ \int_{t=0}^{\infty} e^{-\rho t} u(yE^{\alpha}) dt \right\} \tag{43}$$

subject to

$$\dot{T} = \lambda E,\ T(0) = T_0, \tag{44}$$

for the dynamics of global average temperature, where y is an exogenous productivity function, and

$$\begin{aligned} &\dot{S} = -\dot{T} = -\lambda_{\max} E,\ S(0) = T_c - T_0 \\ &S(t) \equiv T_c - T(t). \end{aligned} \tag{45}$$

This problem is just a standard exhaustible resource problem with the initial reserve set equal to

$$T_c / \lambda_{\max}. \tag{46}$$

We will see this same theme appear in the simple "toy" robust control problem with multiplicative uncertainty that we discuss below.

3.1.2 Robust Emission Control with Multiplicative Uncertainty

We introduce uncertainty and concerns about model misspecification in the temperature dynamics of model (43)-(44). Assuming that the drift distortion - which is chosen by the adversarial (i.e. the minimizing) agent - enters temperature dynamics multiplicatively, the deterministic approximation of the stochastic robust control problem discussed above requires solving the robust control problem[8]

$$\max_{E} \min_{v} \left\{ \int_{t=0}^{\infty} \mathrm{e}^{-\rho t} (u(yE^{\alpha}) + (1/2)\theta v^2) dt \right\} \tag{47}$$

subject to

$$\dot{T} = (\lambda + v)E, \; T(0) = T_0. \tag{48}$$

The ODE (48) describes the dynamics of global mean temperature (GMT), denoted here by $T(t)$ for each date t, with v denoting the distortion of the temperature dynamics due to model misspecification concerns which is chosen by the minimizing agent, and E denoting GHGs emissions which by a suitable choice of units can become equivalent to fossil fuel usage. Furthermore,

$$\begin{aligned} &\dot{R} = -E, \; R(0) = R_0 \\ &\dot{S} = -\dot{T} = -(\lambda + v)E, \; S(0) = T_c - T_0 \\ &S(t) \equiv T_c - T(t) \end{aligned} \tag{49}$$

describe the dynamics of fossil fuel usage and the dynamics of the "safety reserve" $S(t) \equiv T(t) - T_c$. The idea here is that the planner feels that he/she knows from reading Gillett et al. (2013), Matthews et al. (2009) and Matthews et al. (2012) that there is some true value of the CCR parameter[9] and sets for example,

$$\lambda = 1.5^{o}C/1000PgC, \tag{50}$$

based on the mean value reported by Matthews et al. (2012, page 4369), but wishes to, for example, robustify its choice against a possible choice of Nature in the 5-95% range of 1 to 2.1°C reported by Matthews et al. (2012, page 4369).

Specifying the utility function as $u(yE^{\alpha}) = \ln y + \alpha \ln E$, and noting that since $\ln y$ is exogenous it does not affect optimization, the current value Hamiltonian is given by

$$H = \max_{E} \min_{v} \{\alpha \ln E + \mu_S(-(\lambda + v)E) + \theta v^2/2\}. \tag{51}$$

[8]As we will see below, the multiplicative uncertainty problem is much easier to solve than the additive uncertainty case.

[9]The CCR parameter λ is expressed in terms of degrees Celsius per 1000 PgC. 1 PgC (petagram of Carbon) = 1 GtC (gigatonne of carbon). 1 GtC = 10^9 tonnes C = 3.67 Gt carbon dioxide.

The FONCs for a Nash equilibrium imply

$$\begin{aligned} &E = \alpha/(\mu_S(\lambda + v)) \\ &v = \mu_S E/\theta = \alpha/(\theta(\lambda + v)). \end{aligned} \tag{52}$$

The solution of (52) for v is:

$$\begin{aligned} &v^{*}= (-\lambda + D^{1/2})/2, \\ &D \equiv \lambda^2 + 4\alpha/\theta. \end{aligned} \tag{53}$$

We have chosen the positive root since a larger CCR is worse for the welfare of the maximizing agent. Since the path for the co-state variable is given by

$$\mu_S(t) = \mu_S(0)e^{\rho t}, \tag{54}$$

using the constraint

$$\begin{aligned} &S_0 = \int_{t=0}^{\infty}(\lambda + v^*)E(t)dt = \int_{t=0}^{\infty}(\lambda + v^*)(\alpha/(\mu_S(t)(\lambda + v^*))dt = \\ &\Rightarrow \mu_S(0) = \alpha/(\rho S_0) \end{aligned} \tag{55}$$

allows us to solve for $\mu_S(0)$. The solutions for energy use and the dynamics of $S(t)$ are given by

$$E(t) = \alpha \mathrm{e}^{-\rho t}/(\mu_S(0)(\lambda + v^*)) = (\rho S_0 e^{-\rho t})/(\lambda + v^*) \tag{56}$$

$$\begin{aligned} &S(t) = S_0 - \int_{s=0}^{t}(\lambda + v^*)E(s)ds \\ &= S_0 - \int_{s=0}^{t}[(\lambda + v^*)(\rho S_0 \mathrm{e}^{-\rho s})/(\lambda + v^*)]ds = S_0 e^{-\rho t}. \end{aligned} \tag{57}$$

Note that in the non-robust case, i.e. when there are no concerns about model misspecification so that $\theta \to \infty$ and $v \to 0$, energy use is independent of energy's share in production, i.e. energy use is independent of α. But in the robust case, energy use decreases in every period when energy's share increases. Another useful result is the time consistency of the equilibrium solution which is easy to show from (57). An extremely important property of the multiplicative uncertainty case is that $S(t) \geq 0$ holds for all positive dates for solution (57). Thus our solution procedure has actually produced an equilibrium solution to the zero sum game.

The value of the equilibrium of this zero sum dynamic game for the maximizing player is

$$\int_{t=0}^{\infty} \mathrm{e}^{-\rho t}\alpha \ln(E(t))dt = \int_{t=0}^{\infty} \mathrm{e}^{-\rho t}\alpha \ln(\rho S_0 e^{-\rho t}/(\lambda + v^*))dt. \tag{58}$$

Although the case of logarithmic utility is popular, the equilibrium solution has many useful properties that are special to this case. We investigate the more general case,

$$u(yE^{\alpha}) = [(yE^{\alpha})^{1-\gamma} - 1]/(1-\gamma). \tag{59}$$

The Hamiltonian for this case is

$$H = [(yE^{\alpha})^{1-\gamma} - 1]/(1-\gamma) + (1/2)\theta v^2 - \mu_S((\lambda + v^*)E). \tag{60}$$

The FONCs give us the equations (suppressing the dependence upon t except when needed for clarity):

$$v^*(t) = \frac{\mu_S(t)\,E(t)}{\theta} \tag{61}$$

$$E^*(t) = \frac{1}{\alpha}[\mu_S(\lambda + v^*(t))y^{\gamma-1}]^{1/[\alpha(1-\gamma)-1]} \tag{62}$$

$$\mu_S(t) = \mu_S(0)\mathrm{e}^{\rho t}. \tag{63}$$

Equations (61)-(63) imply that the solution for $v(t)$ must satisfy

$$v^*(t) = (1/\theta)[\left(\mu_S(t)^{(\alpha(1-\gamma))}\right)\frac{(\lambda + v^*(t))y(t)^{(\gamma-1)}}{\alpha}]^{1/[\alpha(1-\gamma)-1]}. \tag{64}$$

We see right away from (61)-(63) that for the logarithmic case, $\gamma = 1$, the time dependent terms drop out of (64) and we obtain equation (52) for $v(t) = v^*$, which is constant in time and is independent of the shadow price $\mu_S(t)$ of the state variable, $S(t)$. Since $\alpha(1-\gamma) - 1 < 0$, we see that the RHS of (64) is decreasing in v and the LHS is increasing in v. It is easy to see that for each date t there is a unique $v^*(t)$ that solves (64). Even if $y(t)$ is constant, there will still be time dependence of $v^*(t)$ unless $\gamma = 1$. The constancy of $v^*(t)$ as a function of time is important for time consistency of the equilibrium solution of the game.

For the nonlogarithmic case ($\gamma \neq 1$), the system (61)-(63) needs to be solved numerically. Assuming $y(t) = y_0\mathrm{e}^{gt}$, where $g > 0$ denotes an exogenous growth rate and replacing the discount rate ρ with $\omega = \rho - g(1-\gamma)$, an algorithm for numerically solving the nonlinear system can be described as follows:

1. Take a discrete time horizon $t = 0, ..., T$ for sufficiently large T and calculate the discrete approximation $\hat{\omega}$ of the continuous time discount rate ω, using $\omega = \ln(1+\hat{\omega})$;

2. Choose an $\mu_s(0)$ and numerically solve (61),(62) for E_t^* and v_t^*, $t = 0, ..., T$;

3. Calculate the sum $S_0^* = \sum_t (\lambda + v_t^*)E_t^*$;

4. Repeat steps 2 and 3 for different values of $\mu_s(0)$. Select the value for $\mu_s(0)$ for which the paths (E_t^*, v_t^*) result in a sum S_0^* that approximates the true S_0.

3.1.3 Cumulative Carbon Budgeting and Climate Changes Damages

The cumulative carbon budgeting framework can be combined with an explicit damage function associated with climate change to determine robust optimal emission policy. Writing the utility function as $y(t)\,E(t)^a\,\mathrm{e}^{-DT(t)}$, with the term $\mathrm{e}^{-DT(t)}$ reflecting climate change damages and assuming logarithmic utility, the robust control problem for the regulator can be written as:

$$\max_{E}\min_{v}\left\{\int_{t=0}^{\infty}\mathrm{e}^{-\rho t}(\alpha\ln E-DT+(1/2)\theta v^2)dt\right\} \tag{65}$$

subject to

$$\dot{R}=-E,\ R(0)=R_0 \tag{66}$$

$$\dot{T}=(\lambda+v)E,\ T(0)=0 \tag{67}$$

$$T_c-\int_0^{\infty}(\lambda+v)Edt\geq 0. \tag{68}$$

The current value Hamiltonian for this problem is

$$H=\alpha\ln E-DT+(1/2)\theta v^2-\mu_R E+\mu_T(\lambda+v)E+\phi\mathrm{e}^{\rho t}\left(T_c-\int_0^{\infty}(\lambda+v)Edt\right), \tag{69}$$

with optimality conditions

$$E(t)=\frac{\alpha}{\mu_R+(\phi\mathrm{e}^{\rho t}-\mu_T)(\lambda+v(t))} \tag{70}$$

$$v(t)=\frac{1}{\theta}\left(\phi\mathrm{e}^{\rho t}-\mu_T\right)E(t) \tag{71}$$

$$\phi\left(T_c-\int_0^{\infty}(\lambda+v)Edt\right)=0,\phi\geq 0 \tag{72}$$

$$\dot{\mu}_R=\rho\mu_R \tag{73}$$

$$\dot{\mu}_T=\rho\mu_T+D. \tag{74}$$

Note that taking the forward solution of (74), $\mu_T(t)=\int_{s=t}^{\infty}\mathrm{e}^{-\rho(s-t)}(-D)\,ds=-D/\rho$ for all t, which is constant. Now assume R_0 is large enough that some of the initial reserve is left in the ground in the non-robust case. Then we can try for a solution where $\mu_R(t)=0$ for all $t\geq 0$. Using (70) to substitute $E(t)$ into (71), we obtain

$$v(t)=\frac{\alpha}{\theta(\lambda+v(t))}. \tag{75}$$

Thus $v(t) = v^*$ constant and

$$E^* = \frac{\alpha}{(\phi + D/\rho)(\lambda + v^*)}. \tag{76}$$

Substituting E^* into the isoperimetric constraint (67), a ϕ^* that satisfies the constraint can be determined. The procedure can easily be extended to the case where $T(0) = T_0 > 0, T_0 < T_c$.

3.2 Climate change policy with multiple life time for greenhouse gases

Pierrehumbert (2014) has raised the important point that policy needs to take into account not only the GWP of a GHG but also the atmospheric lifetime of the gas. For example, methane has a much shorter lifetime but a higher GWP than CO_2. This part of our climate economics section explores policy on multiple life time gases in the context of a simple energy balance model with forcing determined by stocks of two GHGs, #1 with infinite life and #2 with short life but larger GWP, modelled as in the equations (78)-(79) below which abstract from the spatial considerations presented in section 3.3. We represent the problem and the different effects of the two gases introduced below in a very stark way and caution the reader thus. Notation is close to that in section 3.3. Furthermore, instead of a limit temperature T_c, we have introduced a damage function in (77). The positive value of $\beta > 0$ penalizes increases in global mean temperature by loss of consumable output.

Consider the following problem:

$$\max_{\{E_1(t),E_2(t)\}} \left\{ \int_{t=0}^{\infty} \mathrm{e}^{-\rho t} u(y(t)(E_1(t) + E_2(t))^{\alpha} \mathrm{e}^{-\beta(T(t)-T_0)}) dt \right\} \tag{77}$$

subject to

$$\begin{aligned} &C\dot{T}(t) = -BT(t) + \xi \ln[(M_1(t) + NM_2(t))/(M_1(0) + NM_2(0))], \\ &T(0) = 0 \end{aligned} \tag{78}$$

$$\begin{aligned} &\dot{M}_1(t) = b_1 E_1(t)\ ,\ M_1(0) = M_{10} > 0 \text{ given} \\ &\dot{M}_2(t) = -m_2 M_2(t) + b_2 E_2(t)\ ,\ M_2(0) = M_{20} > 0, \text{ given} \\ &\dot{R}_1(t) = -E_1(t), R_1(0) = R_{10}, \text{ given} \\ &\dot{R}_2(t) = -E_2(t), R_2(0) = R_{20}, \text{ given.} \end{aligned} \tag{79}$$

For clarity of focus we study a polar case where GHG #1 stays in the atmosphere forever and GHG #2 decays according to $m_2 > 0$, but the GWP of #2 is $N > 1$ times that of #1. Initially it might be thought that, in order to simplify the problem, we could also assume that the known reserves of the two sources of energy that produce emissions of the two GHGs are so large that the optimal costates will turn out to be zero. However, under the damage function above, this causes contradictions for any finite reserve, no matter how large, as we will show below.

The Hamiltonian and FONCs for this problem for the log utility case are, putting $M \equiv M_1 + NM_2$ and dropping terms that are irrelevant for optimization,

$$H = \ln y + \alpha \ln(E_1 + E_2) - \beta(T - T_0) + \mu_{M_1}(b_1 E_1) + \mu_{M_2}(-m_2 M_2 + b_2 E_2) \\ +\mu_T[(-B/C)T + (\xi/C)] \ln(M_1 + NM_2)) - \mu_{R_1} E_1 - \mu_{R_2} E_2. \tag{80}$$

The FONCs of optimal control for (80) associated with the evolution of the co-state variables which are interpreted as the shadow values of the corresponding stocks are:

$$\dot{\mu}_T = (\rho + B/C)\mu_T + \beta \tag{81}$$
$$\mu_T(t) = -\beta/((\rho + B/C)) \equiv \bar{\mu}_T, \text{ for all } t \tag{82}$$
$$\dot{\mu}_{M_1} = \rho\mu_{M_1} - ((\xi\bar{\mu}_T/C)(1/M)) \tag{83}$$
$$\dot{\mu}_{R_1} = \rho\mu_{R_1} \tag{84}$$
$$\mu_{M_1}(t) = \int_{s=t}^{\infty} e^{-\rho(s-t)}(\xi\bar{\mu}_T/C)(1/M(s))ds \tag{85}$$
$$\dot{\mu}_{M_2} = (\rho + m_2)\mu_{M_2} - ((\xi\bar{\mu}_T/C)(N/M)) \tag{86}$$
$$\mu_{M_2}(t) = N \int_{s=t}^{\infty} e^{-(\rho+m_2)(s-t)}(\xi\bar{\mu}_T/C)(N/M(s))ds \tag{87}$$
$$\dot{\mu}_{R_2} = \rho\mu_{R_2}. \tag{88}$$

In (81)-(88) we write the differential equations and their forward solutions for the co-state variables for the two gases as well as for the temperature co-state variable. These are part of the solution for the optimal control problem. We always impose the usual transversality conditions which help pick out these solutions for the co-state variables. Note that the co-state variable solution for temperature turns out to be a constant in time.

The Hamiltonian (80) can be written, after setting $E := E_1 + E_2$, as

$$\begin{aligned} H \;=\; & \ln y + \alpha \ln(E) + \mu_{M_2} b_2 E - \mu_{R_2} E + \\ & \left(\mu_{M_1} b_1 - \mu_{R_1}\right) E_1 - \left(\mu_{M_2} b_2 - \mu_{R_2}\right) E_1 \\ & -\beta(T - T_0) - \mu_{M_2} m_2 M_2 \\ & +\mu_T[(-B/C)T + (\xi/C) \ln(M_1 + NM_2)]. \end{aligned} \tag{89}$$

Since the Hamiltonian is linear in E_1 we obtain the switching rule,

$$\begin{array}{l} E_1 = 0, E_2 = E, \text{ if } b_1\mu_{M_1} - \mu_{R_1} < b_2\mu_{M_2} - \mu_{R_2} \\ E_1 = E, E_2 = 0, \text{ if } b_1\mu_{M_1} - \mu_{R_1} > b_2\mu_{M_2} - \mu_{R_2}. \end{array} \tag{90}$$

One might think it is intuitive that the shadow prices of the reserves, $\mu_{R_i}(0; \rho, R_{0i}),\ i = 1, 2$ would go to zero as $R_{0i} \to \infty,\ i = 1, 2$, but it is not quite so simple. Equations (85) and (86) imply that the shadow prices grow at rate ρ from any positive initial value, no matter how small, so we must proceed with care. For example, at first glance, one might think that as $\rho \to 0$, it would be obvious that the long-lived GHG, #1, would not be used, but the fact that the initial shadow prices of the reserves $\mu_{R_i}(0; \rho, R_{0i}),\ i = 1, 2$ depend upon ρ and may even increase as ρ decreases, makes it difficult to actually prove precise results for this system. Nevertheless we may obtain some results and stay within the scope of this chapter. We do enough here to make a strong case that policy analysis for multiple lifetime GHGs is a very fruitful and important area for future research.

Put $\zeta \equiv -\xi\bar{\mu}_T/C$. Since $\mu_{M_i} < 0,\ i = 1, 2$, the switching rule (90) may be written in the more transparent "cost" form

$$\begin{array}{l} E_1(t) = 0, E_2(t) = E(t) \\ \text{if } b_1\{\int_{s=t}^{\infty} e^{-\rho(s-t)}(\zeta/M(s))ds\} + \mu_{R_1}(0; \rho, m_2, R_{10})e^{\rho t} > \\ b_2N\{\int_{s=t}^{\infty} e^{-(\rho+m_2)(s-t)}(\zeta/M(s))ds\} + \mu_{R_2}(0; \rho, m_2, R_{20})e^{\rho t} \\ E_1(t) = E(t), E_2(t) = 0 \\ \text{if } b_1\{\int_{s=t}^{\infty} e^{-\rho(s-t)}(\zeta/M(s))ds\} + \mu_{R_1}(0; \rho, m_2, R_{10})e^{\rho t} < \\ b_2N\{\int_{s=t}^{\infty} e^{-(\rho+m_2)(s-t)}(\zeta N/M(s))ds\} + \mu_{R_2}(0; \rho, m_2, R_{20})e^{\rho t}. \end{array} \tag{91}$$

The rule (91) says to use the GHG that is cheapest in terms of social marginal cost at each point in time. Pierrehumbert (2014) stresses that policy focus should be on mitigating emissions of the long-lived gas #1 which plays the role of CO_2 in our model in contrast to the short-lived gas #2, which plays the role of methane, even though the short-lived gas has a larger GWP. Even though the switching rule in (91) is somewhat complicated, we may still draw some conclusions without too much work.

First, if some of R_{10} is not used, if the shadow price of #2 is positive, then it is not optimal to eventually specialize in using #2. To show this, by way of contradiction, note that the shadow price of #1 is zero, $\mu_{R_1}(t) = 0$, if it is not all used, and if the shadow price of #2 is positive it must grow at the rate ρ. But since, $M_{10} > 0$, we must have $1/M(t) \leq 1/M_{10}$ for all dates t. Thus the social marginal cost of #1 is bounded above by $\zeta/(\rho M_{10})$ while the social marginal cost of #2 is eventually growing at least by rate ρ. Hence eventually it will be cheaper to switch to #1. This is a contradiction.

Second, here is a corollary to the above argument : $\mu_{R_2}(0; \rho, m_2, R_{20}) - \mu_{R_1}(0; \rho, m_2, R_{10}) \leq 0$. To prove this by way of contradiction, subtract $\mu_{R_1}(0; \rho, m_2, R_{10})e^{\rho t}$ from both sides of (91). Note that if our claim does not hold repeat the same type of argument as above to get a contradiction.

Third, since $\dot{R}_i = -E_i$, $i = 1, 2$, the shadow price of one of the GHGs, can't be zero if the other is positive. We prove this assertion by way of contradiction. If the shadow price of gas i were zero at some point in time, then it must be zero at date $t = 0$, since $\dot{\mu}_{R_i} = \rho \mu_{R_i}$. But the shadow price of the other gas, call it j, being positive, grows at rate ρ. Eventually the social marginal cost of GHG j, i.e. $-\mu_{M_j}(t) + \mu_{R_j}(t)$, because it grows at least at rate ρ, must exceed the social marginal cost of GHG i, by an argument similar to that above, which causes a switch to i and usage of i until the reserve is exhausted which then causes the shadow price to become positive. This is a contradiction.

Fourth, at first we thought it would be simple to show that as $\rho \to 0$ eventually it would be optimal to specialize in using GHG #2 since it decays in the atmosphere but GHG#1 does not decay. However, since the shadow prices depend upon the decay rate m_2 as well as the discount rate ρ it is not so straightforward.

While we have barely scratched the surface of the interesting interaction of economics and climate science stimulated by Pierrehumbert (2014), we think that we have done enough to indicate that this is a promising area for future research. Of course we have neglected adjustment costs of switching and other complexities of the real world. This analysis has been pushed far enough to suggest that it is important to develop this kind of analysis of management policies toward multi-lived GHGs emissions in more realistic models. There are many directions in which this analysis can be taken. Implementation by differential taxes on different GHGs, for more general specifications of utility functions, general production functions, etc., as well as extension to robustness to treat specification doubts on the part of the planner as in Anderson et al. (2014) are all promising research directions.

We turn now to a very short illustrative discussion of implementation of equilibrium solutions by decentralized market institutions.

4 Implementation

The framework above assumes that the economist's job is just to implement the best way to satisfy the cumulative carbon budget constraint recommended by climate scientists. A natural way (at least for economists) is to try to find an optimal energy tax that implements the optimal solution to problem (47) in competitive equilibrium. A representative consumer solves the problem

$$\max_{c} \left\{ \int_{t=0}^{\infty} e^{-\rho t} u(c) dt \right\}, \tag{92}$$

subject to

$$c + \dot{b} = \pi + pE + rb + Tr, \; b(0) = 0. \tag{93}$$

Here $Tr(t)$ is lump sum redistribution to consumers of the energy taxes imposed on the representative firm at each date t, and pE is the lump sum redistribution of revenues from the representative energy firm to consumers.

The representative firms solve the problem

$$\pi = \max_{\{x,E\}} \{ yE^{\alpha} - (\tau + p)E \}. \tag{94}$$

Let $E^*(t)$ denote the equilibrium function of energy use in the robust control problem (47), which is a function of date t. Define

$$p(t) + \tau^*(t) = y\alpha E^*(t)^{\alpha-1}. \tag{95}$$

A representative energy firm solves

$$\Pi = \max_{\{E\}} \left\{ \int_{t=0}^{\infty} \mathrm{e}^{-\int_{s=0}^{t} r(s) ds} p(t) E(t) + \mu_{R0} (R_0 - \int_{t=0}^{\infty} E(t) dt \right\}. \tag{96}$$

The FONCs for the representative energy firm are

$$\begin{array}{l} \mathrm{e}^{-\int_{s=0}^{t} r(s) ds} p(t) = \mu_{R_0} \Rightarrow \\ \frac{\dot{p}(t)}{p(t)} = r(t) \;,\; p(0) = \mu_{R_0}. \end{array} \tag{97}$$

This "Hotelling's Rule" is just what we would expect since this is standard economics. An interesting wrinkle here occurs when the robust control problem recommends leaving some of R_0 in the ground. In this case the energy price $p(t) = 0$ for all dates t, and the value of the representative energy firm is zero.

Proposition 1 *Assume that the solution to problem (92) leaves some of* R_0 *in the ground. Suppose firms solving problem (94) face energy tax function,*

$\tau^*(t) = y(t)\alpha E^*(t)^{\alpha-1}$, then they will pick their profit-maximizing level of energy use to be $E^*(t)$ at each date t. If the profits and taxes distributed lump sum to the consumer solving problem (92) subject to the constraint (93) are evaluated at the $(^*)$-solution, then in equilibrium where borrowing and lending $b(t) = 0$ for all dates t, the consumption will be $c(t) = c^*(t) = y(t)E^*(t)^\alpha$ for all dates t.

Proof. Facing $\tau^*(t) = y(t)\alpha E^*(t)^{\alpha-1} + p(t)$, the representative firm's FONC is

$$y(t)\alpha E(t)^{\alpha-1} = \tau^*(t) + p(t) = y(t)\alpha E^*(t)^{\alpha-1} + p(t). \tag{98}$$

We will show that $p(t) = 0$ for all dates, t. Hence, in this case, by (98) it must be the case that $E(t) = E^*(t)$ for all dates t. This shows that the firms pick the $(^*)$-level of energy use at each date t. Suppose by way of contradiction that $\mu_{R_0} > 0$. Then by the FONCs for the representative energy firm we will have $p(t) > 0$ for all dates t and the energy firm will exhaust all of R_0, because $\mu_{R_0} > 0$. However if $p(t) > 0$ for all dates t in (96) (actually if it is positive for any date t), then less energy will be used in the $(^*)$-solution since the price is higher. Thus the total energy use must be less than the total energy used in the $(^*)$-solution. This contradicts $\mu_{R_0} > 0$. Hence $\mu_{R_0} = 0$ and thus $p(t) = 0$ for all dates t. This ends the proof for the case of the representative firm.

Turning to the consumers, the budget constraint when there is no borrowing and lending (which must be the case in equilibrium) is

$$\begin{aligned} &c(t) = \pi(t) + Tr(t) = \\ &\pi^*(t) + Tr^*(t) \\ &= yE^*(t)^\alpha - \tau^*(t)E^*(t) + \tau^*(t)E^*(t) = yE^*(t)^\alpha. \end{aligned} \tag{99}$$

■

Remark 2 *Notice that the function of time $E^*(t)$ can be an arbitrary function of time and the method above can still be used to find a tax function $\tau^*(t)$ that will implement it, provided that the sufficient condition, $\lambda R_0 > T_c$, for $\mu_{R_0}(0) = 0$ holds.*

Note that the equilibrium return on assets is given by

$$r^*(t) = \rho - \frac{u''(c^*(t))(dc^*(t)/dt)}{u'(c^*(t))}, \tag{100}$$

which can be worked out in closed form for this particular example, if needed.

5 Energy Balance Climate Models and Spatial Transport Phenomena

EBCMs are a useful and tractable way to model spatial effects on local temperatures of "outside" forcing of the climate system (North et al., 1981). Consider the following EBCM,

$$\begin{aligned} C\dot{T}_b(x,t) = QS(x)a(x) - (A + BT_b(x,t)) \\ +D\partial[D(x)(1-x^2)\partial T_b(x,t)/\partial x]/\partial x], \end{aligned} \tag{101}$$

where the notation follows North et al. (1981).That is,

$$C, Q, x, S(x), A, B, T_b(x,t), \partial[D(x)(1-x^2)\partial T_b(x,t)/\partial x]/\partial x,$$

denote respectively heat capacity per unit area (a constant), solar constant, sine of latitude x, solar energy received at latitude x, empirical constants A, B, "baseline" temperature of the climate system at latitude x with no outside human-induced forcing, and spatial energy transport operator. Now assume that for $t \geq 0$ a human-induced forcing $h(t)$ is "switched on" where we imagine that date zero is pre-industrial, e.g. date zero is 1750 and $h(t)$ is produced by humans using fossil fuels. This produces an "anomaly", i.e. a departure from the solution of (101), which we denote by $T(x,t)$ which satisfies the equation,

$$\begin{aligned} C\dot{T}(x,t) &= -BT(x,t)+\frac{\partial}{\partial x}[D(x)(1-x^2)\partial T(x,t)/\partial x] + h(t), \\ T(x,0) &= 0. \end{aligned} \tag{102}$$

In writing $h(t)$ instead of $h(x,t)$ for the human injections into the dynamical system, we are assuming that the effects of the emissions at each latitude are rapidly distributed across the globe's atmosphere relative to the time scale of the dynamics (102).

In order to show the relationship between the spatial climate dynamics (93) and the dynamics of global temperature, integrate both sides of (102) over latitudes, $x = -1$(minus 90 degrees) to $x = +1$ (plus 90 degrees), to obtain

$$C\dot{T}(t) \equiv \int_{x=-1}^{1} \dot{T}(x,t)dx = \tag{103}$$

$$= -B\int_{x=-1}^{1} dxT(x,t) + \int_{x=-1}^{1} dx\frac{\partial}{\partial x}[D(x)(1-x^2)\partial T(x,t)/\partial x] + \int_{x=-1}^{1} dxh(t) =$$

$$= -BT(t) + 2h(t),$$

$$T(0) = 0.$$

The usual formulation of human induced forcing $h(t)$ is

$$\begin{aligned} &h(t) = \xi \ln(M(t)/M_0), \\ &\dot{M}(t) = -mM(t) + bE(t), M(0) = M_0 \text{ given}, \end{aligned} \tag{104}$$

where $\xi, M(t), m, b$ denote respectively the climate sensitivity, the concentration of GHGs, e.g. CO_2 in the atmosphere (in ppm), removal rate parameter, and unit effect parameter of each unit of fossil fuels used. Allen et al. (2009), Matthews et al. (2009), Matthews et al. (2012), and Gillett et al. (2013) suggest that when the reaction of the carbon cycle (e.g. the response of the ocean and land to increased human-induced injections into the atmosphere) is taken into account, the dynamics (103) and (104), which we rewrite below as

$$\begin{aligned} &\dot{T}(t) = -(B/C)T(t) + (2/C)\xi \ln(M(t)/M_0), \\ &T(0) = 0, \\ &\dot{M}(t) = -mM(t) + bE(t), M(0) = M_0 \text{ given}, \end{aligned} \tag{105}$$

might be quite closely approximated by the dynamics (if an "ocean" is added as in Pierrehumbert (2014, Equation (4)),

$$\dot{T}(t) = \lambda E(t), T(0) = 0, \tag{106}$$

for an appropriate value of the climate carbon response parameter, λ, of Matthews et al. (2009) and Matthews et al. (2012). Pierrehumbert (2014, Equation (4), Figure 3) obtains the near linearity of the increase in global mean temperature as cumulated emissions increases with a simple two box model that includes an ocean as well as a shallow mixed layer like the above simple energy balance "one box" model (96). Cai et al.'s (2012b) DSICE model has a two layer temperature dynamics and a three layer carbon cycle. It is plausible that approximate linearity of the increase in global mean temperature with cumulated emissions might occur in their model too.

The papers of Brock, Engström, and Xepapadeas (2014b) and Brock et al. (2013) use the spatial energy balance approach like equation (102), together with specification of damage functions at each spatial location, to discuss the impact of energy transport as well as to offer a framework for allocating the burden of mitigation across space. If one abandons the attempt to specify damage functions and allocates the job of specifying a safe limit for global average temperature to climate scientists we can adapt the simple logarithmic utility example above to illustrate some approaches to allocating the burden of mitigation across locations.

Consider the problem

$$\begin{array}{l} \max\left\{\int_{t=0}^{\infty} \mathrm{e}^{-\rho t}(\int_{x=-1}^{1} w(x)\ln(y(x,t)E(x,t)^{\alpha})dx)dt\right\} \\ \text{subject to (106) and} \\ \int_{x=-1}^{1} E(x,t)dx = E(t). \end{array} \tag{107}$$

The statement of problem (107) leads us to the key issue raised by spatial concerns and that is how to specify the welfare weights, $\{w(x)\}$. In a series of works, Chichilnisky and Heal (1994, 2000) and Chichilnisky and Sheeran (2009) have long argued that poorer countries and countries that have not polluted as long as the developed countries should bear less of the burden of mitigation. While it is somewhat standard to give larger weights to countries or locations with larger populations, there are other considerations for assigning weights besides population weighting.

A recent paper by Saez and Stantcheva (2013) proposed an approach to choosing welfare weights in optimal tax theory that could be adapted here. Their approach is to use empirical evidence on attitudes towards redistribution and who should bear the biggest tax burden to discipline the choice of welfare weights. We believe that it would be useful to adapt their approach to specify welfare weights as a function of each country's historical emissions, current yearly emissions, yearly emissions per capita, growth in yearly emissions per capita, etc. Of course it is beyond the scope of this chapter to do the empirical work that would be required to data-discipline the choice of welfare weights here.

The simplest illustrative approach is just to base the weights on the "relative blame" for the problem. This could be done by specifying the welfare weight function as a decreasing function of the share of total emissions since 1750 as in (109) below. Figure 2 shows these shares across countries.

A possible allocation of welfare weights that captures the idea that locations that have emitted relatively heavily in the past should be allowed a smaller share of the world carbon budget in the future can be characterized as follows. Let

$$\int_{-\infty}^{0} E(s)ds \tag{108}$$

denote total world emissions up to the reference date zero. Define

$$\begin{array}{l} w(x) = \frac{1}{Z}\left[\exp(-\beta s(x)\int_{-\infty}^{0} E(t)dt)\right], \\ Z \equiv \int_{x'=-1}^{1} dx' \exp(-\beta s(x')\int_{-\infty}^{0} E(t)dt), \end{array} \tag{109}$$

where $\beta > 0$ is an "intensity" tuning parameter for penalization of locations that have emitted heavily in the past, and $s(x)$ denotes the share of location

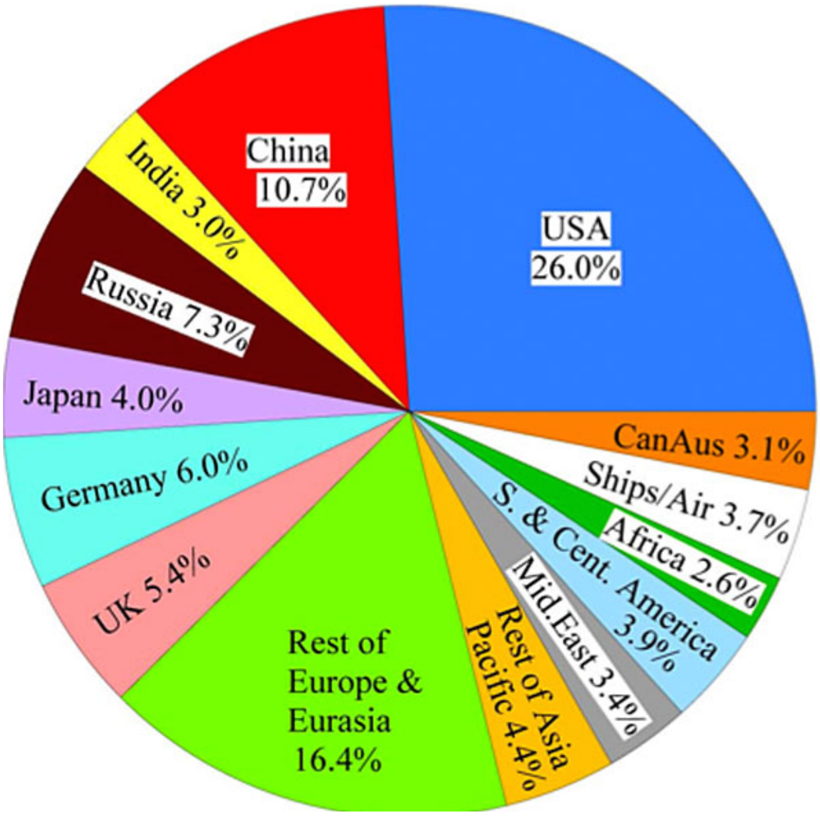


Figure 2: Fossil fuel CO2 cummulative emissions 1751-2012 Source: http://www.popularmechanics.com/science/environment/climate-change/

x in past total world emissions. While x denotes the sine of latitude in our illustrative example here, the approach in (109) could be extended to actual countries by replacing x by (x, y) where x is latitude and y is longitude and integrating over the set of (x, y) that characterizes a country. Even easier is to replace x and the integral with country index k and a sum over countries.

Solving the sub problem

$$\max \int_{x=-1}^{1} w(x) \ln(E(x)^{\alpha})dx, \text{ s.t. } \int_{x=-1}^{1} E(x)dt = E, \tag{110}$$

we obtain

$$\begin{aligned} &E(x) = w(x)E, \\ &U(E) = \textstyle\int_{x=-1}^{1} w(x) \ln(E(x)^{\alpha})dx = \int_{x=-1}^{1} w(x) \ln(w(x)^{\alpha})dx + \ln(E^{\alpha}). \end{aligned} \tag{111}$$

Hence, we see that for the case of logarithmic utility we can easily adapt the above treatment of robust control to get the solution of problem (107) for any limit temperature, T_c, and for any set of welfare weights. This brings us to implementation.

One way to implement the solution is to allocate at base date zero, to each x, a number of "rights to emit" equal to

$$w(x)\int_{t=0}^{\infty} E^{*}(t)dt = w(x)((T_c - T_0)/\lambda), \tag{112}$$

where example values of T_c, T_0, λ are 2°C, 0.85°C (IPCC working group 1 contribution to AR5), and 1.5°C per 1000 PgC emitted (Matthews et al., 2012). Then, as in Chichilnisky and Sheeran (2009), a world trading market for emission permits would set the world market price. Of course, in the real world, settling on the weights, i.e. the initial allocation of rights across countries, the number of rights to be allocated, etc., is a very difficult problem. Chichilnisky and Sheeran (2009) suggest possible ways of getting around such political problems, e.g. by bargaining over the distribution of marketable rights.

Weitzman (2014) is a very recent paper which suggests that negotiating over a uniform price for carbon can induce a large free riding emitter to support a higher price when it considers that everyone else being bound to pay that price would reduce negative externalities imposed by the rest upon that free riding emitter. The Chichilnisky-Sheeran type bargaining over the distribution of rights might be harnessing similar type incentives since the trading market will strike a uniform world-wide price.

5.1 Discounting for Climate Change

There is a substantial literature on the choice of the discount rate, or the consumption discount rate, which is appropriate for discounting future costs and benefits which are associated with environmental projects (e.g., Arrow et al. 1996; Weitzman, 1998b, 2001; Newell and Pizer, 2003). In this section we show how the approach of cumulative carbon budgeting can be used to adjust the consumption discount rate in order to take climate change into account. The consumption discount rate can be defined by the equilibrium condition in two equivalent ways: (i) following Arrow et al. (2012, 2014) and considering a social planner who would be indifferent between \$1 received at time t and \$$\varepsilon$ received today when the marginal utility of \$$\varepsilon$ today equals the marginal utility of \$1 at time t, or (ii) following Gollier (2007) and considering a marginal investment in a zero coupon bond which leaves the marginal utility of the representative agent unchanged.

Assuming that the utility function of the representative agent depends on consumption and damages associated with the time path of global average temperature

$$U = u\left(c\left(t\right), T\left(t\right)\right), \tag{113}$$

the equilibrium condition associated with the Arrow et al. (2014) approach implies that

$$\begin{aligned} \varepsilon u_c\left(c\left(0\right), T\left(0\right)\right) &= \mathrm{e}^{-\rho t} u_c\left(c\left(t\right), T\left(t\right)\right), \ \textit{or} \quad (114) \\ \varepsilon &= \frac{\mathrm{e}^{-\rho t} u_c\left(c\left(t\right), T\left(t\right)\right)}{u_c\left(c\left(0\right), T\left(0\right)\right)} = \mathrm{e}^{-r_t t}, \quad (115) \end{aligned}$$

where r_t denotes the annual consumption discount rate between periods 0 and t, and ρ is the utility discount rate. The equilibrium condition associated with the Gollier (2007) approach implies that

$$u_c\left(c\left(0\right), T\left(0\right)\right) = \mathrm{e}^{-\rho t} u_c\left(c\left(t\right), T\left(t\right)\right) \mathrm{e}^{r_t t}, \tag{116}$$

where r_t is interpreted as per period rate of return at date 0 for a zero coupon bond maturing at date t. Both approaches are equivalent for determining the consumption discount rate. Assume, as it is common in this case, a constant relative risk aversion utility function

$$u\left(c\left(t\right), T\left(t\right)\right) = \frac{1}{1-\eta}\left[c\left(t\right) \mathrm{e}^{-DT(t)}\right]^{1-\eta}, \ 0 < \eta < \infty, \tag{117}$$

where η is both the coefficient of relative risk aversion and (minus) the elasticity of marginal utility with respect to consumption. Then using equilibrium condition (116), the Matthews et al. (2009) and Matthews et al. (2012) framework where $\dot{T}\left(t\right) = \lambda E\left(t\right)$ implies that

$$\begin{aligned} r_t &= \rho - \frac{d}{dt} \ln u_c\left(c\left(t\right), T\left(t\right)\right) = \rho - \frac{d}{dt}\left[-\eta c\left(t\right) - \left(1-\eta\right) DT\left(t\right)\right] \quad (118) \\ r_t &= \rho + \eta g\left(t\right) + \left(1-\eta\right) D\lambda E\left(t\right), \quad (119) \end{aligned}$$

where $g\left(t\right) = \acute{c}\left(t\right)/c\left(t\right)$ is the consumption rate of growth, $\rho + \eta g\left(t\right)$ is the standard Ramsey discount rate, and the term $\left(1-\eta\right) D\lambda E\left(t\right)$ is the climate change adjustment to the Ramsey rule for discount rate. The sign of the adjustment depends on the value of η. Regarding this value, Mehra and Prescott (1985) suggest that a value above 10 is not justifiable, while Dasgupta (2008) suggests that values of η in the region of 1.5 to 3 would be reasonable.[10] Thus

[10] Note that Cline (1992) uses $\rho = 0, \eta = 1.5$, Nordhaus (1994) $\rho = 3\%$, and $\eta = 1$, Stern (2006) $\rho = 0.1\%$, $\eta = 1$. See Dasgupta (2008) for a detailed discussion of these assumptions.

values of η greater than 1 are plausible, and therefore in such cases climate damage effects cause market discount rates to be smaller than the Ramsey rule. Since the effect is larger the larger are D and λ and the emissions path $E(t)$, the plausible assumption that the world will continue increasing emissions before they finally start to decrease (e.g., see Pierrehumbert, 2014, Figure 1), implies that effect of climate change on market discounting in (119) could be quite large for η greater than 1.

Consider now the case where spatial considerations are explicitly introduced by allowing for spatially dependent welfare weights $w(x)$ along the lines of welfare weights introduced in section 5, and spatial differentiation of damages, so that the utility function is defined as

$$U = u\left(c\left(x,t\right), T\left(x,t\right)\right) = w\left(x\right)\left[\frac{c\left(x,t\right)\mathrm{e}^{-D(x,T(t))}}{1-\eta}\right]^{1-\eta}. \tag{120}$$

The term $D\left(x, T\left(t\right)\right)$ can be regarded as shorthand for the impact of the global heating of the Earth on damages at location x, which includes heat transport effects as in the EBCMs discussed in Brock et al. (2013) and Brock, Engström, and Xepapadeas (2014b). The equilibrium condition associated with the Arrow et al. (2014) approach implies that

$$\frac{\int_X \mathrm{e}^{-\rho t} u_c(c\left(x,t\right), T\left(x,t\right))\, dx}{\int_X u_c(c\left(x,0\right), T\left(x,0\right))\, dx} = \mathrm{e}^{-r_t t}. \tag{121}$$

Using the specific utility function we obtain

$$r_t = \rho - \frac{1}{t}\ln\left\{\frac{\int_X \left[w\left(x\right)c\left(x,t\right)^{-\eta}\mathrm{e}^{-(1-\eta)D(x,T(t))}\right]dx}{\int_X \left[w\left(x\right)c\left(x,0\right)^{-\eta-\eta}\mathrm{e}^{-(1-\eta)D(x,T(0))}\right]dx}\right\}, \text{ or} \tag{122}$$

$$r_t = \rho - \frac{d}{dt}\ln\left\{\int_X \left[w\left(x\right)c\left(x,t\right)^{-\eta}\mathrm{e}^{-(1-\eta)D(x,T(t))}\right]dx\right\}. \tag{123}$$

This can be regarded as an average global consumption discount rate between periods 0 and t, that a social planner will use for cost benefit calculations over the entire spatial domain. The location specific discount rate can be determined by using, the equilibrium condition (119) to obtain

$$r_t\left(x\right) = \rho - \frac{d}{dt}\ln u_c\left(c\left(x,t\right), T\left(x,t\right)\right), \text{ or} \tag{124}$$

$$r_t\left(x\right) = \rho + \eta g\left(t,x\right) - \left(1-\eta\right)\frac{\partial D\left(x,T\left(t\right)\right)}{\partial T}\lambda E\left(t\right). \tag{125}$$

In this case the consumption discount rate in a certain location depends on the location specific consumption rate of growth and the damages due to climate change associated with the location.[11] It should be noted however that if there is a world capital market, arbitrage would force the local rates $r_t(x)$ equal to a global market rate r_t. A thorough analysis of this would need to model borrowing and lending in each location in world capital markets. Of course if for some reason each x is is treated as a closed economy then $r_t(x)$ would be the equilibrium rate inside that "country" x. This analysis could be an interesting area for future research.

6 Spatial Aspects in Economic/Ecological Modelling

In section 5 we introduced EBCMs as a tractable way to model spatial effects in local temperatures. While spatial effects are a very important aspect of climate change economics, their importance is extended to a large number of areas related to environmental and resource economics (e.g. Wilen, 2007; Brock and Xepapadeas, 2010; Xepapadeas, 2010; Kyriakopoulou and Xepapadeas, 2013; Brock, Xepapadeas and Yannacopoulos, 2014a, 2014b, 2014d) but also to other areas.

Biology has been an area where spatial effects in the context of mechanisms generating form, or spatial patterns, have been extensively studied. The question of 'how the leopard got its spots' has been central to this type of analysis (e.g., Levin and Segel, 1985; Okubo and Levin, 2001; Murray, 2003).

In economics, spatial patterns in a static framework have been extensively studied in the context of new economy geography (e.g., Krugman, 1996; Fujita et al., 1999; Baldwin et al., 2001; Fujita and Thisse, 2002). Recent research in this area explicitly studies spatial dynamics agglomeration formation in models of competitive industries and models of economic growth (Boucekkine et al., 2009; Boucekkine et al., 2013; Brock, Xepapadeas, and Yannacopoulos 2014a, 2014d).

In environmental and resource economics, the spatial dimension has been introduced mainly in the context of fishery management with the use of metapopulation models to study harvesting rules or reserve creation (e.g., Sanchirico and Wilen, 1999, 2001, 2005; Smith and Wilen, 2003; Sanchirico, 2005; Wilen, 2007; Costello and Polasky, 2008). More recently Brock and Xepapadeas (2005, 2008, 2010) and Brock, Xepapadeas, and Yannacopoulos (2014c) by using continuous spatial dynamic processes (see also Smith et

[11] The utility discount rate ρ and elasticity of marginal utility η are assumed to be the same across locations. This assumption can easily be relaxed.

al., 2009) generalized the concept of Turing diffusion-induced instability to dynamic optimization problems and studied pattern formation and agglomeration emergence in optimal control problems with applications to resource management. It is interesting to note that the spatial pattern of local temperatures in models of climate and the economy can be studied in the same general context with models studying interactions in natural, economic or unified systems of ecosystems and the economy, which evolve in time and space.

Uncertainty or ambiguity related to concerns about model misspecification and robust control approaches discussed in the previous sections can be naturally extended to spatial settings. In this case, a situation emerges in which a decision maker or a regulator distrusts his model and wants good decisions over a cloud of models that surrounds the regulator's approximating or benchmark model, but these concerns have a spatial structure and may different across locations given the characteristics and structure of the problem. The problem of spatially structured uncertainty has been studied by Brock, Xepapadeas, and Yannacopoulos (2014c) where a central result is the development of spatial robust control regulation, and the potential emergence of spatial hot spots, which are locations where the spatial structure of uncertainty causes regulation to break down.

In this section we use methods that allow us to obtain a deterministic control problem (Campi and James, 1996; Anderson et al., 2012; Anderson et al., 2014) to study spatially extended models of ecosystems and economy. The purpose is to show how models with spatially structured ambiguity can be developed and explicitly solved in order to obtain spatially dependent robust control rules, and explore the potential emergence of hot spots.

6.1 Spatially extended deterministic robust control problems

We develop a spatially extended linear – quadratic (LQ) robust control problem, which can be regarded as an LQ version of problem (6) with spatial transport related to the state variable. In particular we consider a bounded spatial domain. Then $c(t,z)$, $x(t,z)$, and $h(t,z)$ denote control, state and distortion at time t and location z respectively. Spatial transport can be introduced in the following way. Assume that the mass or substance associated with the state variable which is located at point z moves to nearby locations and that the direction of the movement is such that mass from locations where mass is abundant, i.e., locations of high mass concentration, moves toward locations of low mass concentration. This is the assumption of Fickian diffusion, or Fick's first law, and is equivalent to stating that the flux of mass denoted by $x(t,z)$ is proportional to the gradient of the mass concentration, i.e., the spatial derivative of concentration, or

$$J(t,z) = -D\frac{\partial x(t,z)}{\partial z}, \tag{126}$$

where D is the diffusion coefficient or diffusivity measuring how fast mass moves from locations of high concentration to locations of low concentration. In terms of the spatial EBCM the state variable can be interpreted as heat moving from the equator to the Poles. In terms of ecosystem modeling the state variable can be interpreted as concentration of a resource or pollution at a specific location. Following Brock, Xepapadeas, and Yannacopoulos (2014b, 2014d), the spatiotemporal evolution of a state variable under Fickian diffusion and concerns about model misspecification reflected in a Hansen-Sargent entropic constraint can be written as

$$dx = \left(\lambda c(t,z) - \delta x(t,z) + D\frac{\partial^2 x(t,z)}{\partial z^2} + \varepsilon^{1/2}\sigma h(t,z)\right)dt + \varepsilon^{1/2}\sigma x(t,z)\,dW \tag{127}$$

where W is a Hilbert space-valued Wiener process.[12] Thus the LQ version of problem (6) is

$$\max_{c(t,z)}\min_{h(t,z)} \int_0^\infty \int_Z \mathrm{e}^{-\rho t}\left[\alpha c(t,z) - \frac{\beta}{2}c(t,z)^2\right. \tag{128}$$

$$\left. -\frac{\gamma}{2}x(t,z)^2 + \frac{\theta(\varepsilon)}{2}h(t,z)^2\right] dzdt \tag{129}$$

$$\text{subject to } (127)\,,\ x(0,z) = x(z)\,,$$

and appropriate spatial boundary conditions.

Let

$$g(x,c,h) = \alpha c(t,z) - \frac{\beta}{2}c(t,z)^2 - \frac{\gamma}{2}x(t,z)^2 + \frac{\theta(\varepsilon)}{2}h(t,z)^2 \tag{130}$$

$$f(x,c,h,x_{zz}) = \lambda c(t,z) - \delta x(t,z) + D\frac{\partial^2 x(t,z)}{\partial z^2} + \varepsilon^{1/2}\sigma h(t,z)\,. \tag{131}$$

The HJBI equation for problem (128) can be written as

$$\rho V(x) = \max_c \min_h \left\{\int_Z \left[g(x,c,h)\,dz + V'(x)\,f(x,c,h,x_{zz})\right.\right. \tag{132}$$

$$\left.\left. +V''(x)\,\varepsilon(\sigma h)^2\right]dz\right\},$$

[12] For definitions, see for example da Prato and Zabczyk (2004).

where $V'(x)$, are $V''(x)$ Frechet differentials of the value function.[13]

Following Campi and James (1996), Anderson et al. (2012) and Anderson et al. (2014) let $\theta(\varepsilon) = \theta\varepsilon$ and scale the θ with ε in such a way that as $\varepsilon \to 0$ we obtain the HJBI equation which is associated with the deterministic robust control problem in the spatiotemporal domain, or

$$\rho V(x) = \max_c \min_h \left\{ \int_Z \left[g(x, c, h)\, dz + V'(x)\, f(x, c, h, x_{zz}) \right] dz \right\}. \tag{133}$$

Equation (133) can be associated with the spatial deterministic robust control problem

$$\max_{c(t,z)} \min_{h(t,z)} \int_0^\infty \int_Z \mathrm{e}^{-\rho t} \left[\alpha c(t,z) - \frac{\beta}{2} c(t,z)^2 - \frac{\gamma}{2} x(t,z)^2 + \frac{\theta}{2} h(t,z)^2 \right] dz\, dt \tag{134}$$

subject to

$$\frac{\partial x(z,t)}{\partial t} = \lambda c(t,z) - \delta x(t,z) + \sigma h(t,z) + D \frac{\partial^2 x(t,z)}{\partial z^2}.$$

Problem (134) which has been studied by Derzko et al. (1984), Brock and Xepapadeas (2008), and Brock, Xepapadeas, and Yannacopoulos (2014b), has a Hamiltonian representation:

$$\max_{c(t,z)} \min_{h(t,z)} \left[\alpha c(t,z) - \frac{\beta}{2} c(t,z)^2 - \frac{\gamma}{2} x(t,z)^2 + \frac{\theta}{2} h(t,z)^2 \right. \tag{135}$$

$$\left. p(t,z) \left(\lambda c(t,z) - \delta x(t,z) + \sigma h(t,z) + D \frac{\partial^2 x(t,z)}{\partial z^2} \right) \right]. \tag{136}$$

The maximum principle implies the following FONCs for the controls:

$$c(t,z) = \frac{\alpha + \lambda p(t,z)}{\beta} \tag{137}$$

$$h(t,z) = -\frac{\sigma}{\theta} p(t,z). \tag{138}$$

Then the Hamiltonian system is a system of backward-forward PDEs:

$$\frac{\partial p(t,z)}{\partial t} = (\rho + \delta) p(t,z) + \gamma x(t,z) - D \frac{\partial^2 p(t,z)}{\partial z^2} \tag{139}$$

$$\frac{\partial x(t,z)}{\partial t} = \frac{\lambda \alpha}{\beta} + \left(\frac{\lambda^2}{\beta} - \frac{\sigma^2}{\theta} \right) p(t,z) - \delta x(t,z) + D \frac{\partial^2 x(t,z)}{\partial z^2}. \tag{140}$$

[13] A heuristic derivation of the HJBI equation is presented in the Appendix.

As shown analytically in Brock, Xepapadeas, and Yannacopoulos (2014b), by using solutions of the form

$$\begin{aligned} x(t,z) &= \sum_n x_n(t) \sin\left(\frac{n\pi}{L} z\right) \\ p(t,z) &= \sum_n p_n(t) \sin\left(\frac{n\pi}{L} z\right) \end{aligned} \tag{141}$$

where n is the number of Fourier modes. The system of backward-forward PDEs (139)-(140) can be transformed into a countable set of linear ODEs. This system can be written as:

$$\begin{aligned} &\frac{dp_n(t)}{dt} = (\rho + \delta + \phi_n) p_n(t) + \gamma x_n(t) \\ &\frac{dx_n(t)}{dt} = \frac{\lambda\alpha}{\beta} + \left(\frac{\lambda^2}{\beta} - \frac{\sigma^2}{\theta}\right) p_n(t) - (\delta + \phi_n) x_n(t) \\ &x_n(0) = \xi_n \,, \ \lim_{t\to\infty} e^{-\rho t} p_n(t) x_n(t) = 0 \\ &x(z) = \sum_n \xi_n \sin\left(\frac{n\pi}{L} x\right) \,, \ \xi_n = \frac{2}{L}\int_0^L x(z) \sin\left(\frac{n\pi}{L} x\right) \\ &\phi_n = \frac{D\pi^2}{L^2} n^2. \end{aligned} \tag{142}$$

Solving system (142) for a sufficient number of Fourier modes and substituting back the solutions for into (137),(138), and (141) we can obtain the optimal robust spatiotemporal paths for the state, costate and control variables.

6.1.1 An Example

To provide a worked out example of the approach we use the following parameterization of the linear quadratic pollution control problem.[14]

Parameter	Value
α	224.26
β	1.9212
γ	0.0223
δ	0.0083
σ	0.2343
ρ	0.03
λ	1
D	1

[14] This parametrization has been used by Karp and Zhang (2006), Athanassoglou and Xepapadeas (2012) for the study of linear quadratic climate change models. We use the same parametrization here, although we are not calibrating a spatial climate change model, to show how the spatially dependent solution for the states and the controls can be constructed

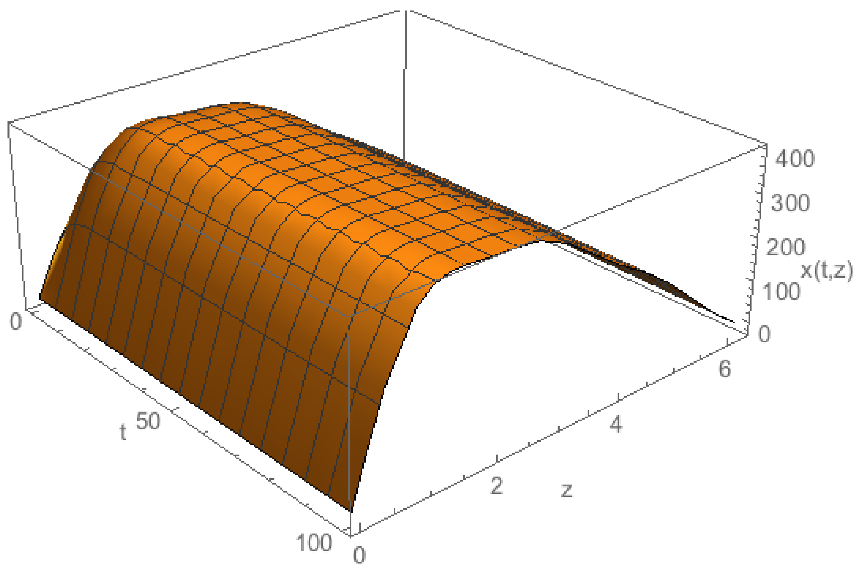
The spatial dimension is introduced by considering a spatial domain $Z = [0, 2\pi]$, allowing for spatial transport with diffusion parameter $D = 1$, and considering the following initial spatial distribution for the stock variable:

$$x(0, z) = 100 \exp\left[-(z - \pi)^2\right] \ , \ z \in [0, 2\pi] . \tag{143}$$

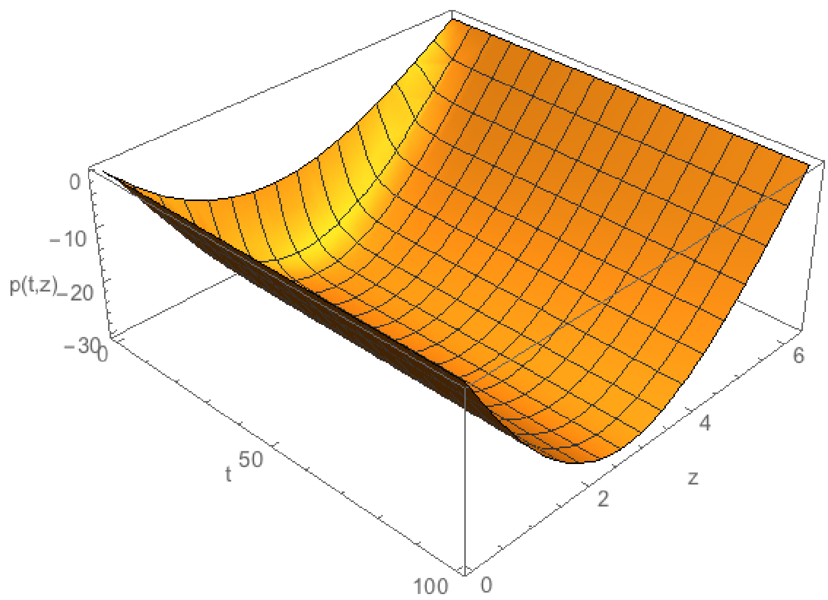
We solved the system (142) for the first six modes $n = 1, ..., 6$ and for two different values of $\theta = \{1, 10\}$. The steady state of the Hamiltonian system corresponding to each mode was a saddle point. Setting the constant associated with the positive eigenvalue equal to zero, the solutions for the state and the costates were of the general form

$$p_n(t) = p_n^* + \upsilon_n^1 C_n \mathrm{e}^{s_n t} \tag{144}$$
$$x_n(t) = x_n^* + \upsilon_n^2 C_n \mathrm{e}^{s_n t}, \tag{145}$$

where (x_n^*, p_n^*) is the steady state for mode n, s_n is the negative eigenvalue, $(\upsilon_n^1, \upsilon_n^2)$ is the eigenvector corresponding to the negative eigenvalue, and C_n is a constant determined by initial conditions on $x_n(0)$. The mode-solutions (144), (145) are substituted into (141) to obtain the spatiotemporal paths for the state, costate and control functions. Figures 4 - 7 show these paths for $\theta = 10$.



3.pdf

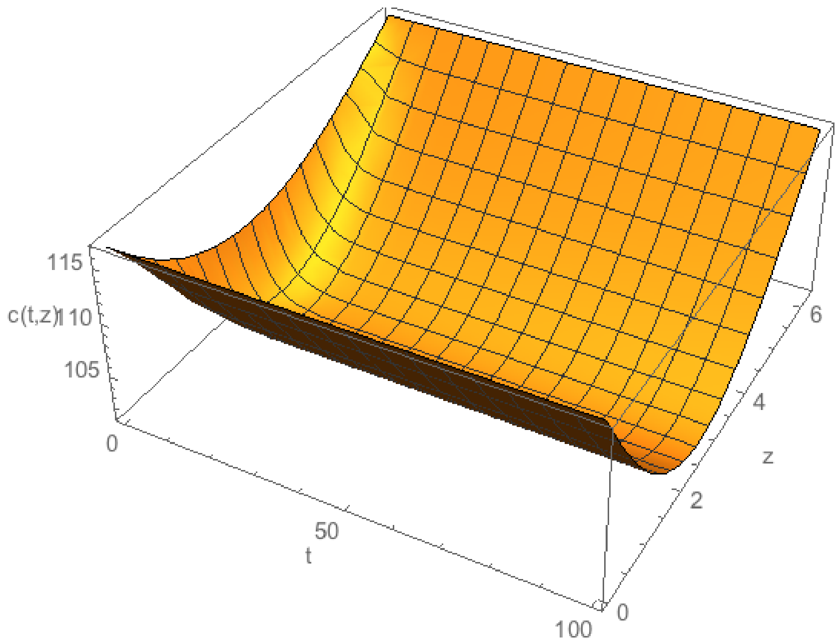


7 Future Directions

The literature on climate economics models is huge; this chapter has reviewed only a relatively narrow slice of work on climate economics models where space and distributional questions play a major role. We think that the tools and methods presented in this review, using climate economics as the main vehicle, provide useful insights about the way to model coupled ecosystems and economic systems. In this section we become even more speculative and discuss potential future directions that research might take.

7.1 Bottoms Up Implementation Rather than Top Down Implementation

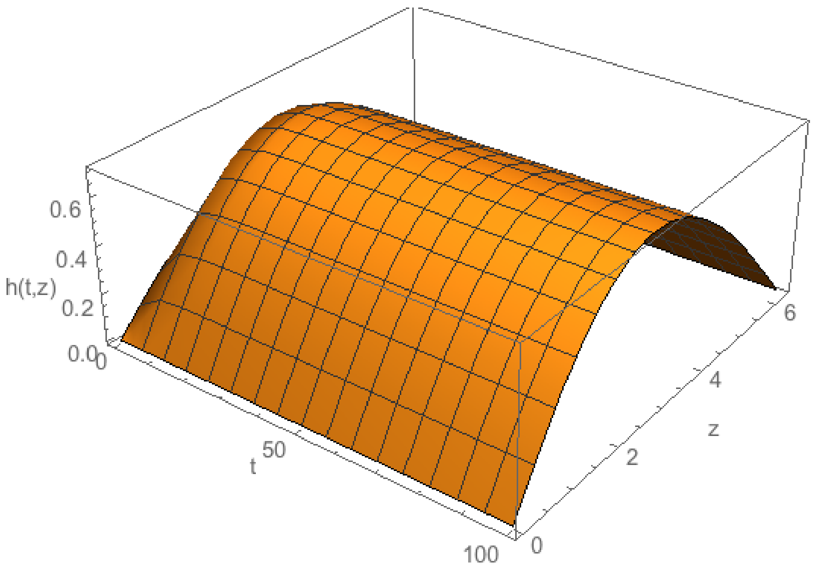
Ostrom (2009) has stressed the problems inherent in any kind of top down central authority organizing effective adaptation and policy measures in a massive collective action problem like global climate change. Indeed the climate management problem could be labeled, “The Mother of All Collective Action Problems”. This is so because the spillover effects are world-wide. Given the difficulties of organizing any kind of collective action at such a large scale, Ostrom argues for a bottoms up approach in what she calls a polycentric approach, in which entities at multiple scales adapt and respond individually. For example, California, which is bigger than a lot of countries, has instituted strong responses to climate change on its own. Since California



tends to be a world leader, its example may prompt others to take action.

This is a good place to discuss adaptation as well as mitigation in dealing with climate change. Adaptation has the attractive feature that its benefits tend to be local so that any unit of government at any scale that bears costs of adaptation will also capture its benefits at the local scale. For example, a unit of government at the scale of a beach front that taxes beach front owners to pay for adaptation to storm surges serves a public that benefits directly from that adaptation when the next storm surge hits that beach front community. This "scale matching" of benefits and costs of adaptation is absent in the case of mitigation. In the case of mitigation a nation or state that pays costs of mitigation ends up benefiting the whole planet and, hence, failing to capture the full benefits of its contribution.

As units of government at different scales struggle to organize collective action on adaptation, this may lead to formation of effective institutions that might be leveraged to organize collective action on large scale issues such as mitigation. We see this kind of work, given the lead of Ostrom, as especially promising in the study of potentially effective policy actions to deal with global climate change. In particular the benefits from adaptation can be quite large relative to costs, e.g. it can be quite dramatic as documented by de Bruin et al. (2009) in the context of the DICE model modified for adaptation, which they dub AD-DICE. At smaller scales than DICE or even RICE's regional scales, Deschênes and Greenstone (2007, 2011) document



climate damages at smaller scales. The very recent report "Heat In The Heartland" (Gordon, 2015), projects climate damages at smaller scales for the U.S. Midwest. As coping and adaptation develops at smaller scales, it is feasible that units of government will form at scales of spillover externalities (e.g. river authorities to manage increased flooding from climate change and water management authorities to manage increased stress on aquifers). As these units of government form, it is plausible to imagine that they might cooperate to manage externalities that spill across their boundaries. Thus we believe that an especially fruitful area of future research is to draw on work such as Ostrom (2009) for endogenous institution formation at various scales, work like Deschênes and Greenstone (2007, 2011) for documentation of climate damages at various scales, and work on adaptation at various scales to try to understand what institutions are likely to form at what scales.

7.2 Stochastic Modelling and Computational Approaches

Climate models come in a hierarchy ranging from the simplest models of energy balance (North, 1975a, 1975b; North et al., 1981), energy and moisture balance (Fanning and Weaver, 1996), which can be solved analytically, to models that are larger but still small enough that their mechanisms can be comprehended with a combination of analytical and computational work. Examples of the latter are Nordhaus's (2008, 2013) DICE and RICE models and the DSICE model of Cai et al. (2012a, 2012b, 2012c, 2013a, 2013b) and Cai et al. (2014), which contrast with the big General Circulation Models

(Weaver et al., 2001).

An important area for future research would be to extend work on the coupled heat balance EBCMs and economic models of Brock et al. (2013) and Brock, Engström, and Xepapadeas (2014a, 2014b) to coupled heat and moisture balance EBCM's like the Fanning and Weaver (1996) model. This approach adds to the PDE describing the dynamic evolution of temperature across latitudes, and a second one describing the evolution of surface specific humidity at latitude x. Human actions take the form of emissions of GHGs and geoengineering that blocks incoming short-wave radiation. The advantage of this approach would be to provide more insights into the spatial impacts of climate change and the associated policies in terms of temperature precipitation and evaporation. This research would fit well with the work recently developed by Brock, Xepapadeas and Yannacopoulos (2014b, 2014c), on spatial hot spots which are locations where regulation breaks down due to deep structural uncertainty. Hot spots indicate locations where damages might be excessive and extra attention is required by regulators. Hot spot research regarding changes in precipitation and evaporation at localized scales may be more important than hot spot research regarding temperature changes at smaller scales. Local damages can be modelled by damage functions of the form $\exp[-D(x,t)T(t)]$, which have been useful in characterizing spatial discounting.

7.3 Bifurcations and Tipping Points

Cai et al. (2012b) extend their DSICE model to include tipping points which may or may not be caused by bifurcations. However, they do not treat spatial transport or space itself in their model. We see a particularly promising line of research to be that of extending the work of Cai, Judd, and Lontzek to include spatial transport phenomena. This kind of framework would be very useful for economic analysis of the risk of bifurcations and how much it might be worth to society to avoid such risks. A major concern in spatial settings when compensatory transfers are not available is inequalities of burdens of future climate change across the globe.

Very recent papers have appeared on potential bifurcations of Arctic Sea ice (Eisenman and Wettlaufer, 2009; Abbot et al., 2011). One potential value of spatial transport modeling is that it might augment the case for different policies treating GHGs that have short lives in the atmosphere but much higher GWPs compared to GHGs like CO_2 emitted from coal and oil usage that have very long lives in the atmosphere. To explain further if there is a relatively short-term immediate "damage reservoir" threat like a potential bifurcation of Arctic Sea ice, it might make sense to tax short-lived GHGs with very large GWPs like methane emissions at a temporary higher rate to slow down current global warming even though the extra consumption of fossil fuels with long-lived GHGs will result in more damaging warming in the long term future. The framework used by Brock, Engström, and Xepapadeas(2014b) and Brock et al. (2013) that couples economic models with energy balance models with spatial energy transport could be extended to GHGs with different lives in the atmosphere and different GWPs. Brock et al. (2013) includes the phenomenon of polar amplification, while Brock, Engström, and Xepapadeas(2014a) includes a damage reservoir that is present because of an endogenous ice line as in the papers by North (1975a, 1975b). North's bifurcations are not the same as the bifurcations discussed in Eisenman and Wettlaufer (2009) and Abbot et al. (2011), but are related in mathematical structure. We believe that this kind of framework could shed light on the temporal spatial structure of optimal policy intervention on emissions of GHGs with different lives in the atmosphere and different GWPs. For example, while there are strong arguments for uniform taxes on a unit of emissions independent of location, especially if costless compensatory transfers are available for poorer areas, Pierrehumbert's (2014) argument which we discussed earlier reminds us that short-lived GHGs with relatively large GWPs should be treated differently than long-lived GHGs like CO_2.

Local bifurcations and local tipping points might be sources of extreme local events that are accentuated and magnified by climate change. Leeds et al. (2013) have done work on simulating future climate under changing covariance structures. They discuss forcing by spatial stochastic processes with thicker tails than the spatial white noise processes used by, for example Kim and North (1992) in their spherical energy balance model. Later versions of spherical spatial energy balance models used by Brock et al. (2013) have polar amplification effects. We believe that a useful direction for future research would be to include the bifurcation possibility of Abbot et al. (2011) and the resulting impact on damages across space and to compare the changing spatial covariance structure that results with the findings of Leeds et al. (2013). In the same context designing an optimal "Tech Fix" path to a sustainable low carbon economy is an area of promising future research (David and van Zon, 2014)

8 Appendix

8.1 Appendix I: The Case of Additive Uncertainty

While we believe that the case of multiplicative uncertainty that we treated in the main text does a better job of reflecting the model uncertainty, i.e. the variation in the CCR parameters across respected climate models discussed by Matthews et al. (2009) and Gillett et al. (2013), for completeness we treat an example of additive uncertainty in this Appendix.

Consider the following deterministic robust control problem which is a drastic simplification of the model of Daniel et al. (2014) but with robustness added:

$$\max_E \min_v \left\{ \int_{t=0}^{\infty} e^{-\rho t}(u(yE^\alpha) + (1/2)\theta v^2)dt \right\} \tag{146}$$

subject to

$$\dot{T} = \lambda E + Cv,\ T(0) = T_0 \tag{147}$$

for the dynamics of GMT, denoted here by $T(t)$ for each date t, and,

$$\begin{aligned} &\dot{R} = -E,\ R(0) = R_0 \\ &\dot{S} = -\dot{T} = -(\lambda E + Cv),\ S(0) = T_c - T_0 \\ &S(t) \equiv T_c - T(t) \end{aligned} \tag{148}$$

for the dynamics of fossil fuel usage and the dynamics of the "safety reserve" $S(t) \equiv T(t) - T_c$. Although robustness is typically associated with stochastic models, deterministic models can be derived from stochastic robustness models by scaling the robustness parameter with the standard deviation of stochastic forcing shocks and then taking the standard deviation to zero in a type of "small noise" approximating procedure (Anderson et al., 2014). This type of procedure leads to problem (146)-(148). We say more about this later but treat deterministic problems for now. Note that the requirement that $S(t) \geq 0$ for all dates t translates into equation (149) for the central case, $T_c = 2°\text{C}$. Here $y = y(t)$ is an exogenously given function which is augmented by fossil fuel input $E^\alpha, 0 < \alpha \leq 1$ to give total consumption, $y(t)E(t)^\alpha$ at each date t. The utility function $u(c)$ is assumed to be strictly concave, strictly increasing, twice continuously differentiable and to satisfy the usual Inada conditions, $u'(0) = \infty, u'(\infty) = 0$.

At the risk of repeating, what we have done with (148) and the requirement $S(t) \geq 0$ is this. Rather than attempting to specify a detailed parameterized damage function, we simply let the climate scientists specify T_c and impose the constraint

$$T_c \geq \int_{t=0}^{\infty} (\lambda E + Cv) dt. \tag{149}$$

As Hansen and Sargent (2008) explain in their book, the presence of the minimizing agent is simply a device to help the maximizing agent design a policy that works well for a set of deviations around a baseline and the role of the parameter θ is to index the width of the set of deviations the maximizing agent wishes to robustify against. The equilibrium value of v in the problem (146) is the drift distortion that is most consequential for the robust planner. We refer the reader to Hansen and Sargent (2008) for detailed exposition of robustness in economic modeling.

The idea here is that if $T(t) > T_c$ happens at any date t, then catastrophic climate change may occur and the level of risk is unacceptable at a GMT equal to, T_c, e.g. 2°C as in Held (2013). Hence we impose the constraint $S(t) \geq 0$ to keep $T(t) \leq T_c$ for all dates t. When we want to illustrate with a particular value of T_c, we will use the focal 2°C benchmark for catastrophic climate change because it is the one commonly used in the literature. Of course when we solve the problem above for a candidate equilibrium, we need to check that $S(t) \geq 0$ actually holds for all dates $t \geq 0$ before we can actually proclaim that it *is an* equilibrium. We can use a smaller or larger benchmark and the same analysis used here will apply. Some climate scientists might select a larger (smaller) threshold temperature if they were doing cumulative carbon budgeting. We interpret the literature as saying that many researchers have the same concerns about the levels of deep uncertainty in the layers upon layers of assumptions built into the IAMs that Pindyck (2013a, 2013b) has in economics and researchers like Curry and Webster (2011) have in climate science.

The robust control specification (146) and (147) is an attempt to capture the model uncertainty emphasized by Matthews et al. (2009) and Matthews et al. (2012) in their discussion of the CCR parameter. An alternative specification of (147) is

$$\dot{T} = (\lambda + Cv)E, \; T(0) = T_0, \tag{150}$$

where the term $(\lambda + Cv)$ is a more direct representation of the uncertainty of the CCR parameter discussed in Matthews et al. (2009) and Matthews et al. (2012). We will call (150) multiplicative uncertainty and (147) additive uncertainty. If specification (150) is used, equation (148) and the constraint (149) is replaced by

$$\begin{array}{l} \dot{S} = -(\lambda + Cv)E, \; S(0) = S_0 = T_0 - T_c \\ T_c \geq \int_{t=0}^{\infty} (\lambda + Cv) E dt. \end{array} \tag{151}$$

We solved the robust control problem with specification (151) in the main text. Note that equation (148) basically says that we have two reserves, a fossil fuel reserve and a safety reserve. Since both of these reserve dynamics in equation (148),

$$\dot{R} = -E,\ R(0) = R_0, \dot{S} = -(\lambda E + Cv),\ S(0) = S_0, \tag{152}$$

imply the integral constraints

$$\int_{t=0}^{\infty} E dt \leq R_0, \int_{t=0}^{\infty} (\lambda E + Cv) dt \leq S_0, \tag{153}$$

we could replace the dynamics (152) by the integral constraints (153) and treat our problem as a robust isoperimetric control problem. This approach may be easier in some applications. However, we use the reserve dynamics equations (152) here because for the non-robust case we see that our problem is just a standard exhaustible resource problem but with two reserves rather than just one.

The FONCs for a dynamic Nash equilibrium of problem (146) are given from optimal control theory as

$$\begin{aligned} &H \equiv u(yE^{\alpha}) + (1/2)\theta v^2 + \mu_R(-E) + \mu_S(-\lambda E - Cv) \\ &0 = H_v = \theta v - \mu_S C \\ &0 = H_E = u'^{\alpha})y\alpha E^{\alpha-1} - \mu_R \\ &\dot{\mu}_R = \rho\mu_R - H_R = \rho\mu_R, \\ &\dot{\mu}_S = \rho\mu_S - H_S = \rho\mu_S. \end{aligned} \tag{154}$$

Here is a simple result that follows directly from the FONCs (154).

Result 1: *If $\lambda R_0 > T_c$, then some of R_0 is left in the ground. That is some of the world's fossil fuel reserves become worthless, i.e. the shadow price function $\mu_R(t)$ of fossil fuels must be zero.*

Proof. The proof is by way of contradiction. From (154) $\mu_R(t) = \mu_R(0)e^{\rho t}$. Hence if $\mu_R(0) > 0$, then all of the reserve R_0 is exhausted as $t \to \infty$, hence, if we use specification (149), recalling that we show below that the FONCs for $v(t)$ imply $v(t) \geq 0$ for all t, we have

$$T_c \geq \int_{t=0}^{\infty} (\lambda E + Cv) dt = \lambda R_0 + C \int_{t=0}^{\infty} v dt > \lambda R_0. \tag{155}$$

But (155) is a contradiction to $\lambda R_0 > T_c$. If we use specification (151) and modify the FONCs (154), we have

$$T_c \geq \int_{t=0}^{\infty} (\lambda + Cv) E dt = \lambda R_0 + C \int_{t=0}^{\infty} E v dt > \lambda R_0 \tag{156}$$

which is a contradiction just as above. ■

The requirement that $\lim_{t\to\infty} T(t) \leq T_c$ and the dynamics of global average temperature specified in (147), above as well as the FONCs requiring $v(t) \geq 0$ for all dates t, are what really drive this result. Note that the result is independent of the presence of a minimizing agent, i.e. it holds even if $C = 0$ or $\theta = \infty$. The presence of the minimizing agent just makes the bound on how much of the initial reserve can be used tighter.

Recently some organizations have received a lot of publicity over the idea that known reserves of fossil fuels are already so large that some of the known reserves are at risk of becoming wasted assets (e.g. Carbon Tracker, http://www.carbontracker.org/). Hence, it is useful, as a thought experiment, to analyze the case where some of the known reserves will be left in the ground. Here we focus on the case where Assumption 1 below holds.

Assumption 1: $\lambda R_0 > T_c$, *i.e. Result 1 holds.*

If the utility function $u(c) = \ln(c)$, we may solve the FONCs for a closed form solution. The FONCs for this special case become,

$$\begin{aligned}
&H \equiv \ln(y) + \alpha \ln(E) + (1/2)\theta v^2 + \mu_R(-E) + \mu_S(-\lambda E - Cv)\\
&0 = H_v = \theta v - \mu_S C\\
&0 = H_E = \alpha/E - \mu_R - \lambda\mu_S\\
&\dot{\mu}_R = \rho\mu_R - H_R = \rho\mu_R,\\
&\dot{\mu}_S = \rho\mu_S - H_S = \rho\mu_S.
\end{aligned} \tag{157}$$

We start the procedure of solving equations (157) under Assumption 1 in order to uncover sufficient conditions that need to be imposed to reach a solution, recalling that Assumption 1 implies $\mu_R(0) = 0$,

$$\begin{aligned}
&S_0 = \int_{t=0}^{\infty} (\lambda\alpha/(\mu_R + \lambda\mu_S) + C^2\mu_S/\theta)dt\\
&= \int_{t=0}^{\infty} (\alpha/\mu_S + C^2\mu_S/\theta)dt = \int_{t=0}^{\infty} e^{-\rho t}(\alpha/\mu_S(0)dt + \int_{s=0}^{\infty} (e^{\rho t}C^2\mu_S(0)/\theta)dt\\
&= \alpha/(\rho\mu_S(0)) + \mu_S(0)\int_{s=0}^{\infty} (e^{\rho t}C^2/\theta)dt.
\end{aligned} \tag{158}$$

Hence, in order to obtain a solution, we need to impose conditions that imply,

$$\int_{s=0}^{\infty} (e^{\rho t}C^2/\theta)dt < \infty. \tag{159}$$

One route is to assume θ is constant and to require that the function $C(t)$ satisfy

$$\int_{s=0}^{\infty} e^{\rho t}C^2 dt < \infty. \tag{160}$$

Motivated by (160) we assume the following.

Assumption 2: $C(t) = C_0 e^{-\phi t},\ \rho - 2\phi < 0.$

Under Assumptions 1 and 2 we arrive at the equation

$$S_0 = \alpha/(\rho\mu_S(0)) + C_0^2\mu_S(0)/(\theta(2\phi - \rho)), \tag{161}$$

which can be written in the equivalent form,

$$C_0^2\mu_S(0)^2/(\theta(2\phi - \rho)) - \mu_S(0)S_0 + \alpha/(\rho) = 0 \tag{162}$$

with roots

$$\begin{aligned} &\mu_S(0) = [S_0 \pm D^{1/2}]/[2(C_0^2/\theta(2\phi - \rho))] \\ &D \equiv S_0^2 - 4(\alpha/\rho)(C_0^2/\theta(2\phi - \rho)). \end{aligned} \tag{163}$$

We next make the following assumption.

Assumption 3: *The roots of* (162) *are real. Therefore*

$$D \geq 0, i.e., \; S_0^2 - 4(\alpha/\rho)(C_0^2/(\theta(2\phi - \rho)) \geq 0. \tag{164}$$

Note that given the values of the other parameters, (164) holds if $\theta > \theta_c$ where θ_c solves the equation $D = 0$. In analogy with work on robust control in economics (e.g., Hansen and Sargent, 2008), we call θ_c the breakdown point. We select the negative root since it agrees with the non-robust solution

$$\mu_S(0) = \alpha/(\rho S_0), \tag{165}$$

i.e. the solution when $\theta = \infty$. This can be seen by applying L'Hospital's Rule to the limit by taking $1/\theta \to 0$. The dynamics of the candidate solution for the safety reserve are given by

$$\begin{aligned} &\dot{S}(t) = -\dot{T}(t) = -\alpha/\mu_S(t) - C^2(t)\mu_S(t)/\theta, \\ &= -(\alpha/\mu_S(0))e^{-\rho t} - C^2(0)\mu_S(0)e^{(\rho-2\phi)t}/\theta \\ &S(0) = T_c - T_0 \equiv S_0 \\ &S(t) \equiv T_c - T(t). \end{aligned} \tag{166}$$

We call the solution (166) a candidate solution because one must check that the solution satisfies $S(t) \geq 0$ for all positive dates. It can be shown that for θ large enough, solution (166) satisfies $S(t) \geq 0$ for all positive dates. The proof uses (165) and the negative root goes to zero as $\theta \to \infty$.

The distortion to the dynamics of temperature induced by the robust planner is

$$C^2(0)\mu_S(0)e^{(\rho-2\phi)t}/\theta \tag{167}$$

i.e., the robust planner twists the temperature dynamics towards higher temperatures to induce the economy towards smaller use of fossil fuels compared to the non-robust case $\theta = \infty$. Note that a solution to the simple non-robust case requiring that $S(t) \geq 0$ for all dates t is analytically equivalent to an exhaustible resource problem where the reserve, R_0, is replaced by the adjusted reserve, T_c/λ. We must deal with one more issue before turning to a discussion of implementation and that is time consistency.

8.1.1 Time Consistency Issues of Solutions to Zero Sum Robust Control Games

Let, for any date t, $S(t|S_0)$ denote the solution to the dynamic zero sum game (146) starting with initial condition S_0. If $t_2 > t_1 > 0$ are any two dates, for time consistency we must check the property

$$S(t_2|S(t_1|S_0)) = S(t_2|S_0). \tag{168}$$

That is to say that the players will choose to play the same equilibrium value at date t_2 starting from initial condition $S(t_1|S_0)$ at date t_1 with $t_2 - t_1$ periods "to go" as the players will choose to play with the full t_2 periods to go starting from date zero. We compute (168) to check if and when it holds.

$$\begin{aligned} &S(t_2|S_0) = S_0 - (\alpha/\mu_S(0)) \int_{r=0}^{t_2} dte^{-\rho r} - C^2(0)\mu_S(0) \int_{r=0}^{t_2} dte^{(\rho-2\phi)r}/\theta \\ &= S_0 - (\alpha/(\rho\mu_S(0))[1 - e^{-\rho t_2}] - (C^2(0)\mu_S(0)/(\theta(2\phi - \rho))[1 - e^{(\rho-2\phi)t_2}] \end{aligned} \tag{169}$$

$$\begin{aligned} &S(t_2|S(t_1|S_0)) = S(t_1|S_0) - (\alpha/\mu_S(t_1)) \int_{r=t_1}^{t_2} dte^{-\rho r} - C^2(0)\mu_S(t_1) \int_{r=t_1}^{t_2} dte^{(\rho-2\phi)r}/\theta \\ &= S(t_1|S_0) - (\alpha/(\rho\mu_S(t_1))[1 - e^{-\rho(t_2-t_1)}] - C^2(0)\mu_S(t_1)/(\theta(2\phi - \rho))[1 - e^{(\rho-2\phi)(t_2-t_1)}]. \end{aligned} \tag{170}$$

If (169) and (170) behaved like solutions to an ODE (168) would hold because it is a basic property of solutions of ODEs. However, here, the shadow price

$$\begin{aligned} &\mu_S(0) \equiv f(S_0) = [S_0 - D^{1/2}]/[2(C_0^2/(\theta(2\phi - \rho))] \equiv [S_0 - D^{1/2}]/(2a) \\ &a \equiv C_0^2/(\theta(2\phi - \rho)) \\ &D \equiv S_0^2 - 4(\alpha/\rho)(C_0^2/\theta(2\phi - \rho)) = S_0^2 - 4ac \\ &c \equiv \alpha/\rho \end{aligned} \tag{171}$$

$$\begin{aligned} &\mu_S(t_1) \equiv f(S(t_1|S_0)) = [S(t_1|S_0) - D^{1/2}]/[2(C_0^2/\theta(2\phi - \rho))] \\ &D \equiv S(t_1|S_0)^2 - 4(\alpha/\rho)(C_0^2/\theta(2\phi - \rho)). \end{aligned} \tag{172}$$

It seems pretty clear that unless $C_0 = 0$ or $1/\theta = 0$, i.e. we are back in the pure non-robust, non-distorted dynamics case, the time consistency condition $S(t_2|S(t_1|S_0)) = S(t_2|S_0)$ will not hold.

This problem is a common difficulty with the open loop concept of Nash equilibrium used here. Unless one is willing to take the view that this kind of equilibrium can be used as a rolling plan where the planner re-solves the system at each date t and does not worry about whether what it had planned to do at a later time based upon a plan made at an earlier time is actually desirable when a new plan is drawn up at an intermediate time, then the open loop concept of equilibrium here is not satisfactory.

As shown in Kossioris et al. (2008), re-optimization might also be necessary in order to reach the best steady state, even if we consider non linear feedback Nash equilibrium strategies. Re-optimization takes place in the following sense. Given an initial state, the feedback Nash equilibrium strategy is calculated. This strategy will lead to a steady state which is not the best, after some time has elapsed the state of the system is estimated and the feedback strategy is recalculated using this state as an initial state. The process is continued until the calculated feedback strategies lead to the best steady state. We view our results here as a very rough preliminary insight into what conclusions from robust planning under carbon budgeting make look like. Future research should attempt to develop time consistent concepts of dynamic Nash equilibrium for use in robust planning.

It is also useful to investigate the equilibrium value of the planner's objective for the log utility example worked out above. We have:

$$\begin{aligned} &V(S_0) \equiv \int_{t=0}^{\infty} e^{-\rho t}\alpha \ln(e^{-\rho t}\alpha/(\lambda\mu_S(0))dt = (\alpha/\rho)\{\ln(\alpha/\lambda) - \ln(\mu_S(0)\} \\ &- \int_{t=0}^{\infty} e^{-\rho t}\alpha\rho t e^{-\rho t}dt. \end{aligned} \tag{173}$$

Recall from (173) that $\mu_S(0) = f(S_0)$. We explore the shape of the equilibrium value function by computing, $V'(S_0), V''(S_0)$,

$$\begin{aligned} V'(S_0) &= -(\alpha/\rho)f'(S_0)/f(S_0) \\ V''(S_0) &= -(\alpha/\rho)\{f''(S_0)/f(S_0) - f'(S_0)^2/f(S_0)^2\}. \end{aligned} \tag{174}$$

It is easy to check that $f'(S_0) < 0$. Hence, $V'(S_0) > 0$, which is what would be expected from the economics. It appears that $V''(S_0)$ might have either sign. For the case $\theta = \infty$ it is easy to check that

$$\begin{aligned} &\mu_S(0) = \alpha/(\rho S_0) \\ &V(S_0) \equiv \int_{t=0}^{\infty} e^{-\rho t}\alpha \ln(e^{-\rho t}\alpha/(\lambda\mu_S(0))dt = (\alpha/\rho)\{\ln(\alpha/\lambda) - \ln(\mu_S(0)\} \\ &- \int_{t=0}^{\infty} e^{-\rho t}\alpha\rho t e^{-\rho t}dt = k_0 + (\alpha/\rho)\ln(S_0), \end{aligned} \tag{175}$$

where k_0 is a constant. Hence $V(S_0)$ is concave increasing which is what would be expected in this non-robust case, because it is essentially the same as a standard exhaustible resource problem.

8.2 Appendix II: A heuristic derivation of the HJBI equation for the spatial robust control problem

The problem is to determine admissible, that is, piecewise continuous controls, $c(\cdot,\cdot)$, $h(\cdot,\cdot)$ on $[0,\infty) \times [0, L]$ which extremize

$$\int_0^{\infty}\int_Z \mathrm{e}^{-\rho t} g(x, c, h)\, dz dt \tag{176}$$

where x is governed by the stochastic PDE

$$\frac{\partial x}{\partial t} = f(x, c, h, x_{zz})\, dt + \varepsilon^{1/2}\sigma h dW, \tag{177}$$

with initial conditions $x(0, z) = x(z)$ and appropriate boundary conditions. Following Fond (1979) in the development of a dynamic programming approach to optimization in the context of distributed-parameter problems, let the value function be

$$V(x(z)) = \max_{c(t,z)} \min_{h(t,z)} \mathcal{E}_x \left\{ \int_0^{\infty} \int_Z \mathrm{e}^{-\rho t} g(x, c, h)\, dz dt \right\}. \tag{178}$$

The principle of optimality states that

$$\begin{aligned} V(x(z)) \quad = \quad & \max_{\substack{c(t,z) \\ 0\leq t\leq \Delta t}} \min_{h(t,z)} \mathcal{E}_x \left\{ \int_0^{\Delta t} \int_Z \mathrm{e}^{-\rho t} g(x, c, h)\, dz dt \right\} + \qquad (179) \\ & \max_{\substack{c(t,z) \\ \Delta t\leq t\leq \infty}} \min_{h(t,z)} \mathcal{E}_x \left\{ \int_{\Delta t}^{\infty} \int_Z \mathrm{e}^{-\rho t} g(x, c, h)\, dz dt \right\}. \end{aligned}$$

Following Chang (2004) equation (179) can be written as:

$$0 = \max_{\substack{c(t,z) \\ 0\leq t\leq \Delta t}} \min_{h(t,z)} \mathcal{E}_x \left\{ \mathrm{e}^{-\rho\omega\Delta t} g(x_{\omega\Delta t}, c_{\omega\Delta t}, h_{\omega\Delta t})\, \Delta t + \mathrm{e}^{-\rho t} V(x + \Delta x) - V(x) \right\} \tag{180}$$

where $0 \leq \omega \leq 1$ such that $\omega\Delta t \in [0, \Delta t]$ and $v_{\omega\Delta t} \to v$ as $\Delta t \to 0$, $v = x, c, h$. For sufficiently small Δt we have $\mathrm{e}^{-\rho\Delta t} = 1 - \rho\Delta t + o(\Delta t)$ and

$$\mathrm{e}^{-\rho t} V(x + \Delta x) - V(x) = [V(x + \Delta x) - V(x)] - \rho\Delta t V(x + \Delta x) + o(\Delta t). \tag{181}$$

Assuming that the value function satisfies the requirements for the application of Ito's lemma in infinite dimensions (see Curtain and Falb, 1970), then

$$V(x + \Delta x) - V(x) = V'(x)\, \Delta x + \frac{1}{2} V''(x)\, (\Delta x)^2 + o(\Delta t). \tag{182}$$

Taking the conditional expectation we obtain

$$\mathcal{E}_x [V(x + \Delta x) - V(x)] = \tag{183}$$

$$\left\{ V'(x)\, f(x, c, h, x_{zz}) + \frac{1}{2} V''(x)\, \varepsilon (\sigma x)^2 \right\} \Delta t + o(\Delta t). \tag{184}$$

Dividing by Δt and letting $\Delta t \to 0$, we obtain the HJBI equation (132).

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