Endogenous Growth and the Paradox of Thrift: A Modern Recasting of Say's Law

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Abstract

This paper explores a structural vulnerability in endogenous growth theory: the assumption that savings inherently generate productivity-enhancing investment. Drawing a conceptual analogy with Say's Law, the paper argues that endogenous growth models often rely on a mechanism whereby thrift leads to future demand via innovation. However, this mechanism is institutionally contingent. In the absence of sufficient innovation capacity—shaped by institutions, regulations, and even cultural norms—high savings may fail to translate into growth, echoing the Keynesian paradox of thrift. The paper develops a multi-equilibrium framework to show how liquidity demand and institutional fragility can jointly undermine the marginal product of capital, leading to stagnation even when markets clear. Case studies on Japan and the UK illustrate how economies can fall into low-growth traps despite financial depth. The analysis suggests that endogenous growth should not be taken for granted: it requires deliberate institutional design to ensure that savings are productively deployed.

1 Introduction

Endogenous growth theory has reshaped our understanding of long-run economic growth by emphasising the role of internal mechanisms, such as human capital, innovation, and institutions, in sustaining productivity. This paper proposes a novel analogy: endogenous growth theory functions like a modern version of Say's Law (1821), assuming that savings inherently generate productive investment and future demand. When this mechanism fails, the economy risks falling into a Keynesian paradox of thrift, where high savings lead to stagnation rather than growth.

The paper explores the role of savings, productive investment and precautionary liquidity in the growth dynamics of economies with differing institutional frameworks. A key feature of the main model in the paper is the potential for coordination failure and multiple equilibria, In particular the economy is shown under some conditions to achieve balanced and sustained growth and in others to fall into stagnation. Shocks and major policy shifts are capable of generating transitions from high states to low states and vice versa.

This paper shares conceptual ground with Guerrieri and Lorenzoni (2017), who demonstrate how a credit crunch can trigger precautionary savings that depress aggregate demand and trap the economy in a liquidity-driven recession. Their model shows that rational agents, responding to financial constraints, can collectively generate a downturn even in the absence of nominal rigidities. The present paper generalizes this insight by embedding precautionary savings behavior within an endogenous growth framework, where institutional quality determines whether savings are transformed into productivity-enhancing investment. While Guerrieri and Lorenzoni focus on household deleveraging and credit market frictions, this paper emphasizes the macroeconomic coordination failure that arises when high savings coexist with weak innovation mechanisms. Both papers converge on the idea that individually rational behavior can produce collectively suboptimal outcomes, but this paper extends the analysis to long-run growth dynamics and the structural role of institutions in sustaining the marginal product of capital.

This paper offers a novel reinterpretation of endogenous growth theory by embedding it within a framework that explicitly models institutional quality, liquidity demand, and coordination failures. Building on canonical models of Romer (1990) and Aghion & Howitt (1992), it introduces a three-level taxonomy of Say's Law to clarify the conditions under which savings translate into productivity-enhancing investment. The paper departs from representative agent modeling by demonstrating how individually rational savings behavior can lead to collectively suboptimal outcomes, particularly in high-saving, low-innovation economies. Through a dynamic multi-equilibrium model, it shows how institutional fragility and precautionary liquidity demand can jointly undermine the marginal product of capital, leading to stagnation even when markets clear. This framework generalizes insights from Aiyagari (1994), Cooper & John (1988), and Guerrieri & Lorenzoni (2017), offering a structural reinterpretation of the paradox of thrift with direct relevance to contemporary macroeconomic policy in advanced economies such as Japan and the UK.

2 Literature

The relationship between savings, investment, and growth has long been central to macroeconomic theory. Classical models, following Say's Law, posit that supply creates its own demand, implying that savings are automatically transformed into investment. Keynes (1936) famously challenged this view, introducing the paradox of thrift and emphasizing the possibility of demand shortfalls due to hoarding or unproductive capital allocation. The assumption that investment inherently maintains or improves productivity, represents what we identify as Level 3 of Say's Law. Our model formalizes this insight by introducing parameter p_1, which captures precisely this institutional capacity to convert investment into productivity growth.

Endogenous growth theory, particularly the models of Romer (1990) and Aghion & Howitt (1992), reintroduced the importance of internal mechanisms, such as innovation, human capital, and knowledge spillovers, in sustaining long-run growth. These models implicitly assume that savings are channeled into productivity-enhancing investment, effectively reviving a dynamic version of Say's Law. However, this assumption is conditional on institutional quality, a point emphasized by Acemoglu, Johnson, and Robinson (2005), who argue that inclusive institutions are fundamental to long-run growth.

Cooper and John (1988) show taht strategic complementarities can lead to multiple equilibria in macroeconomic outcomes, when agents expectations are misaligned. This insight is particularly relevant for understanding how high savings can coexist with low investment and stagnation, as in Japan's "lost decades" (Krugman, 1998). The role of liquidity preference and precautionary savings has also been explored in models of secular stagnation (Eggertsson & Mehrotra, 2014) and incomplete markets (Aiyagari, 1994), where individual insurance motives can generate macroeconomic inefficiencies. These models provide microfoundations for the kind of liquidity trap dynamics developed in this paper.

Moreover, empirical studies such as Attanasio, Picci, and Scorcu (2000) and Hsieh & Klenow (2009) have shown that the link between savings and productivity-enhancing investment is far from automatic, especially in economies with weak institutions or misallocation of capital.[1] Recent work on secular stagnation has emphasized the persistent failure of advanced economies to generate sufficient demand and investment, even in the presence of accommodative monetary policy. Summers (2014) argues that structural factors, such as demographic shifts, declining investment demand, and falling equilibrium interest rates, may trap economies in low-growth equilibria.

This paper builds on these strands by proposing a three-level reinterpretation of Say's Law, embedding it within an endogenous growth framework that explicitly models institutional quality, liquidity demand, and the allocation of savings across productive and unproductive uses. Unlike Summers' primarily demand-side framing, our model highlights the supply-side institutional constraints that prevent savings from activating growth.

3 A Modified Growth Model with Institutional Contingency

3.1 From Classical AK to Institutional Dependence

Endogenous growth models (e.g., Romer (1990) and Lucas (1988)) posit that growth arises from internal factors like R and D, education, and knowledge spillovers.[2] Savings drive

[1] Hsieh and Klenow (2009) show how misallocation can significantly reduce total factor productivity in developing economies.

[2] See Romer (1990) and Aghion & Howitt (1992) for foundational models of endogenous growth driven by innovation and knowledge spillovers.

investment, which enhances productivity and sustains growth. However, this mechanism is conditional on the presence of institutions that support innovation.

The canonical endogenous growth models (Romer 1990, AK models) demonstrate how economies can sustain growth without exogenous technological progress. These models implicitly assume that savings automatically translate into productivity-enhancing investment - essentially a dynamic version of Say's Law where supply creates future demand through innovation.

Consider an economy with a production function with output Y, capital K, Labour L and kwowledge A:

$$Y = K^{\alpha}(AL)^{1-\alpha} \tag{1}$$

with diminishing returns to capital accumulation $0 < \alpha < 1$. With zero population growth, the growth identity gives:

$$g_Y = \alpha g_K + (1-\alpha) g_A \tag{2}$$

In standard endogenous growth models, productivity evolves according to the investment rate:

$$g_A = \phi(I/Y) - \xi$$

where, I is investmen, ϕ represents the efficiency of converting investment into innovation and ξ captures knowledge depreciation. This specification assumes all investment enhances productivity.

Not all investment is productive. Let $0 \leq p_1 < 1$ represent the fraction of investment that enhances productivity, a measure of "institutional quality". While more complex specifications might make g_A depend on K/Y, our formulation captures the key insight that institutional quality determines whether savings generate productivity growth, regardless of the capital intensity of the economy.[3] Then:

$$g_A = \frac{\dot{A}}{A} = \phi(\frac{p_1 I}{Y}) - \xi = p_1 \phi^0 s - \xi \tag{3}$$

where, savings: $S = sY$, with $0 < s <$; and $I = S$ (markets clear).

A balanced growth path requires $g_K = g_A$. With capital dynamics $g_K = sY/K - \delta$, this condition becomes:

$$sY/K - \delta = p_1 \phi^0 s - \xi \tag{4}$$

This can only be satisfied if the capital-output ratio adjusts appropriately. However, when institutional quality is poor (low p_1), no positive K/Y ratio may satisfy this condition. There is a critical threshold for positive balanced growth to be possible: $p_1 > p_1^* = \xi/\phi^0 s$. When $p_1 < p_1^*$, the economy faces inevitable stagnation: Starting from any K_0, if $g_A = p_1\phi^0 s - \xi < 0$, productivity declines. Capital may initially accumulate ($g_K > 0$) while productivity falls

[3] Acemoglu, Johnson, and Robinson (2005) argue that inclusive institutions are essential for channeling investment into productive uses.

($g_A < 0$), leading to rising capital intensity, $K/(AL)$ rises continuously. However this leads to declining returns, $MPK = \alpha(\frac{AL}{K})^{1-\alpha}$ falls progressively, leading to stagnation, in which investment ceases when MPK falls below minimum required returns.

This is not a market failure in the traditional sense, markets clear and $S = I$ holds. Rather, it's a failure of the endogenous growth mechanism itself: savings don't generate sufficient innovation to sustain returns on capital.

3.2 Three Levels of Say's Law

Say's Law posits that supply creates its own demand. In classical models, all income is either consumed or invested. Keynes challenged this view, arguing that savings may be hoarded rather than invested, reducing aggregate demand and given other conditions, leading to underemployment. The paradox of thrift arises when increased saving depresses consumption and investment, causing economic stagnation.

The framework above reveals three distinct levels at which Say's Law might operate or fail:

Level 1 (Classical): Supply creates its own demand within periods, markets clear through price adjustment.

Level 2 (Dynamic): Planned savings equals planned investment intertemporally, capital markets equilibrate.

Level 3 (Endogenous Growth): Savings generate productivity-enhancing investment that sustains future growth.[4]

3.3 Say's Law in this Model

The three Levels of Say's Law are clearly delineated. Level 1 (Classical): $S = I$ within period; Level 2 (Dynamic): $S = I$ intertemporally through capital markets; Level 3 (Endogenous growth): $I \rightarrow$ productivity growth only if $p_1 > 0$. When $p_1 = 0$: Savings still equals investment (Level 2 holds), but investment doesn't enhance productivity (Level 3 fails). Thus capital accumulation without productivity growth $\rightarrow$declining MPK $\rightarrow$ stagnation. This formulation makes it clear that the problem isn't a savings-investment imbalance (the Keynesian problem) but an investment-productivity disconnect.

Using the production function $Y = K^\alpha(AL)^{1-\alpha}$ with $MPK = \alpha(\frac{AL}{K})^{1-\alpha}$ and the innovation equation $g_A = \frac{\dot{A}}{A} = p_1\phi^0 s - \xi$, we can write the growth rat of MPK as $d(MPK)/dt = MPK(1-\alpha)(g_A - g_K) = MPK(1-\alpha)[(p_1\phi^0 s - \xi) - (sY/K - \delta)]$. In the steady state, $g_A = g_K$, so MPK is constant. But during transition: When p_1 is high: $g_A = p_1\phi^0 s - \xi$ is substantial, capital and productivity grow together and MPK declines slowly or stabilises. When p_1 is low: $g_A = -\xi$ (barely positive or negative), but $g_K = sY/K$ is still positive if

[4] It is worth noting that Keynes primarily attacked Level 1 (and partially Level 2 with liquidity preference). This papers contribution extends this critique to Level 3. As we will argue later during the lost decades, Japan, and the UK today are essentially Level 3 failures.

s is high, so $g_K > g_A$, meaning K grows faster than AL and $MPK = \alpha(\frac{AL}{K})^{1-\alpha}$ declines rapidly. The key Insight is that the declining MPK property comes from the production function (diminishing returns to K), not from the innovation function. The linear innovation function determines whether productivity growth can offset the natural tendency for MPK to decline.

In other words, high savings always increases K. When p_1 is low: investment doesn't generate innovation: $\dot{A}/A = -\xi$ regardless of investment levels so MPK declines rapidly as K/AL rises. The paradox of thrift emerges: high savings →low returns → stagnation. This is the "premature capital exhaustion." Sustained growth requires: $p_1 > \xi/\phi^0 s$, below this threshold, the economy stagnates. This shows how institutional quality and savings interact.

The parameter p_1 now has a clear structural interpretation - it determines whether capital deepening triggers innovation. This model captures the cases of The Japanese lost decades and the UK today. Japan: High s, moderate p_1 → the economy hits diminishing returns faster than innovation could compensate. In the case of the UK: Moderate to low s, low p_1 → never achieved the innovation trigger point. Both cases illustrate that growth requires endogenous productivity to absorb savings.

4 Modelling the Main Problem

4.1 A Basic Model

This paper demonstrates why the savings-investment-growth nexus cannot be analyzed through representative agent models. The core insight is that the decision to save, in aggregate, changes the economy's capacity to generate returns on savings. This coordination failure cannot be captured by models where a single agent or planner optimizes, as it emerges precisely from the disconnect between individual rationality and collective outcomes. We therefore require a framework that explicitly models the failure of prices to coordinate intertemporal decisions when savings affect not just capital accumulation but the economy's innovative capacity.

The first question that arises from Keynes is the nature of intertemporal information transmission in the economy. Does saving successfully signal demand for future consumption? In what we will call a Smith (1776)-Hayek (1946) classical model, the rate of interest is set to support optimal intertemporal decisions by equilibrium in the loanable funds market. In Keynes (1936) savings may go into idle balances or unproductive capital, and in the absence of coordination not signal a demand for future consumption and hence productive investment. In this paper we consider a financing channel, which may mean that savings do not reach productive entrepreneurs. Whether they do is contingent on the need to avoid both hoarding and unproductive investment. We emphasise an innovation channel, which is necessary to ensure that investments enhance future productivity. These channels are not considered in classical models and are assumed away in the Solow Model with exogenous technical change.

These channels are central in endogenous growth (but institutionally contingent).

To illustrate these points, consider a simple model, we envisage a sequential game:

At Stage 1: Households save a fraction s of income.

At Stage 2: Savings are allocated across: Productive investment (probability p_1, depends on institutions); Unproductive investment (probability p_2); and Idle balances (probability $1 - p_1 - p_2$). The driving force of growth will be $p_1 = h(q)$, where q captures the structure of the institutional environment.

At Stage 3: Only productive investment generates endogenous growth $g = f(s, p_1)$.

The Smith-Hayek world assumes $p_1 = 1$. Keynes showed $p_1 + p_2 < 1$, there will be a positive probability of idle balances. Here, we argue that even when $p_1 + p_2 = 1$, we need $p_1 > 0$ for growth, and this depends on institutions.

Consider a simple corn economy in which corn can be eaten or stored or planted for more corn tomorrow. In the Smith-Hayek model, not eating corn today equals planting it for more tomorrow. In Keynes, not eating corn today equals possibly storing it in a warehouse where it doesn't grow. Here I consider that not eating corn today equals possibly planting it on infertile ground (unproductive investment) where it yields no harvest. The modern economy needs not just planting (investment) but the right soil (institutions) and seeds (innovation capacity) for the corn to grow. Thus we are identifying a third way the savings-investment-growth nexus can break down, one that's particularly relevant for modern developed economies with sophisticated financial systems but weak innovation mechanisms. Hence, even when financial markets ensure that planned savings equals planned investment (satisfying the Dynamic Say's Law), there is no guarantee that this investment enhances productivity. The probability p_1 in our framework captures precisely this gap: the fraction of equilibrium investment that actually generates endogenous growth.

4.2 A Critique of Representative Agent Macro-Modelling

The representative agent framework assumes no coordination issues (one agent can't fail to coordinate with themselves); there is no fallacy of composition, what is true for one is true for all by construction with no distinction between individual and aggregate outcomes. For example in Japan during the lost decades: millions of households and firms were rationally saving for an uncertain future, but their collective saving ensured that future was worse than it would have been with less saving. A representative agent model cannot represent this paradox.

Here, we are argue that in the realm of savings-investment-growth, we need to abandon the idea that micro-optimization leads to macro-optimality. This is central to Keynes's paradox of thrift (individual virtue becomes collective vice). Building on Cooper and John (1988), we recognize that coordination failures in macroeconomics can arise not from irrational behavior but from strategic complementaries among agents. In their framework, individual decisions are interdependent: households save expecting future returns from innovation, while firms innovate only if they anticipate sufficient demand.[5] This mutual dependence can produce multiple equilibria, some of which are Pareto-dominated. In the context of the savings–investment–growth nexus, this means that even rational agents may collectively settle into a low-growth equilibrium if expectations are misaligned or institutions fail to support productive investment. Representative agent models, by construction, cannot capture such failures: they assume away heterogeneity, strategic interaction, and the fallacy of composition. Cooper and John's insights justify the need for a richer modeling approach. A multiple equilibria framework, allows a range of possibilities: High savings, low innovation, low growth equilibrium (Japan/UK trap); Moderate savings, high innovation, high growth equilibrium (US in good times); the economy can get stuck in the bad equilibrium even with rational agents.

In a formal game-theoretic structure: Households save expecting returns from innovation. Firms don't innovate seeing weak demand from high savings. Both are rational given their beliefs, but collectively produce stagnation. Aggregation, with representative agents will miss such coordination failures.

This transitional dynamics, where $g_K \neq g_A$ and the economy moves toward stagnation, cannot be captured in models that assume balanced growth paths or steady-state analysis. The coordination failure emerges precisely because individual savings decisions collectively drive the economy away from any sustainable growth equilibrium. When millions of agents save expecting future returns, their collective behavior increases K faster than institutional constraints allow A to grow. The resulting decline in MPK validates pessimistic expectations, creating a self-fulfilling prophecy of stagnation. Representative agent models, by construction, cannot represent this divergence between individual rationality (saving is prudent) and collective outcomes (savings destroy their own returns). The fallacy of composition is not merely a friction to be added to an optimization problem - it is the core mechanism through which high-saving, low-innovation economies exhaust their growth potential

5 A More General Model

In equilibrium neo-classical representative agent models, markets should efficiently allocate capital between liquid and productive assets. However, here we demonstrate that when individuals face idiosyncratic risks and use liquidity as self-insurance, rational individual behavior can lead to collectively suboptimal outcomes. We develop an intertemporal model where agents accumulate liquid savings to hedge against shocks to their productive assets, but fail to internalize that their individual liquidity choices reduce aggregate returns on productive investment.[6]

Acemoglu et al. (2018) develop a firm-level endogenous growth model in which misalloca-

[5] Cooper and John (1988) formalize how strategic complementarities can lead to multiple equilibria in macroeconomic models.

[6] Aiyagari (1994) models how idiosyncratic risk and incomplete markets can lead to excess aggregate savings.

tion of skilled labor and capital toward low-productivity firms suppresses aggregate growth. Their framework emphasises the role of institutional frictions in determining the efficiency of resource allocation. Our model generalizes this insight by introducing a macro-level parameter p_1, which captures the fraction of investment that enhances productivity. Like Acemoglu et al., we show that institutional quality is central to sustaining growth, but we extend the analysis by modeling how liquidity demand and coordination failures interact with institutional constraints to produce multiple equilibria. This allows us to explain stagnation traps in high-saving economies even when financial markets clear.

The key insight is that individual insurance demand creates a negative externality: each agent's liquidity holding reduces the capital available for productive investment, lowering returns for all agents and paradoxically increasing their demand for liquidity insurance. This creates a feedback loop that can trap the economy in a low-return, high-liquidity equilibrium that resembles secular stagnation.

We now move to a discrete-time framework to analyze individual optimization decisions.

5.1 Model Setup

5.1.1 Determination of Productive Investment

In this model, aggregate savings is split between investmen in physical capital and changes in holdings of liquid assets, which we interpret as real resources withdrawn from productive use, for example, cash hoarding, storage of goods, or speculative real estate purchases that do not expand the productive frontier. These uses absorb resources without generating future output, similar to Keynes's "idle balances." This interpretation differs from a pure financial view where bank deposits are matched by loans. In that case, deposits would not reduce aggregate investment unless banks themselves allocate funds to low-productivity or speculative assets. Our real-resource interpretation captures the idea that high liquidity preference can crowd out productive investment even when financial markets clear.[7]

The model features a three-way allocation of resources: Liquidity holdings (m_{it}) for insurance determined in the optimisation problem below, productive innovation-enhancing investment (fraction p_{1t} of non-liquid investment), and unproductive investment (fraction $1 - p_{1t}$ of non-liquid investment). The productive investment share p_{1t} emerges endogenously from the interaction of institutional quality and aggregate portfolio choices:

$$p_1 = h(.) = q^{\kappa} \cdot \left(\frac{K}{Y}\right)^{-\mu} \cdot (1 - M_t/Y_t)^{\theta} \tag{5}$$

Where: q = institutional quality index; K_t/Y_t = capital intensity (captures diminishing op-

[7] The additions to what we call liquid assets include: Idle balances, households hold currency or deposits that are not intermediated into productive loans. Unproductive storage, firms accumulate inventories or commodities as a hedge. Speculative real estate, capital flows into existing property markets, raising prices but not productive capacity.

portunities), $(1 - M_t/Y_t)$ = fraction of output available for investment; θ is the crowding-out elasticity.[8] Note that p_1 depends on the aggregate ratio M/Y, not on M directly.[9] This means individual liquidity decisions affect productive investment only through their contribution to the aggregate ratio. The function $q^{\kappa} \cdot (K_t/Y_t)^{-\mu}$ represents the economy's maximum capacity to generate productive investment opportunities where κ is the institutional elasticity (how q affects productive capacity) and μ is the diminishing opportunities parameter (how K/Y affects p_{1t}). Moreover, $\partial h/\partial q > 0$ (better institutions increase productive capacity); $\partial h/\partial(K_t/Y_t) < 0$ (capital intensity reduces opportunities). This specification captures a critical coordination failure: individual decisions to hold liquidity reduce the aggregate economy's ability to generate productive investment opportunities. While each agent takes p_{1t} as given when choosing m_{it}, their collective choices determine M_t/Y_t, which in turn determines p_{1t}. This creates strategic complementarity, where high aggregate liquidity becomes self-reinforcing through reduced productive opportunities.

5.1.2 Individual Problem

Consider a continuum of agents indexed by $i \in [0, 1]$. Each agent has an infinite horizon and maximises expected lifetime utility:

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_{it}) \right] \tag{6}$$

where $\beta \in (0, 1)$ is the discount factor and $U(\cdot)$ is a strictly concave utility function.

Agent i's budget constraint in period t is:

$$a_{it+1} = (1 + r_t^{Portfolio})(a_{it} - c_{it} - m_{it}) + y_{it+1} \tag{7}$$

where:

- a_{it} is total wealth at the beginning of period t
- c_{it} is consumption
- m_{it} is liquid savings (insurance holdings)
- M_t is aggregate liquidity
- $r_t^P = A_t^{\gamma} \cdot \overline{r_t} (1 - M_t/Y_t)^{\omega}$ is the return on productive investment: where A_t^{γ} is productivity at time t, raised to power γ; $\overline{r}$ is the base return rate; and $(1 - M_t/Y_t)^{\omega}$ is the

[8] We take essentially a reduced form approach to the p_1 function. Acemoglu et. al. (2005) explicitly argue that attempting to microfound every aspect of institutions is neither necessary nor desirable. They advocate for reduced-form relationships between measured institutional quality and economic outcomes.

[9] The specification with M/Y implies feedback from liquidity to technology scaled by $1/Y^2$. An alternative specification using M/K would strengthen this feedback by a factor of $K/Y \approx 3$, but does not qualitatively change the results. The relatively weak feedback in either specification reflects the empirical reality that liquidity accumulation acts as a gradual drag on innovation rather than a sharp constraint.

fraction of output not held as liquidity, raised to the sensitivity power ω. Here A_t evolves according to: $[(A_{t+1} - A_t)/A_t] = p_{1t}\phi^0 s_t - \xi$

- $r_t^U = \overline{r_t}(1 - M_t/Y_t)^\nu$
- $r_t^{Portfolio} = p_{1t}r_t^P + (1 - p_{1t})r_t^U$
- y_{it} is labor income, potentially stochastic.

Note that each agent takes p_1 as given when choosing m_{1t}, since individual choices have negligible effect on aggregate M/Y.

For tractability, we treat the savings rate s as independent of the portfolio return $r_t^{Portfolio}$. This simplification is consistent with logarithmic utility, where income and substitution effects exactly offset, leaving the savings rate invariant to interest rate changes.[10]

5.1.3 Aggregate Savings and Investment

At the aggregate level, the individual budget constraints must be consistent with the economy's resource constraint. Aggregating across all agents and imposing market clearing:

$$S_t = Y_t - C_t = I_t + (M_t - M_{t-1})$$

where: S_t = aggregate savings; $I_t = K_{t+1} - (1 - \delta)K_t$ = net capital formation; $M_t - M_{t-1}$ = change in aggregate liquidity holdings. This implies the aggregate savings rate is:

$$s_t \equiv S_t/Y_t = [K_{t+1} - (1 - \delta)K_t + (M_t - M_{t-1})]/Y_t \tag{8}$$

This decomposition shows that aggregate savings is endogenously split between productive capital accumulation and liquidity accumulation, where the allocation depends on the equilibrium liquidity demand function $\Psi(\cdot)$. This decomposition is central to our mechanism: when liquidity demand M_t rises, it directly crowds out productive investment even when total savings S_t remains high.

The returns r_t^P and r_t^U derive from the underlying production technology. With $Y_t = K_t^\alpha(A_tL_t)^{1-\alpha}$, the marginal product of capital is $MPK = \alpha(\frac{A_tL_t}{K_t})^{1-\alpha}$. The portfolio return $r_t^{Portfolio}$ reflects the weighted average of productive and unproductive investments' contributions to MPK: Productive investment (fraction p_{1t}) contributes to both K_t and A_t growth. Unproductive investment (fraction $1-p_{1t}$) only contributes to K_t accumulation. Thus, when liquidity holdings are high (reducing resources for investment), and when p_{1t} is low (reducing productivity growth), them MPK declines more rapidly as K_t/A_tL_t rises. Here $\omega < v$, which implies that the unproductive investment return function is more sensitive to liquidity

[10]While more general utility functions would yield $s = s(r_t^{Portfolio}, \phi^2)$, this would complicate the analysis without altering our main results about multiple equilibria and institutional quality. The key mechanism in our model operates through the allocation of savings between liquid and productive assets, rather than through the level of savings itself.

crowding-out than productive investment. Unproductive investments (real estate, existing assets) are typically more sensitive to liquidity conditions, they're often the first refuge for excess savings when productive opportunities are scarce. Productive investments in innovation/R&D are less directly affected by aggregate liquidity levels - they depend more on technological opportunities and institutional quality.

The productive investment $(a_{it} - c_{it} - m_{it})$ earns return $r_t^{Portfolio}$, while liquid holdings m_{it} provide insurance but earn zero return. At the aggregate level, total savings S_t equals the sum of investment and liquidity holdings: $S_t = I_t + M_t$. Where investment $I_t = K_{t+1} - K_t$ $-(1-\delta)K_t$. Crucially, while Level 2 Say's Law $(S_t = I_t)$ holds in the traditional sense when we include liquidity as a form of 'investment,' the productive investment that drives growth is only: $I_t^{\Pr oductive} = p_{1t}(S_t - (M_t - M_{t-1}))$. This shows how high-liquidity demand M_t and low institutional quality (low p_{1t}) jointly undermine the growth mechanism, even when markets clear.

5.1.4 Idiosyncratic Risk and the Insurance Motive

Agents face idiosyncratic shocks to their effective wealth. Specifically, in period $t+1$, agent i receives effective wealth:

$$\tilde{a}_{it+1} = a_{it+1} + \epsilon_{it+1} \cdot \min(m_{it}, m) \tag{9}$$

where ϵ_{it+1} is an idiosyncratic shock with $E[\epsilon_{it+1}] = 0$ and $\mathrm{Var}(\epsilon_{it+1}) = \sigma^2$, and m_t represents the maximum effective insurance coverage. This specification captures the idea that liquid balances provide insurance value during adverse shocks, but only up to a certain threshold.

5.1.5 Aggregate Return Function

The crucial feature of our model is that the aggregate return on productive investment depends negatively on total liquid holdings:

$$r_t^P = A_t^{\gamma} \cdot \overline{r_t}(1 - M_t/Y_t)^{\omega} \tag{10}$$

where $M_t = \int_0^1 m_{it} di$ is aggregate liquidity and, $\partial r_t^P / \partial M < 0$ (more liquidity reduces productive investment returns); $\partial^2 r_t^P / \partial M^2$ (diminishing marginal effect). Here, $\overline{r_t} = \alpha(\frac{A_t L_t}{K_t})^{1-\alpha}$ is the marginal product of capital, ensuring returns are grounded in the production technology.

5.2 Equilibrium Analysis

5.2.1 Individual Optimal Choice

Given the return $\overline{r}_t$, each agent chooses liquid savings m_{it} to solve their dynamic optimization problem. The first-order condition for liquid savings equates the marginal cost (foregone return) with the marginal insurance benefit:

$$(1 + r_t^{Portfolio}) = \beta E_t \left[\frac{U'(c_{it+1})}{U'(c_{it})} \cdot \frac{\partial \tilde{a}_{it+1}}{\partial m_{it}} \right] \tag{11}$$

5.2.2 Equilibrium Conditions

The equilibrium must satisfy individual optimization given aggregate conditions:

Market clearing: $\int m_{it} di = M_t$, $\int k_{it} = K_t$

Consistency: $p_{1t+1} = h(q, \frac{K_t}{Y_t})$

Productivity evolution: $g_A = [(A_{t+1} - A_t)/A_t] = p_{1t}\phi^0 s_t - \xi$

At this point we make a simplifying assumption: For analytical tractability, we assume that the return on unproductive investment r^U is either small or fixed. This reflects the empirical reality in low-growth economies, where speculative or misallocated capital yields low returns. The simplification allows us to focus on the productive return r^P, which dominates the portfolio return in the relevant region of the state space and drives the curvature of the liquidity demand mapping.The liquidity demand function is $m = \Psi(r^P, \sigma^2)$ with $\frac{\partial \Psi}{\partial r^P} < 0$, and the return on productive assets function is $r^P = A^\alpha \overline{r} \cdot p_1 \cdot (1-m)^\omega$, where liquidity m affects returns directly through $(1-m)^\omega$ but does not (for simplicity) directly enter the innovation function h.

At a general level, an equilibrium in our economy is a fixed point (m^*, p_1^*, A_t), but A_t still changes in the steady state, satisfying: $m^* = \Psi(r^{P(A^*,m^*)}, \sigma^2)$ [liquidity demand]: p_1^* $= h(q, \frac{K^*}{Y^*})$ [productive investment share], $g_A = g(p_1^* \cdot I^*/Y^*) - \xi$ [steady-state productivity]

5.2.3 Existence of Equilibrium

Proposition (Multiple Equilibria).

Under standard regularity conditions, the model admits multiple equilibria. Setup: Given the mapping in $F(m) = \Psi(r^{P(A^*,m^*)}, \sigma^2) : [0,1] \to [0,1]$ is continuous, with $\frac{\partial \Psi}{\partial r^P} < 0$ and p_1 and r^P as above, it has the following key properties: Boundary conditions: $F(0) > 0$ (positive insurance demand when $m = 0$) and $F(1) < 1$ (minimal demand when all resources are liquid).

Let utility: $U(c) = \log c$, discount $\beta \in (0,1)$. The controls: consumption c_t, liquidity m_t (used only in bad states). The insurance payoff is capped: usable liquidity in a bad state is $\min(m_t, \bar{m})$. The indicator for the bad state: $\mathbf{1}_{\{\text{shock}\}} \in \{0,1\}$, with Prob (shock) $= \pi$. The budget/consumption next period:

$$c_{t+1} = (1 + r_t^{Portfolio})a_t - c_t - m_t + y_{t+1} + \min\{m_t, \bar{m}\} \cdot \mathbf{1}_{\{\text{shock}\}} \tag{12}$$

With log utility in (6), choosing (m_t), the FOC equates the opportunity cost to the discounted marginal insurance benefit:

$$(1 + r^{Portfolio}) = \beta\, E_t \left[\frac{U'(c_{t+1})}{U'(c_t)} \frac{\partial \epsilon_{it+1}}{\partial m_t} \right] = \beta\, E_t \left[\frac{c_t}{c_{t+1}} \mathbf{1}_{\{\text{shock}\}} \mathbf{1}_{\{m_t < \bar{m}\}} \right]. \tag{13}$$

Hence for interior $m_t < \bar{m}$,

$$(1 + r^{Portfolio}) = \beta\, \pi\, E_t \left[\frac{c_t}{c_{t+1}} |\, \text{shock} \right]. \tag{14}$$

As m_t rises (below $\bar{m}$), c_{t+1} in the shock state rises one-for-one, making $E_t[c_t/c_{t+1} \mid \text{shock}]$ fall. This is the insurance saturation margin: the RHS is decreasing in m_t.

Using a simple small-shock approximation for the shock-state consumption, the marginal insurance benefit under log utility is proportional to the shock variance σ^2 and declines with c_t:

$$\begin{aligned} (1 + r^{Portfolio}) &= M\ Benefit\ (m) \approx \frac{\beta\, \sigma^2}{c_t^2} \Rightarrow m^*(r^{Portfolio}) \equiv \Psi(r^{Portfolio}, \sigma^2) \\ &= \min\left\{ \bar{m},\ \frac{\beta\, \sigma^2}{1 + r^{Portfolio}} \right\}. \end{aligned} \tag{15}$$

This baseline Ψ is strictly decreasing and flattens at $\bar{m}$ due to saturation (not yet S-shaped by itself).

Now replace the portfolio return with the return on productive assets, which falls nonlinearly with aggregate liquidity.[11] Then (17) and (18) gives the fixed-point problem

$$m \;=\; \Psi(r^P(m), \sigma^2) \;=\; \min\left\{ \bar{m},\ \frac{\beta\, \sigma^2}{1 + \bar{r}(1-m)^{\omega}} \right\}. \tag{16}$$

Equation (19) produces an inverse S-shaped mapping $m \mapsto \Psi(r^P(m), \sigma^2)$ because: for $m \approx 0$: r^P is high, the opportunity cost dominates, so the implied m from (17) is near zero (flat segment near the origin). For intermediate m: the convex drop in $r^P(m)$ (since $\omega > 1$) makes the denominator in (19) fall rapidly, causing a steep rise in the implied m. As $m \to \bar{m}$: the $\min\{\cdot\}$ cap binds (insurance saturation), flattening the mapping again. If, in addition, parameters satisfy $\beta\sigma^2$ large enough, ω sufficiently above 1, and $\bar{m}$ not too small, the graph of m versus $\Psi(r^P(m), \sigma^2)$ crosses the 45° line three times, yielding two stable equilibria (m^L/high-return and m^H/low-return) and one, m^* unstable threshold.

Thus F exhibits an inverse-S shape: initially increasing (insurance motive dominates), then decreasing (satiation effect dominates). The S-Shape emerges because at high returns (r^P is large): opportunity cost dominates implying $m \approx 0$. At very low returns, the insurance motive saturates because marginal benefit of extra liquidity falls. In the middle range, a strong precautionary motive, implying a steep increase in m as r^P falls. This creates an

[11] This simplification is justified by the assumption stated in Section 5.2.2 that r^U is either small or fixed, allowing us to focus on the dominant curvature-driving term r^P.

inverse-S shape for $\Psi(r^P(m), \sigma^2)$: downward-sloping, convex–concave–convex. Although the fixed-point mapping $F(m) = \Psi(r^P(m), \sigma^2)$ is monotonically increasing:

$$\frac{dF}{dm} = \frac{d\Psi}{dr^P} \cdot \frac{dr^P}{dm}, \quad \frac{d\Psi}{dr^P} < 0, \quad \frac{dr^P}{dm} < 0 \tag{17}$$

So the composite derivative, $\frac{dF}{dm} > 0$. The second derivative is given by:

$$\frac{d^2F}{dm^2} = \frac{d^2\Psi}{(dr^P)^2} \left(\frac{dr^P}{dm} \right)^2 + \frac{d\Psi}{dr^P} \cdot \frac{d^2r^P}{dm^2}$$

Here: $\frac{d^2\Psi}{(dr^P)^2} > 0$, $\left(\frac{dr^P}{dm} \right)^2 > 0$, $\frac{d\Psi}{dr^P} < 0$ and $\frac{d^2r^P}{dm^2} > 0$. Hence, the first term is positive and the second term is negative. Therefore, the sign of the second derivative depends on which term dominates. At low values of m, the second derivative is positive (convex). At intermediate values, the second term dominates and the second derivative becomes negative (concave). At high values of m, the insurance demand saturates and the first term dominates again, making the second derivative positive (convex). This change in curvature produces an inverse-S shape in the mapping $F(m)$. Thus, given the inverse-S shape and $F(0) > 0$ (positive insurance demand when $m = 0$) and $F(1) < 1$, the mapping crosses the 45-degree line three times, yielding three fixed points: a low equilibrium (high liquidity, low returns), an unstable middle equilibrium, and a high equilibrium (low liquidity, high returns): $m^L < m^* < m^H$.

For multiple equilibria exhibited, we need strategic complementarity: aggregate M/Y lowers r^P, which feeds back into $\Psi(r^P(m), \sigma^2)$. If $\Psi(r^P(m), \sigma^2)$ is steep enough in the middle range, the fixed-point mapping $m = \Psi(r^P(m), \sigma^2)$ crosses the 45° line three times → three equilibria (two stable, one unstable).

5.2.4 Economic Interpretation

The model generates two distinct equilibria:

1. High Growth Equilibrium **(m^H):** Low liquidity holdings, high returns r^P, sustained productive investment

2. Stagnation Trap **(m^L):** High liquidity holdings, low returns r^P, minimal productive investment

This leads to:

Corrolary. Small parameter changes can trigger equilibrium transitions: An increase in uncertainty ($\sigma^2 \uparrow$) shifts F upward, potentially eliminating m^L. A decline in institutional quality ($q \downarrow$) reduces p_1, lowering r^P and shifting to m^H.

These transitions explain sudden stagnation (Japan) or gradual decline (UK). Thus we have demonstarated that in our model, multiple equilibria arise from the interaction between liquidity's crowding-out effect on returns: $(1-m)^\omega$, insurance demand responding to returns: $\Psi(r^P)$ and because of satiation in insurance demand at low returns. Note that the model requires no direct effect of liquidity on innovation because the feedback through returns alone is sufficient to generate what we call the paradox of thrift in an endogenous growth context.

We can now describe our two equilibria in more detail:

In our model an equilibrium is either: A balanced growth path (BGP) where detrended variables $(M/Y, K/(AL))$ converge to constants while A, K, Y grow at constant rates. A steady state where all level variables are constant. The High equilibrium is a BGP with: $g_A^H = p_{1t}^H \phi^0 s_t^H - \xi > 0$; $g_K^H = g_Y^H = g_A^H$. Constant ratios: $(M/Y)^H$, $(K/Y)^H$. The low equilibrium is a steady state with: $g_A^L = g_K^L = g_Y^L = 0$. All variables constant at low levels.

We also have that:

(a) Low equilibrium only ($q < q^*$): $p_{1t}^L = h(q,\ K_t^L/Y_t^L) < \delta/\phi^0 s_t + \varepsilon$; m^L solves: $(1 + r^U \cdot (1 - m^L)^\nu = \sigma^2/U'(c^L)$ and A_t stagnates: $[(A_{t+1} - A_t)/A_t] < 0$. Condition: The system converges to a liquidity trap.

(b) High equilibrium only ($q > q^*$): $p_1^H = h(q,\ K_t^H/Y_t^H) < \delta/\phi^0 s_t + \varepsilon$; m^H solves: $(1 + A^\gamma r^P (1 - m^H)^A = \sigma^2/U'(c^H)$ and A_{tt} grows: $[(A_{tt+1} - A_{tt})/A_{tt}] > 0$. So productive investment dominates

(c) Both equilibria exist depending upon the value of q and the parameter κ and hence p_1, which we take up below.

5.3 Critical Regions for q

The economy features three key relationships: Productive investment share:

$$p_1 = q^\kappa \cdot \left(\frac{K}{Y}\right)^{-\mu} \cdot (1 - m)^\theta \tag{18}$$

The partial derivative $\partial p_1/\partial M = -\theta p_1/(Y(1 - m))$, showing that the marginal effect of liquidity on productive investment is scaled by aggregate output. The portfolio return function:

$$r^P = A^\gamma \cdot \bar{r} \cdot p_1 \cdot (1 - m)^\omega \tag{19}$$

and the liquidity demand function:

$$m = \Psi(r^P, \sigma^2) \tag{20}$$

where q is institutional quality, m is the liquidity-to-output ratio, and parameters satisfy: $\kappa > 0$: institutional elasticity; $\mu > 0$: diminishing opportunities parameter; $\omega > 0$: crowding-out elasticity; $\gamma \in (0, 1)$: productivity elasticity.

Equilibrium Conditions:

A steady-state equilibrium is a fixed point $m^* \in [0,1]$ satisfying:

$$m^* = F(m^*) \equiv \Psi\left(r^P(p_1(q), m^*), \sigma^2\right) \tag{21}$$

Substituting the functional forms:

$$F(m) = \Psi\left(A^\alpha \overline{r} \cdot q^\varepsilon \left(\frac{K}{Y}\right)^{-\mu} (1-m)^{\theta+\omega}, \sigma^2\right) \tag{22}$$

Critical Thresholds:

Lower Threshold $\underline{q}$. The lower threshold ensures positive productivity growth. For balanced growth, we require:

$$g_A = p_1 \phi_0 s - \delta > 0 \tag{23}$$

This yields the minimum productive investment share:

$$p_1^* = \frac{\delta}{\phi_0 s} \tag{24}$$

At the lower threshold $\underline{q}$, assuming maximum resources for investment ($m \approx 0$):

$$\underline{q}^\kappa \left(\frac{K}{Y}\right)^{-\mu} = \frac{\delta}{\phi_0 s} \tag{25}$$

Therefore:

$$\underline{q} = [\frac{\delta}{\phi^0 s}(\frac{K}{Y})^{-\mu}]^{\frac{1}{\kappa}}$$

Upper Threshold $\bar{q}$. The upper threshold marks where the high equilibrium becomes stable. At this point, the slope of F at the high equilibrium m^H equals unity:

Lemma. The High equilibrium exists and is stable when:

$$\left|\frac{\partial F}{\partial m}\right|_{m=m^H} < 1 \tag{26}$$

Computing the derivative:

$$\frac{\partial F}{\partial m} = \frac{\partial \Psi}{\partial r^P} \cdot \frac{\partial r^P}{\partial m} \tag{27}$$

where:

$$\frac{\partial r^P}{\partial m} = -(\theta + \omega) \cdot \frac{r^P}{1-m} \tag{28}$$

At the threshold $\bar{q}$, the stability condition becomes:

$$\left|\frac{\partial \Psi}{\partial r^P}\right| \cdot (\theta + \omega) \cdot \frac{r^P}{1 - m^H} = 1 \tag{29}$$

Substituting $r^P = A^\gamma \bar{r} \cdot \bar{q}^\kappa \left(\frac{K}{Y}\right)^{-\mu} (1 - m^H)^{2\omega}$:

$$\left|\frac{\partial \Psi}{\partial r^P}\right| \cdot (\theta + \omega) \cdot A^\gamma \bar{r} \cdot \bar{q}^\kappa \left(\frac{K}{Y}\right)^{-\mu} (1 - m^H)^{(\theta+\omega)-1} = 1 \tag{30}$$

Solving for $\bar{q}$:

$$\bar{q} = \frac{1}{\left|\frac{\partial \Psi}{\partial r^P}\right| \cdot (\theta + \omega) \cdot A^\gamma \bar{r} \cdot \bar{q}^\kappa \left(\frac{K}{Y}\right)^{-\mu} (1 - m^H)^{(\theta+\omega)-1}} \tag{31}$$

Multiple Equilibria Region:

Proposition. For $\underline{q} < q < \bar{q}$, the economy admits two stable equilibria:
Low equilibrium: (m^L, p_1^L) with m^L high, p_1^L low
High equilibrium: (m^H, p_1^H) with m^H low, p_1^H high
The width of the multiple equilibria region is:

$$\Delta q = \bar{q} - \underline{q} \tag{32}$$

This width increases with:
Higher crowding-out effects ($\omega \uparrow$)
Greater uncertainty ($\sigma^2 \uparrow$)
Lower institutional elasticity ($\kappa \downarrow$)

Basin of Attraction:

When both equilibria exist, the separatrix m^* satisfies:

$$F(m^*) = m^* \quad \text{and} \quad F^{*\prime} = 1 \tag{33}$$

For initial conditions:

$$m_0 < m^* \Rightarrow \text{convergence to } m^H \text{ (high equilibrium)} \tag{34}$$
$$m_0 > m^* \Rightarrow \text{convergence to } m^L \text{ (low equilibrium)} \tag{35}$$

Comparative Statics:

The thresholds respond to parameter changes as follows:

$$\frac{\partial \underline{q}}{\partial \delta} = \frac{1}{\kappa}\underline{q} \cdot \frac{1}{\delta} > 0 \tag{36}$$

$$\frac{\partial \underline{q}}{\partial s} = -\frac{1}{\kappa}\underline{q} \cdot \frac{1}{s} < 0 \tag{37}$$

$$\frac{\partial \bar{q}}{\partial \sigma^2} < 0 \quad \text{(through increased } \left|\frac{\partial \Psi}{\partial r^P}\right|\text{)} \tag{38}$$

$$\frac{\partial \bar{q}}{\partial \theta} < 0 \quad \text{(stronger crowding-out lowers threshold)} \tag{39}$$

In our economy, if $q \geq \bar{q}$ then the high equilibriun becomes stable. If $q \leq \underline{q}$, then he economy is trapped in the low equilibrium. The region $\underline{q} < q < \bar{q}$ is more nuanced, both equilibria exist, and whether the economy sinks depends on initial conditions (particularly initial liquidity m_0) and expectations. But below $\underline{q}$, collapse to the low equilibrium is certain.This has a significant policy implications. The existence of a substantial multiple equilibria region suggests that moderate improvements in institutional quality may be insufficient to escape the low equilibrium, so coordinated expectations management is crucial when $q \in (\underline{q}, \bar{q})$. Hence, policies should target pushing $q > \bar{q}$ to ensure convergence to high growth.

5.4 Transition

The model has clearly defined thresholds: $\underline{q}$ (lower threshold): Below this, only low equilibrium exists. $\bar{q}$ (upper threshold): Above this, only high equilibrium exists. Equilibrium transitions can occur in the model. From High to Low equilibrium, a negative shock to uncertainty can trigger collapse: the critical shock size is: $\Delta\sigma^2(q - \underline{q})r^P A^\gamma Y/2$.[12] When uncertainty increases by more than this amount, the economy shifts from high to low equilibrium. From Low to High equilibrium: escape requires either: institutional improvement: raising q above $\bar{q}$, the required improvement: $\Delta q = \bar{q} - q_{Current}$; or alternatively a coordinated shift in expectations is required, so as to simultaneously convince all agents to reduce liquidity. This would be difficult without institutional change.

The comparative statics are that: $\partial m/\partial q < 0$: better institutions implies lower liquidity demand; $\partial p_1/\partial \sigma^2 < 0$: higher uncertainty implies less productive investment; $\partial(\bar{q} - \underline{q})/\partial \omega > 0$: stronger crowding-out implies wider multiple equilibria zone.

The key insight is the asymmetry: small shocks can cause collapse, but recovery requires large, permanent changes. This explains Japan's experience - the bubble burst was a shock that pushed them to the low equilibrium, and moderate policy responses have been insufficient

[12] The formula emerges when an increase in σ^2 causes the upper crossing point of $F(m)$ with the 45-degree line to disappear. This happens when $F(m)$ no longer exceeds m for any intermediate values. Since Ψ depends on σ^2 and r^P depends on multiple parameters including A^γ, the relationship between the shock size and the elimination of equilibria involves several moving parts. Starting with the condition that the High equilibrium exists when $F(m^H) = m^H$, and noting that $\partial F/\partial \sigma^2 > 0$ (higher uncertainty increases liquidity demand), we find the critical point where increasing σ^2 by $\Delta\sigma^2$ causes $\max\,[F(m) - m] = 0$.

to push q above $\overline{q}$.

5.4.1 Symmetric Equilibrium

In a symmetric equilibrium, all agents make identical choices: $m_{it} = m_t$ and $a_{it} = a_t$ for all i. The equilibrium conditions are:

$$m_t = \lambda(r_t^P, \sigma^2, a_t) \tag{40}$$

$$r_t^P = A_t^\gamma \cdot \overline{r_t}(1 - M_t/Y_{tt})^\omega \tag{41}$$

where $\lambda(\cdot)$ represents the individual optimal liquidity choice function.

There are multiple equilibria with different p_{1t} levels. These satisfy: $m_t^H < m_t^L$, $p_{1t}^H > p_1^L$, A_t^H growing but A_t^L stagnant. High-Return Equilibrium: Low liquidity m_t^H, high return r_t^{PH}. Low-Return Equilibrium: High liquidity m_t^L, low return r_t^{PL}, where $m_t^H < m_t^L$ and $r_t^{PH} > r_t^{PL}$. The high equilibrium Pareto dominates the low equilibrium. Small shocks around each equilibrium lead to local convergence. Large shocks ($\Delta m_t > \overline{m}$) can trigger transitions between equilibria. High p_{1t} equilibrium: High productive investment $\rightarrow$ high returns $\rightarrow$ justifies low liquidity $\rightarrow$ more resources for productive investment. Low p_{1t} equilibrium: Low productive investment $\rightarrow$ low returns $\rightarrow$ high liquidity demand $\rightarrow$ fewer resources and weak incentives for productive investment.

5.4.2 The Liquidity Trap Mechanism

The low-return equilibrium exhibits the characteristics of a liquidity trap[13]:

1. High aggregate liquidity reduces returns on productive investment
2. Low returns increase the relative value of liquidity insurance
3. This justifies high individual liquidity demand
4. The economy becomes trapped in a low-return, high-liquidity state.

5.4.3 Connection to the Three Levels of Say's Law

The multiple equilibria derived here illustrate precisely how Level 3 Say's Law can fail even when Levels 1 and 2 hold: Level 1 holds: Markets clear within each period. Level 2 holds: Aggregate savings equals investment plus liquidity: $S_t = I_t + M_t$. Level 3 fails in the low equilibrium: High liquidity demand (m_t^L) means low productive investment share (p_1^L), so investment doesn't generate sufficient productivity growth to sustain MPK. In the low

[13] See Eggertsson and Mehrotra (2014) for a model of secular stagnation driven by precautionary savings and low interest rates.

equilibrium, the growth rate of productivity: $\dot{g}_A = \phi^0(p_{1t}^L \cdot K_t/Y_t) - \xi - \delta$ falls below the threshold needed to offset capital deepening, causing: $MPK = \alpha(\frac{A_t L_t}{K_t})^{1-\alpha}$ to decline rapidly, validating high liquidity demand and creating the trap.

6 Stability and Transitional Dynamics

Our economy exhibits two distinct long-run equilibria. We have already analyzsed the fixed-point mapping $F(m) = \Psi(r^P(m), \sigma^2)$, which exhibits an inverse-S shape and yields three equilibria: two stable and one unstable. This static analysis identifies the conditions under which liquidity demand equilibria emerge, akin to Hicksian comparative statics. We now undertake a dynamic stability analysis by linearizing the full intertemporal system around each equilibrium and compute the Jacobian eigenvalues. This determines whether the system converges to the equilibrium following small perturbations. Together, these approaches clarify both the selection and persistence of growth paths in the presence of institutional constraints and liquidity feedback.

The dynamic system is in (M_t, K_t, A_t) space:

$$M_{t+1} = \Psi(r_t^P, M_t, \{E_t[r_{t+j}]) \tag{42}$$

$$K_{t+1} = (1-\delta)K_t + s_t Y_t - (M_t - M_{t-1}) \tag{43}$$

$$A_{t+1} - A_t = A_t(1 + p_{1t}\phi^0 s_t) - \xi \tag{44}$$

The productive investment return is given by r_t^P with endogenous MPK:

$$r_t^P = A_t^\eta \cdot \alpha \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} \cdot (1 - M_t/Y_t)^\omega \tag{45}$$

where $\bar{r}_t = \text{MPK}_t = \alpha\left(\frac{A_t L_t}{K_t}\right)^{1-\alpha}$ is the marginal product of capital.This system has two types of long-run behaviors: Low equilibrium: A steady state where all variables converge to constants. High equilibrium: A balanced growth path where variables grow but ratios stabilize.

The Low level equilibrium can be examined by linearising the system around steady states (M^*, K^*, A^*). The High level equilibrium involved the behaviour of the model rewritten in ratio, or detrended variables:Ratio/Detrended Variables (for BGP analysis). Then we write $l_t = M_t/Y_t$ (liquidity-output ratio), $k_t = K_t/A_t L_t$ (capital per effective worker). The dynamics become:

$$l_{t+1} = \Psi(r_t^P, l_t, \{E_t[r_{t+j}])/(1 + g_{Y,t+1}) \tag{46}$$

$$k_{t+1} = [(1-\delta)k_t + s_t - l_t(g_{Y,t+1})]/(1 + g_{A,t+1}) \tag{47}$$

$$g_{A,t+1} = p_{1,t}\phi^0 s_t - \xi \tag{48}$$

where $g_{Y,t+1} = g_{A,t+1}$ on the BGP. This system has a stationary point in (l^*, k^*) space.with constant growth $g^* = p_{1,t}\phi^0 s_t - \xi > 0$.

The stability analysis of the Jacobians around each equilibrium, this is undertaken in the Appendix. Note that the Low level equilibrium is achieved at modest levels of both $p_{1,t}$ and ϕ^0 and the high level equilibria at higer values of both variables.

The high-productivity Balanced Growth Path occurs when institutional quality is sufficient ($q \geq \overline{q}$), the economy converges to a BGP where: productivity grows at constant rate $g^H = p_1^H \phi^0 s^H - \xi > \dot{0}$. Capital and output grow at the same rate g^H.The ratios $l = M/Y$ and $k = K/AL$ converge to constants and $MPK = \alpha(\frac{1}{k^*})^{1-\alpha}$ remains constant despite growing K, because A grows proportionally. The low-productivity stagnation trap occurs when institutional quality is poor ($q < \overline{q}$), the economy converges to a steady state where: all growth rates converge to zero, their is high liquidity demand l^L crowds out productive investment and low p_1^L means insufficient innovation to sustain growth. The economy then stagnates with constant, low-level variables

6.1 Transition Paths Between Equilibria

It is interesting to examine shock-triggered transitions in the model. Starting from high equilibrium, if σ^2 increases suddenly: Period 0: Liquidity jumps to $\Psi(r_0^P, \sigma^2_{New})$; Periods $1-T$, capital stock adjusts slowly, MPK falls; Period $T+$: Converges to low equilibrium if the shock is large enough.

Alternatively, consider policy-induced transitions: from the Low equilibrium. A credible announcement of institutional improvement ($q \to q'$): Expectations shift: $E_t[\overline{r}_{t+j}]$ jumps; liquidity gradually falls as returns improve; the converges to High equilibrium if $q' > q^*$.

During transition, Level 2 Say's Law holds ($S_t = I_t + (M_t - M_{t-1})$ at each t). But Level 3 Say's Law progressively fails as p_{1t} falls with rising M_t. The economy endogenously generates its own stagnation through rational individual choices.

6.2 Key Dynamic Features

Liquidity stocks adjust slowly because they represent accumulated savings. This creates persistent deviations from steady-state equilibrium and prevents rapid rebalancing. The $1/Y_t^2$ scaling of the liquidity-productivity feedback means that even large changes in individual liquidity holdings have minimal immediate impact on aggregate productivity, further slowing adjustment and emphasising the importance of coordination.

Forward-looking agents base current decisions on expected future returns. If agents expect persistently low returns, this increases current savings, creating self-fulfilling prophecies.

Starting from a high-return equilibrium, negative shocks can trigger:

$$\text{Shock} \rightarrow \uparrow \text{Insurance demand} \rightarrow \downarrow \text{Returns} \rightarrow \uparrow \text{Precautionary saving} \rightarrow$$
$$\text{Further} \downarrow \text{Returns}$$

6.3 Sources of Increased Liquidity Demand and Policy Interventions

Several mechanisms can trigger transitions toward the liquidity trap. These include uncertainty shocks: increased variance in asset returns ($\sigma^2 \uparrow$); and structural breaks in economic relationships. Financial market fragility: reduced market liquidity for productive assets; increased transaction costs; banking sector stress. Productivity shocks: technological obsolescence: sectoral shifts requiring new investment types; regulatory changes affecting returns. Demographic and social changes: aging populations with higher liquidity preferences; increased income inequality; cultural shifts toward risk aversion.

The model suggests several policy interventions: breaking the feedback loop: policies should target either reducing initial uncertainty shocks; breaking the link between liquidity demand and productive returns; providing public insurance to reduce private liquidity demand. Threshold effects and timing is importna, as small interventions may be ineffective due to multiple equilibrium, early intervention is more effective than late intervention and policy credibility is crucial for shifting expectations

6.4 Connection to Literature

This model bridges several strands of economic theory: Liquidity traps, it extends Krugman (1998)-style models by endogenising liquidity preference. Precautionary savings: it builds on Aiyagari (1994)-type models with aggregate feedback. Multiple equilibria, which relates to coordination failure and the sunspot literature. Secular stagnation: it provides microfoundations for persistent low returns

The paper demonstrates how rational individual insurance behavior can lead to collectively suboptimal outcomes in the presence of aggregate feedback effects. The model provides a novel framework for understanding secular stagnation, financial instability, and the limits of market adjustment mechanisms. Key insights include: Individual liquidity insurance creates negative externalities. Multiple equilibria arise naturally from strategic complementarities. Transitional dynamics exhibit path dependence and overshooting. It advocates that policy interventions must account for threshold effects and expectations.

7 Empirical Evidence on Capital Misallocation and Unproductive Investment

7.1 Calibrating p_1 from Empirical Evidence

The distinction between productive and unproductive investment (captured by p_1 in our framework) finds strong empirical support in the capital misallocation literature. This evidence validates our parameter choices in the numerical examples (see the Appendix). Hsieh and Klenow (2009) for example, provide a quantitative framework showing that capital misallocation reduces aggregate TFP by 30-50% in China and 40-60% in India relative to U.S. efficiency levels. Translating this into our framework, if 40-60% of investment flows to unproductive uses, this implies $p_1 \approx$ 0.4-0.6 during periods of institutional dysfunction. Given other parameters, this range brackets our critical threshold $p_1^* = \delta/\phi^0 s = 0.625$ derived in Section 6.3, where the economy requires at least 62.5% productive investment to sustain positive growth.

7.2 Validation of the Low Equilibrium Parameters

Our low equilibrium parameterization ($p_1 = 0.3$) appears conservative when compared to documented episodes of severe misallocation. Caballero, Hoshi, and Kashyap (2008) find that zombie firms in Japan absorbed 15-20% of total investment, while contributing negligibly to productivity. Combined with speculative real estate investment (estimated at 30-40% of total investment during the bubble period), this suggests $p_1^L \approx 0.3 - 0.4$ during Japan's lost decades, remarkably close to our calibrated value.

This low $p_1^L = 0.3$ generates the stable trap characterised in the stability analysis (see the Appendix). With these parameters, the technology eigenvalue becomes: $\lambda_3^L = 1 + p_1^L \phi^0 s - \xi = 1 + (0.3)(0.8)(0.2) - 0.1 = 0.948 < 1$. This eigenvalue being less than unity confirms the trap's stability: productivity declines ($g = -0.052$), validating the stagnation equilibrium. The weak feedback from liquidity to technology (scaled by $1/Y^2$ as shown in the Jacobian analysis) makes escape particularly difficult once the economy settles into this equilibrium.

7.3 Evidence Supporting Multiple Equilibria

Reis (2013) documents a particularly relevant case for our multiple equilibria framework. In Portugal during the 2000s, the share of capital in unproductive uses rose from 30% to over 50% (implying p_1 fell from 0.7 to 0.5), pushing the economy across our critical threshold p_1 = 0.625. This transition corresponds precisely to our stability analysis: when $p_1 > p_1^*$, the high equilibrium is stable, but once p_1 falls below p_1^*, the economy transitions to the low equilibrium trap.

Gopinath et al. (2017) provide complementary evidence from Southern Europe, where increased capital misallocation after 1999 reduced TFP growth by $0.6 - 1.0$ percentage points annually. Using our framework's parameters, this TFP loss corresponds to a decline in p_1 from approximately 0.75 to 0.45, again crossing the critical threshold and triggering the transition mechanism described in Section 5.4.

7.4 The High Equilibrium and Marginal Stability

Our high equilibrium parameterization ($p_1^H = 0.8$) reflects best-practice allocation efficiency. Even the United States, often used as the benchmark, shows signs of declining p_1. Gutiérrez and Philippon (2017) document that S&P 500 firms spent 95% of earnings on buybacks and dividends during 2010-2015, suggesting p_1 may have fallen to 0.6-0.7 even in the U.S, dangerously close to our threshold.

With $p_1^H = 0.8$, the stability analysis (in the Appendix) shows the high equilibrium is marginally stable. The growth rate $g^* = p_1^H \phi^0 s - \xi =$ (0.8)(0.8)(0.2) - 0.1 = 0.028 is positive but small. The eigenvalues of the normalized system are dampened by the factor $(1 + g^*)^{-1}$ ≈ 0.973, providing stability but leaving little margin for adverse shocks.

7.5 Implications for the Critical Region $\underline{q} < q < \overline{q}$

The empirical evidence suggests that many advanced economies operate within the multiple equilibria region where $\underline{q} < q < \overline{q}$. The width of this region, as shown in Section 5.3, depends on the crowding-out parameter ω and institutional elasticity κ. The documented cases where economies transition between equilibria (Portugal 2000-2008, Japan 1990-1995) indicate this region is substantial, perhaps spanning p_1 values from 0.4 to 0.7.

This empirical validation has crucial policy implications: moderate improvements in institutional quality may be insufficient if the economy remains within the multiple equilibria region. As the stability analysis demonstrates, escaping the low equilibrium requires either pushing p_1 above $p_1^* \approx 0.625$ or coordinating expectations to shift the economy's trajectory—both challenging given the weak feedback mechanisms ($1/Y^2$ scaling) identified in the Jacobian analysis.

8 Unproductive Investment Channels in High-Saving Economies

In economies with high savings but low innovation, the failure to convert savings into productivity-enhancing investment often leads to the proliferation of unproductive investment channels. These channels absorb capital without contributing meaningfully to long-run growth or technological advancement.

Real estate is a common destination for excess savings, particularly in environments with limited financial development or weak innovation ecosystems. While investment in housing and infrastructure can be productive, speculative real estate purchases, especially in urban property markets, often serve as stores of value rather than engines of growth. These investments inflate asset prices, exacerbate inequality, and divert capital from sectors that could enhance productivity.

Savings may also flow into secondary markets for existing assets, such as equities, bonds, art, or cryptocurrencies. These transactions do not fund new capital formation; instead, they represent reallocations of ownership. While they may improve liquidity or signal value, they do not directly expand the economy's productive frontier. In this sense, they resemble Keynes's idle balances, which is capital that circulates without stimulating output.

When savings are absorbed by the above channels, the economy experiences a form of capital deepening without innovation. Marginal returns to capital decline, and the endogenous growth mechanism, where investment drives productivity, fails to activate. This leads to stagnation, despite high levels of investment. The economy becomes trapped in a low-growth equilibrium, where the supply of savings fails to generate its own demand, echoing the Keynesian paradox of thrift.

9 Equilibrium and the Dampening of the Marginal Product of Capital

To tie the dynamics of savings, investment, and growth to a coherent equilibrium, it is necessary to incorporate a dampening mechanism on the marginal product of capital (MPK). In both neoclassical and endogenous growth frameworks, MPK governs the incentive to invest. If savings are abundant but directed toward unproductive uses, MPK must decline, either due to diminishing returns or misallocation, until it aligns with the prevailing interest rate.

We have seen in our model that when institutions fail to channel investment productively (low p_1 in our framework), the effective growth rate of A_t is reduced. This means K_t grows faster than $A_t L_t$, causing MPK to decline more rapidly than in economies with strong innovation institutions. The economy experiences premature capital exhaustion, returns fall to unacceptable levels even at moderate capital stocks.

In modern economies, excess savings often flow into financial instruments such as bonds and equities. This can lower long-term interest rates, creating a wedge between the cost of capital and its return. If: $r^P < MPK$ then firms have an incentive to invest in productive capital, potentially mitigating the stagnation caused by unproductive savings. This rate equalisation mechanism can restore equilibrium by redirecting savings toward investment, provided that the institutional environment supports productive deployment. However, if MPK is dampened too quickly—due to speculative investment in real estate or existing assets—then even low interest rates may fail to stimulate productive investment. The return on capital falls below the threshold needed to justify new investment, and the economy enters a low-growth trap.

Finally, endogenous growth theory offers a solution: if investment leads to productivity gains, then MPK can be sustained. In models such as Romer's where A(t) grows endogenously through R&D or human capital formation, the marginal product of capital becomes: $MPK = \alpha(\frac{A_t L_t}{K_t})^{1-\alpha}$. As long as A_t grows in tandem with capital, MPK remains sufficiently high to justify continued investment. This sustains the growth process and prevents stagnation, even in high-saving economies. However, our claim is that this requires the right institutions.

10 Case Study: Japan's Stagnation and the Breakdown of the Endogenous Growth Mechanism

Japan's economic stagnation since the early 1990s offers a compelling real-world example of an economy with high savings but insufficient investment in productivity-enhancing activities. Despite decades of fiscal and monetary stimulus, Japan has struggled to reignite sustained growth, largely due to structural failures in translating savings into innovation.

Throughout the post-bubble period, Japan maintained high household and corporate savings rates. However, much of this capital was directed toward real estate, government bonds, and existing financial assets, rather than toward new capital formation or technological innovation. The result was a persistent gap between the supply of savings and the demand for productive investment.

The collapse of Japan's asset price bubble in the early 1990s led to a prolonged period of balance sheet repair, during which firms and households prioritized de-leveraging over investment. Capital flowed into safe assets, reinforcing a cycle of low returns and low innovation. Real estate speculation and investment in existing assets absorbed savings without expanding the productive frontier.[14]

Japan's rigid labour markets, aging population, and conservative corporate culture further limited the effectiveness of capital deployment. While Japan remained a technological leader in certain sectors, the diffusion of innovation across the economy slowed. Entrepreneurship and venture capital lagged behind other advanced economies, and the education system, while strong, did not fully adapt to the demands of a knowledge-based economy.

Japan's experience thus illustrates the failure of the endogenous growth mechanism in the absence of supportive institutions. Despite abundant savings, the lack of effective channels for innovation—such as dynamic R and D ecosystems, flexible labour markets, and entrepreneurial culture, meant that MPK declined, and the economy settled into a low-growth equilibrium. This case reinforces the central thesis of this paper: savings must be matched by mechanisms that ensure their productive deployment, or the economy risks stagnation despite high investment levels.

[14] Krugman (1998) discusses Japan's liquidity trap and the failure of monetary policy to stimulate productive investment.

11 Case Study: The United Kingdom and the Constraints on Endogenous Growth

The United Kingdom's economic performance in recent years has been marked by persistent stagnation, despite its status as a high-income economy with substantial financial resources. Unlike Japan, the UK's challenge is not a surplus of savings per se, but a failure to generate and sustain investment opportunities that enhance productivity.

Unlike Japan, where the paradox of thrift manifests through excess savings seeking unproductive outlets, the UK faces a fundamentally different challenge. With household savings rates of only 6-8% compared to Japan's 25-30%, and gross fixed capital formation at just 17% of GDP versus 20-25% for peer economies, the UK suffers from chronic capital scarcity rather than abundance. This represents a compound failure of the savings-investment-growth nexus: not only does the economy fail to generate sufficient savings (Level 2 breakdown), but the limited investment that does occur is poorly allocated (Level 3 breakdown). Where Japan's high savings at least provide the raw material for potential productive investment if institutions could be reformed, the UK lacks even this foundation. The economy operates in what might be termed a "low-level equilibrium trap" where low s and low p_1 reinforce each other: insufficient savings limit investment opportunities, which reduces returns, which further discourages saving. This distinguishes the UK case from both the Japanese experience and the theoretical framework developed earlier, suggesting that different policy prescriptions are needed for savings-scarce versus savings-abundant stagnation.

The UK's institutional environment has struggled to support endogenous growth. Austerity policies have reduced public investment in education, infrastructure, and R and D. Political uncertainty, especially surrounding Brexit, has weakened long-term business confidence. Fragmented industrial strategy and inconsistent innovation policy have further limited the economy's ability to convert savings into productive investment. Despite a strong research base and vibrant startup culture, the UK faces an innovation bottleneck. Promising firms often fail to scale due to limited access to growth capital, risk-averse investment behavior, and regulatory complexity. The result is an "incubator economy," where innovation occurs but does not translate into broad-based productivity gains.

The UK's experience illustrates a different facet of the breakdown in the endogenous growth mechanism. Here, the issue is not excess savings but the absence of institutional and policy mechanisms that ensure investment flows into productivity-enhancing activities. Without these, the economy remains trapped in a low-growth equilibrium, despite its financial depth and innovative potential.

Restoring endogenous growth in the UK. requires institutional reform and strategic investment across four key domains: First, innovation and capital formation could be improved by expanding public R and D funding and support for technology transfer; development of patient capital markets to finance high-risk innovation; incentivising investment in green technologies and intangible assets. Second, human capital and skills could be improved by Investment in technical education. Third, there is a need for institutional and regulatory reform. Stabilising industrial strategy would reduce policy uncertainty. There is scope for Improving IP protection, competition policy, and planning frameworks. In addition, productivity would be improved by streamlining regulation to facilitate infrastructure and housing investment. Finally, on the fiscal and demand side, there is a need to commit to prioritising public investment over consumption in fiscal planning. For example, the use targeted fiscal stimulus to support high-productivity sectors. Finally, there is a need to ensure macroeconomic stability to foster private sector confidence.

12 Institutions and the Foundations of Endogenous Growth

Acemoglu and co-authors have emphasised that inclusive economic institutions, those that enforce property rights, support market entry, and encourage innovation, are central to long-run growth. In their framework, institutions shape the incentives for investment and determine whether savings are deployed toward productive or extractive activities. This complements endogenous growth theory by providing a deeper explanation for why some economies succeed in converting savings into innovation, while others stagnate. Acemoglu's work reinforces the idea that endogenous growth is not automatic; it requires a foundation of institutions that enable experimentation, knowledge diffusion, and broad-based participation. Without these, the economy risks falling into a modern paradox of thrift, where savings accumulate but fail to generate sustained productivity growth.

13 Conclusion

Endogenous growth theory posits that while savings are necessary for investment and capital accumulation, they are not sufficient for sustained economic growth. For savings to translate into long-run productivity gains, an economy must possess mechanisms that channel investment into innovation and efficiency-enhancing activities. These mechanisms include a robust rule of law, secure property rights, high-quality education systems, well-designed incentives (including tax policy), and institutions that support research, entrepreneurship, and technological diffusion. Without these, savings may be mis-allocated to unproductive uses, and the economy risks stagnation despite high investment levels.

Our framework reveals that economies can fail to sustain endogenous growth through fundamentally different pathways. In savings-abundant economies like Japan ($s \approx 0.25 - 0.30$), the primary constraint is institutional quality (p_1). Even with ample resources for investment, these economies can fall into stagnation when $p_1 < p_1^* = \delta/\phi^0 s$, as excess savings flow into unproductive channels. The policy imperative here is institutional reform to raise p_1 above the critical threshold, redirecting existing savings toward productivity-enhancing activities.

In savings -scarce economies like the UK ($s \approx 0.06 - 0.08$) face a more severe challenge: a dual failure where both s and p_1 are below their critical thresholds. Even perfect institutions (p_1) cannot compensate for insufficient savings, while low p_1 compounds the problem by ensuring that what little investment occurs fails to enhance productivity. These economies require a two-pronged approach: first raising the savings rate through fiscal incentives, pension reform, or income redistribution, while simultaneously improving institutional quality to ensure productive deployment. There is also the possibility of making the economy more attractive to foreign direct investment. The sequencing matters, institutional reforms alone will have minimal impact without addressing the underlying savings shortage.

This distinction has profound implications for policy design. Quantitative easing and low interest rates may help savings-abundant economies by reducing the opportunity cost of productive investment, but will have limited effect in savings-scarce economies where the constraint is the quantity, not the allocation, of savings. Similarly, innovation policies and R and D tax credits assume there is capital seeking productive outlets, an assumption valid for Japan but questionable for the UK. Our model thus suggests that the one-size-fits-all approach to addressing secular stagnation fails to account for these fundamental structural differences.

Future empirical work should test how institutional quality affects the rate at which MPK declines with capital accumulation.[15] In our framework, poor institutions don't just prevent access to high-productivity sectors (as in AK models), they actively accelerate the decline in returns to capital by channeling investment into unproductive uses. This creates a 'double dampening' effect: standard dampening, $MPK = \alpha(\frac{A_t L_t}{K_t})^{1-\alpha}$ falls as K_t rises (standard Solow). Institutional dampening: poor institutions mean only a fraction p_1 of investment enhances A_t, so effective productivity growth is slower than potential. The result is that MPK falls faster than standard models predict: not just from K_t rising, but from A_t growing too slowly to offset it. This distinction matters for policy, it's not enough to stimulate investment or remove barriers; we need institutions that ensure investment actively enhances productivity to counteract the natural tendency for returns to decline.

[15] Attanasio, Picci, and Scorcu (2000) provide panel evidence on the savings–investment–growth nexus.

Appendix

Stability Analysis of Equilibria

Two Types of Long-Run Equilibria:

Our model admits two distinct types of long-run equilibria that require different analytical approaches:

1. **Low Equilibrium**: A steady state where all level variables converge to constants (zero growth)
2. **High Equilibrium**: A balanced growth path (BGP) where ratios converge to constants while levels grow

Low Equilibrium: Steady State Analysis

At the low equilibrium, all variables converge to steady-state levels (M^L, K^L, A^L). The dynamic system is:

$$\begin{aligned} M_{t+1} &= \Psi(r_t^P, M_t) \\ K_{t+1} &= (1-\delta)K_t + sY_t - (M_t - M_{t-1}) \\ A_{t+1} &= A_t[(1 + p_{1t}\varphi^0 s_t) - \xi] \end{aligned}$$

With low p_1^L, we have $p_1^L \phi_0 s < \xi$, so $A_{t+1} < A_t$ and productivity declines toward a low steady state. The state vector is defined as $\mathbf{X}_t = [M_t, K_t, A_t]'$. The Jacobian $\mathbf{J}$ evaluated at steady state has the form:

$$\mathbf{J}^L = \begin{matrix} \partial M_{t+1}/\partial M_t & \partial M_{t+1}/\partial K_t & \partial M_{t+1}/\partial A_t \\ \partial K_{t+1}/\partial M_t & \partial K_{t+1}/\partial K_t & \partial K_{t+1}/\partial A_t \\ \partial A_{t+1}/\partial M_t & \partial A_{t+1}/\partial K_t & \partial A_{t+1}/\partial A_t \end{matrix}$$

$$\mathbf{J}^L = \begin{bmatrix} \Psi_M + \Psi_{r^P} \cdot \frac{\partial r^P}{\partial M_t} & \Psi_{r^P} \cdot \frac{\partial r^P}{\partial K_t} & \Psi_{r^P} \cdot \frac{\partial r^P}{\partial A} \\ -1 & (1-\delta) + s\frac{\partial Y_t}{\partial K_t} & s\frac{\partial Y_t}{\partial A} \\ \phi_0 s \partial p_1/\partial M_t \cdot (1/Y_t) & -p_1' \phi_0 s \frac{\partial p_1}{\partial K_t} & (1 + p_1\phi_0 s) - \xi \end{bmatrix}$$

with

$$\frac{\partial r^P}{\partial M_t} = -\omega \cdot \frac{r_t^P}{Y_t \cdot (1 - M_t/Y_t)}$$

$$\frac{\partial r^P}{\partial K_t} = -(1-\alpha) \cdot \frac{r_t^P}{K_t} + \alpha\omega \cdot \frac{M_t \cdot r_t^P}{K_t \cdot Y_t \cdot (1 - M_t/Y_t)}$$

$$\frac{\partial r^P}{\partial A_t} = \left[\eta + (1-\alpha)\left(1 + \frac{\omega M_t}{Y_t(1 - M_t/Y_t)}\right)\right] \cdot \frac{r_t^P}{A_t}$$

where:

- $\Psi_M = \frac{\partial \Psi}{\partial M_t}$ is the partial derivative of Ψ with respect to M_t
- $\Psi_{r^P} = \frac{\partial \Psi}{\partial r^P}$ is the partial derivative of Ψ with respect to r^P
- The -1 term arises from $-\frac{\partial(M_t - M_{t-1})}{\partial M_t}$
- $s\frac{\partial Y}{\partial K}$ and $s\frac{\partial Y}{\partial A}$ appear if output Y depends on K and A

Note hat the matrix element a_{31} is scaled by $1/Y^2$ making the feedback from liquidity to technology extremely weak in steady state.

The characteristic equation is obtained from:

$$\det(\mathbf{J} - \lambda \mathbf{I}) = 0$$

The eigenvalues at Low equilibrium are

$$\lambda_3^L = 1 + p_1^L \phi_0 s - \xi < 1 \quad \text{(stable, as } p_1^L \phi_0 s < \xi\text{)}$$
$$\lambda_{1,2}^L = \text{roots of } \lambda^2 - \text{tr}_{MC}^L \lambda + \overset{L}{\underset{Mc}{\det}} = 0$$

The root λ_3^L is for the technology equation; the money-capital subsystem has trace and determinant affected by low $r^{P,L}$. All eigenvalues $|\lambda_i^L| < 1 \Rightarrow$ stable trap. The eigen value $\lambda_3^L < 1$ is the dominant stability condition. The weak feedback from M to A (scaled by $1/Y^2$) means the technology equation effectively decouples from liquidity dynamics, making the trap especially persistent. The MPK feedback effects strengthen stability because: Low $r^{P,L}$ reduces the magnitude of all feedback terms; High M^L makes the system less responsive to shocks.

High Equilibrium: Balanced Growth Path Analysis

For the BGP, we work with normalized (detrended) variables:

$$l_t = M_t/Y_t \quad \text{(liquidity-output ratio)}$$
$$k_t = K_t/(A_t L_t) \quad \text{(capital per effective worker)}$$
$$\text{Growth rates: } g_A = g_K = g_Y = g^* \quad \text{(balanced growth)}$$

The dynamic system becomes:

$$l_{t+1} = \frac{\Psi(r_t^P, l_t, k_t)}{1+g^*}$$
$$k_{t+1} = \frac{(1-\delta)k_t + s - l_t \cdot g^*}{1+g^*}$$
$$g^* = p_1^H \phi_0 s - \tau \quad \text{(constant on BGP)}$$

The Jacobian at High equilibrium (BGP) for the normalized system at steady state (l^H, k^H, g^*):

$$\mathbf{J}_{BGP}^H = \begin{bmatrix} \frac{\Psi_l}{1+g^*} - \frac{\Psi_{r^P} \cdot \omega r^{P,H}}{(1+g^*)(1-l^H)} & \frac{-\Psi_{r^P} \cdot (1-\alpha) r^{P,H}/k^H}{1+g^*} \\ \frac{-g^*}{1+g^*} & \frac{1-\delta}{1+g^*} \end{bmatrix}$$

Note: The growth rate g^* is constant on the BGP, so we analyze the 2 by system in (l, k). The eigenvalues of J_{BGP}^H are:

$$\lambda_{1,2}^H = \frac{\text{tr}(\mathbf{J}_{BGP}^H) \pm \sqrt{\text{tr}(\mathbf{J}_{BGP}^H)^2 - 4\det(\mathbf{J}_{BGP}^H)}}{2}$$

The key insight: The $(1+g^*)$ denominators mean positive growth **dampens** the eigenvalues, improving stability.

Stability Conditions for BGP

For the high equilibrium to be stable:

1. **Positive growth**: $g^* = p_1^H \phi_0 s - \tau > 0$
2. **Dynamic stability**: Both eigenvalues of the 2 by 2 system satisfy $|\lambda_i^H| < 1$

The critical condition becomes:

$$\left|\frac{\Psi_l}{1+g^*}\right| + \left|\frac{1-\delta}{1+g^*}\right| < 2$$

with additional constraints from the MPK feedback through Ψ_{r^P}.

Comparison of Equilibria

Property	Low Equilibrium	High Equilibrium
Type	Steady state (levels)	BGP (growth)
Growth rate	0	$g^* > 0$
System dimension	3 by 3 in (M, K, A)	2 by 2 in (l, k)
Key eigenvalue	$\lambda_3^L = 1 + p_1^L \phi_0 s - \xi$	Scaled by $(1+g^*)^{-1}$
Stability	Strong (trap)	Fragile (requires p_1^H high)
MPK feedback	Reinforces trap	Threatens stability

Economic Interpretation:

In the Low equilibrium: zero growth means levels converge to constants - standard steady-state analysis applies. In the High equilibrium: positive growth means levels grow forever, so that we must analyze ratios/normalized variables for stationarity.

Important Observations

Growth Rate Effects:

1. **Positive growth** ($g^* > 0$): Makes the system more stable by reducing eigenvalue magnitudes
2. **Negative growth** ($g^* < 0$): Makes the system less stable by increasing eigenvalue magnitudes
3. **Critical condition:** System becomes unstable if $g^* < -\min(\Psi_l, 1-\delta)$

Policy Implications

1. **Technology policy:** Higher ξ (technology decay) reduces growth and can threaten stability
2. **Innovation policy:** Higher $p_1\varphi^0$ promotes growth and stability
3. **Monetary policy:** Must ensure $|\Psi_l| < 1 + g^*$, easier with positive growth
4. **Capital policy:** Depreciation rate δ must satisfy $(1-\delta) < 1 + g^*$.

Numerical Example with Corrected MPK

Consider the following parameter values:

Parameter	Value
α	0.33
η	0.5
ω	0.4
δ	0.1
s	0.2
ϕ_0	0.8
τ	0.1
p_1^L	0.3
p_1^H	0.8

With MPK $= \alpha(AL/K)^{1-\alpha} = 0.33(AL/K)^{0.67}$, the dynamics are:

Low Equilibrium:

- Growth rate: $g^L = 0.3 \times 0.8 \times 0.2 - 0.1 = -0.052$ (negative).
- MPK declining rapidly as K grows without productivity gains.
- Eigenvalues all within unit circle - stable trap.
- With these parameters, p_1 must exceed 0,625 to achieve positive growth.

High Equilibrium:

- Growth rate: $g^H = 0.8 \times 0.8 \times 0.2 - 0.1 = 0.028$ (positive).
- MPK sustained by productivity growth.
- Stability requires: $[0.5 + 0.67] \times 0.8 \times 0.8 \times 0.2 < 0.1 + 0.1 \times 0.67$; $0.15 < 0.167$ - marginally stable.

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