Kalman Filter Learning Versus Bounded Rationality in a Heterogeneous Agent NK Model*

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Abstract

We construct and estimate a New Keynesian (NK) heterogeneous agents model with Rational Expectations (RE) and bounded rationality (BR) agents in fixed proportions. BR agents are anticipated utility learners and use simple heuristic rules to forecast aggregate variables exogenous to their micro-environment. We study two information assumptions for the RE agents: the standard perfect information (PI) case and the imperfect information (II -involving Kalman-Filter learning) case. We show that II generates endogenous persistence of decisions in response to unobserved exogenous shocks. We find (a) in a likelihood race the RE model with II outperforms the pure (homogeneous) BR model which in turn outperforms the pure RE with PI; (b) the composite RE(II)-BR with estimated proportions of RE and BR agents, outperforms its RE(PI)-BR counterpart in terms of both a likelihood race and the fit of the model second moments with those of the data. Our findings highlight the importance of information assumptions in the empirical comparison of RE and BR NK models.

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1 Introduction

Around the same time that the New Keynesian (NK) model was emerging, macro-economists began to explore formal models of bounded rationality (BR). An important early paper is Evans and Ramey (1992), which embeds costly expectation technologies into a simple macroeconomic model. Following this seminal paper a large literature has emerged that offers a learning alternative to rational expectations (RE) and, in empirical applications, shows that this genre of models can provide a better fit to macroeconomic data.

The pioneering work of Evans and Honkapohja (2001) assumes that agents know the minimum state variable (MSV) form of the equilibrium (equivalent to the saddle-path under RE) and use direct observations or VAR estimates of these states to update their estimates each period using a discounted least-squares estimator. Then a statistical learning equilibrium is one where this perceived law of motion and the actual one coincide. Although this form of BR responds to what many regard as an extreme assumption of model-consistent expectations, the departure is often seen as only a modest one in that agents still need to know the MSV form of the equilibrium. The defining characteristic of what we refer to as behavioural macro-models is to limit the cognitive skills of at least a group of agents in the model and this is achieved by introducing simple 'heuristic' learning rules which can be thought of as parsimonious forms of forecasting rules (as in Branch and Evans, 2011). This is the approach to BR adopted in our paper.[1]

We construct a heterogeneous RE-BR model with exogenous proportions or RE and BR households and firms. Where we depart from the literature is in modelling the RE agents and making a comparison between RE and BR models. In particular for the former we relax a crucial information assumption that agents have *perfect information* (PI) of the state vector. We construct and estimate a NK heterogeneous expectations behavioural model with fixed proportions of BR and rational agents. The novelty of the paper lies in our comparisons of different composites including the pure RE and BR cases; in particular we impose what we term *informational consistency* where RE and BR agents in the model share the same *imperfect information* (II) set as the econometrician estimating the model. Under II the RE solution involves learning via a Kalman Filter alluded to in the paper's title. In the absence of PI as an endowment agent observe aggregate data (output and inflation with a lag, and the current nominal interest rate) at time t and form expectations of unobserved current realizations of shocks as a weighted average of their t-1 estimate updated by the Kalman gain of their prediction error.

Apart from information assumptions, an important feature of our model is the choice of learning behaviour for given proportions of RE and BR agents. We follow the *anticipated utility* (henceforth AU) approach of Hommes *et al.* (2019) (which in turn follows the seminal work of Kreps, 1998). Under AU agents are individually rational forming beliefs over the future infinite time horizon of aggregate states and prices which are exogenous to their decisions. AU, also known as the "infinite time-horizon" framework, is closely related to the "internal rationality" (IR) approach of Adam and Marcet (2011) and Gerko (2018). Under both IR and AU agents

[1] The model is in the class of restricted perception equilibria where agents do not know the form of the RE solution and there no E-learning takes place. Nor is it a Stochastic Consistent Expectations Equilibria which seek a fixed point to equate the perceived and actual laws of motion as in Hommes and Zhu (2014) and Hommes *et al.* (2023). Continuing with some other possible BR modelling approaches: k-level thinking in Garcia-Schmidt and Woodford (2019) and Farhi and Werning (2019) proposes an iterated solution of temporary equilibria each based on the AU approach. Another equilibrium concept is that of inattention and cognitive discounting proposed by Gabaix (2020). This is closely related to finite-time horizon optimization in Woodford (2019)

maximize utility, given their constraints and a set of probability beliefs about payoff-relevant variables that are *external*. Then with IR, beliefs are model-consistent and take the form of a well-defined probability measure over a stochastic process (the 'fully Bayesian' plan), whereas AU beliefs treat parameters as random variables when they learn but as constants when they formulate decision rules. See Eusepi and Preston (2011) for an RBC BR model with AU, Preston (2005) and Woodford (2013) who adopt a similar NK framework as in this paper.[2]

Our choice of the AU approach for the behavioural model is just one of many we could have taken to establish the importance of information assumptions in making a comparison with a rational expectations model. In the book De Grauwe (2019) and a series of important papers including De Grauwe (2011), De Grauwe (2012a), De Grauwe and Katwasser (2012) and De Grauwe and Gerba (2018), the assumption of Euler Learning (EL) is made Under EL agents are also individually forecasting their own one-period ahead decisions. Then in a representative agent model these decisions seen by agents to be those in the aggregate economy and therefore exogenous; see Branch and McGough (2018) for a recent discussion of the EL vs AU approaches.[3]

A paper particularly close to ours is Massaro (2013) which presents a calibrated composite heterogeneous expectations model of RE and AU agents with fixed proportions. He emphasizes the need for policymakers to design robust rules that stabilize the economy across different composite models; but here we focus on the informational assumptions made by the two sets of agents and we seek empirical support for the modelling choices. We simultaneously relax the two extreme RE and PI assumptions and examine the empirical evidence in the data for different forms of agent-level learning. To the best of our knowledge, our paper is the first contribution to the learning literature to estimate a version of DSGE heterogenous-agent learning model with II as an additional/alternative source of learning. The central research question of this paper is to study whether Kalman-filtering learning with RE can match bounded-rationality in matching persistence seen in the data without explicitly constructing the further persistence mechanisms such as habit in consumption and price-indexing in the model.

Our empirical findings are in line with the finding that BR and learning improves the model fit and the persistence of the model (see, e.g., Milani, 2007 and De Grauwe, 2012b among others). But RE under II remarkably outperforms all of the belief specifications under both homogeneous and heterogeneous expectations. Our empirical results also demonstrate that our basic NK models are able to generate strong persistence mechanisms via various forms of learning without relying on the backward-looking inertial components in the model (DSGE models with and without mechanical/ad-hoc persistence often struggle to reproduce the persistence in the data). This implies that both hybrid BR models and RE with II can serve as alternative approaches which can account for the persistence mechanisms seen in the data.

In summary, the main contributions of this paper are as follows: (a) we examine empirically the support for a composite RE-BR model of the AU type by Bayesian estimations with fixed proportions of RE and BR(AU) agents; (b) in our comparisons of different composites including the pure RE and BR cases, we impose informational consistency where RE and BR agents in the model share the same imperfect information as the econometrician estimating

[2]Cogley and Sargent (2008) compares the IR vs AU and find that AU can closely approximate the fully Bayesian optimization. There are other agent-level learning alternatives such as shadow value and finite-horizon learning. See Branch *et al.* (2013), Woodford (2018), Evans and McGough (2018) and Jump and Levine (2019) for reviews.

[3]AU also fits into the Agent-Based Modelling (ABM) framework: Sinitskaya and Tesfatsion (2015) introduce forward-looking optimizing agents into an ABM model. They use essentially the AU concept which they refer to as constructive rational decision-making.

the model. We focus on a comprehensive empirical assessment and comparison of alternative learning mechanisms. Furthermore, we carry out our estimations based on the sample period encompassing the Great Moderation and check the robustness of the result against an extended data set including the Great Recession.

The rest of the paper is structured as follows. Section 2 steps back to the non-linear foundations of the model that is ultimately studied in linearized form. Section 3 describes the specific market-consistent environment in which households and firms form their expectations. Section 4 examines the information assumptions that are made explicitly or implicitly in the RE and boundedly rational forms of the NK model. Section 5 presents and discusses some simulation properties of the RE-BR composite models with fixed proportions of RE and BR agents.

For the empirical analysis, Section 6 estimates the latter, alongside the pure BR and RE models by Bayesian methods, and conducts a likelihood race. This section estimates the behavioural model in which the adaptive expectations assumption used by BR agents follows the standard Brock-Hommes heuristic rules. It first assumes RE agents have PI regarding current state variables. Then it adds an additional learning mechanism assuming that RE agents do not observe all current state variables and only have an II set. Section 6.7 examines the ability of these estimated variants of the NK model to match the second moments in the data. Section 6.8 examines the impulse response functions of the estimated model and discusses endogenous persistence. Section 8 concludes the paper.[4]

2 The NK Model under RE and Bounded Rationality

Ultimately our analysis will be conducted in terms of linear RE and Behavioural mddels. But first we step back to the underlying non-linear model and introduce the distinction between internal decisions and aggregate macro-variables. We start with the non-linear RE model and proceed from pure RE to pure BR in stages. The complete model setup and its balanced growth steady state are summarized in Online Appendices A and B.

2.1 Households

Household j chooses savings and between work and labour supply. Let $C_t(j)$ be consumption and $H_t(j)$ be the proportion of this available for work or leisure spent at the former. The single-period utility we choose, compatible with a balanced growth steady state, is

$$U_t(j) = U(C_t(j), H_t(j)) = \log(C_t(j)) - \frac{H_t(j)^{1+\phi}}{1+\phi}$$

and the value function of the representative household at time t dependent on its assets B is

$$V_t(j) = V_t(B_{t-1}(j)) = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s U(C_{t+s}(j), H_{t+s}(j))\right] \tag{1}$$

The household's problem at time t is to choose paths for consumption $\{C_t(j)\}$, labour supply $\{H_t(j)\}$ and holdings of financial savings to maximize $V_t(j)$ given by (1) given its budget

[4] A separate Online Appendix contains details of the model solutions, robustness checks, identification results, the II solution procedure and a further robustness check that re-estimates the models using an extended data set.

constraint in period t

$$B_t(j) = R_t B_{t-1}(j) + W_t H_t(j) + \Gamma_t - C_t(j) - T_t - \frac{\varpi}{2}(B_{t-1}(j) - B)^2 \tag{2}$$

where $B_t(j)$ is the given net stock of real financial assets at the end of period t, W_t is the wage rate, T_t are lump-sum taxes, Γ_t are profits from wholesale and retail firms owned by households. In order to allow for a wealth distribution heterogenous agents introduced later and to achieve a stationary path for bond holdings we introduce a portfolio adjustment cost.[5] R_t is the real interest rate paid on assets held at the beginning of period t given by the Fischer equation

$$R_t = \frac{R_{n,t-1}}{\Pi_t} \tag{3}$$

where $R_{n,t}$ and Π_t are the nominal interest and inflation rates, respectively. W_t, $R_{n,t}$, Π_t and Γ_t are all exogenous to household j. As usual all real variables are expressed relative to the price of final output. The standard first order conditions are

$$\begin{aligned} \mathbb{E}_t\left[\Lambda_{t,t+1}(j)R_{t+1}\right] &= 1 + \varpi(B_t(j) - B) \\ \frac{U_{H,t}(j)}{U_{C,t}(j)} &= -W_t \end{aligned}$$

where $\Lambda_{t,t+1}(j) \equiv \beta\frac{U_{C,t+1}(j)}{U_{C,t}(j)}$ is the stochastic discount factor for household j, over the interval $[t, t+1]$. For our choice of utility function $U_{C,t} = \frac{1}{C_t}$ and $U_{H,t} = -H_t^{\phi}$ so these become

$$\beta\mathbb{E}_t\left[\frac{C_t(j)R_{t+1}}{C_{t+1}(j)}\right] = 1 + \varpi(B_t(j) - B) \tag{4}$$

$$C_t(j)H_t(j)^{\phi} = W_t \Rightarrow H_t(j) = \left(\frac{W_t}{C_t(j)}\right)^{\frac{1}{\phi}} \tag{5}$$

The first-order conditions up to now are suitable for the RE solution. We now express the solution in a form suitable for moving from a RE to a learning equilibrium. We consider the limit as $\varpi \to 0$. Solving (2) forward in time and imposing the transversality condition on debt we can write

$$B_{t-1}(j) = \mathrm{PV}_t(C_t(j)) - \mathrm{PV}_t(W_t H_t(j)) - \mathrm{PV}_t(\Gamma_t) + \mathrm{PV}_t(T_t) \tag{6}$$

where the present (expected) value of a series $X \equiv \{X_{t+i}\}_{i=0}^{\infty}$ at time t is defined by

$$\mathrm{PV}_t(X_t) \equiv \mathbb{E}_t \sum_{i=0}^{\infty} \frac{X_{t+i}}{R_{t,t+i}} = \frac{X_t}{R_t} + \frac{1}{R_t}\mathrm{PV}_t(X_{t+1}) \tag{7}$$

writing $R_{t,t+i} \equiv R_t R_{t+1} R_{t+2} \cdots R_{t+i}$ as the real interest rate over the interval $[t-1, t+i]$.

The forward-looking budget constraint (6) holds for the representative household. If we allow RE and BR agents to borrow from or lend to one another we must allow for $B_{t-1} \neq 0$.

[5]This as a modelling device similar to that used in open economies with home and foreign household as pioneered by Schmitt-Grohe and Uribe (2003). We examine the limit as ϖ becomes very small so our choice of real rather than nominal bond holding costs is immaterial. In fact, the wealth distribution effect does not significantly change the equilibrium.

Then in a symmetric equilibrium with $C_t(j) = C_t$ and $H_t(j) = H_t$, (6) and (5) become

$$\begin{aligned} B_{t-1} &= \mathrm{PV}_t(C_t) - \mathrm{PV}_t\left(\frac{W_t^{1+\frac{1}{\phi}}}{C_t^{\frac{1}{\phi}}}\right) - \mathrm{PV}_t(\Gamma_t) + \mathrm{PV}_t(T_t) \\ H_t &= \left(\frac{W_t}{C_t}\right)^{\frac{1}{\phi}} \end{aligned}$$

Solving (4) forward in time and using the law of iterated expectation we have for $i \geq 1$

$$\frac{1}{C_t} = \beta^i \mathbb{E}_t\left[\frac{R_{t+1,t+i}}{C_{t+i}}\right]\ ;\ i \geq 1 \tag{8}$$

We now express the solution to the household optimization problem for C_t and H_t that are functions of *point expectations* $\{\mathbb{E}_t W_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_t R_{t+1,t+i}\}_{i=1}^{\infty}$ and $\{\mathbb{E}_t \Gamma_{t+i}\}_{i=0}^{\infty}$ treated as exogenous processes given at time t.[6] With point expectations we use (8) to obtain the following optimal decision for C_{t+i} given point expectations $\mathbb{E}_t R_{t+1,t+i}$

$$\begin{aligned} C_{t+i} &= C_t \beta^i \mathbb{E}_t R_{t+1,t+i}\ ;\ i \geq 1 & (9) \\ \mathbb{E}_t(W_{t+i} H_{t+i}) &= \frac{(\mathbb{E}_t W_{t+i})^{1+\frac{1}{\phi}}}{C_{t+i}^{\frac{1}{\phi}}} & (10) \end{aligned}$$

Substituting (9) and (10) into the forward-looking household budget constraint, using $\sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}$ and $\mathbb{E}_t R_{t,t+i} = R_t \mathbb{E}_t R_{t+1,t+i}$ for $i \geq 1$, we arrive at

$$\frac{C_t - R_t B_{t-1}}{(1-\beta)} = \frac{1}{C_t^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i}\left(\frac{\mathbb{E}_t W_{t+i}}{\mathbb{E}_t R_{t+1,t+i}}\right)^{1+\frac{1}{\phi}}\right) + \Gamma_t - T_t + \sum_{i=1}^{\infty} \frac{\mathbb{E}_t(\Gamma_{t+i} - T_{t+i}))}{\mathbb{E}_t R_{t+1,t+i}}$$

which can be written in recursive form as

$$\begin{aligned} \frac{C_t - R_t B_{t-1}}{(1-\beta)} &= \frac{1}{C_t^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \Omega_{1,t}\right) + \Gamma_t - T_t + \Omega_{2,t} & (11) \\ \Omega_{1,t} &\equiv \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i}\left(\frac{\mathbb{E}_t W_{t+i}}{\mathbb{E}_t R_{t+1,t+i}}\right)^{1+\frac{1}{\phi}} = (\beta^{\frac{1}{\phi}})^{-1}\left(\frac{\mathbb{E}_t W_{t+1}}{\mathbb{E}_t R_{t+1,t+1}}\right)^{1+\frac{1}{\phi}} + \frac{\Omega_{1,t+1}}{\beta^{\frac{1}{\phi}} \mathbb{E}_t R_{t+1}} \\ \Omega_{2,t} &\equiv \sum_{i=1}^{\infty} \frac{\mathbb{E}_t(\Gamma_{t+i} - T_{t+i})}{\mathbb{E}_t R_{t+1,t+i}} = \frac{\mathbb{E}_t(\Gamma_{t+1} - T_{t+1})}{\mathbb{E}_t R_{t+1,t+1}} + \frac{\Omega_{2,t+1}}{\mathbb{E}_t R_{t+1}} \end{aligned}$$

Consumption is then given by (11) assuming point expectations or by the symmetric form of the Euler equation (4) under full rationality (i.e. households know symmetric nature of equilibrium with $C_t(j) = C_t$). C_t is a function of *rational point expectations* $\{\mathbb{E}_t W_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_t R_{t,t+i}\}_{i=i}^{\infty}$ and $\{\mathbb{E}_t \Gamma_{t+i}\}_{i=1}^{\infty}$ which can be treated as exogenous processes given at time t or as rational model-consistent expectations. Since $E_t f(X_t) \approx f(E_t(X_t))$; $E_t f(X_t Y_t)) \approx f(E_t(X_t) E_t(Y_t))$ up to a first-order Taylor-series expansion, assuming point expectations is equivalent to using a linear approximation (given below) as is usually done in the literature.

[6] Point expectations are implied in a full linearization of the model. However in our set-up non-linearity in decisions given point expectations is retained which in a second-order perturbation solution allows the computation of the household expected welfare and welfare-optimized Taylor-type rules. See Deak *et al.* (2023).

2.2 Firms, Government Expenditures and Monetary Policy

This section sets out the wholesalers and the retail sector which optimizes using Calvo-pricing contracts. We close the non-linear setup with resource and balanced government budget constraints, a monetary policy rule and by specifying the structural shocks in the economy. Wholesale firms employ a Cobb-Douglas production function to produce a homogeneous output

$$Y_t^W = F(A_t, H_t) = A_t H_t^\alpha$$

where A_t is total factor productivity. Profit-maximizing demand for labour results in the first-order condition

$$W_t = \frac{P_t^W}{P_t} F_{H,t} = \alpha \frac{P_t^W}{P_t} \frac{Y_t^W}{H_t} \tag{12}$$

The retail sector costlessly converts a homogeneous wholesale good into a basket of differentiated goods for aggregate consumption

$$C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta} dm \right)^{\zeta/(\zeta-1)} \tag{13}$$

where ζ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (13) given total expenditure $\int_0^1 P_t(m)C_t(m)dm$. Assuming government services are similarly differentiated, this results in a set of demand equations for each differentiated good m with price $P_t(m)$ of the form

$$Y_t(m) = \left(\frac{P_t(m)}{P_t} \right)^{-\zeta} Y_t \tag{14}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm \right]^{\frac{1}{1-\zeta}}$. P_t is the aggregate price index. C_t and P_t are Dixit-Stigliz aggregates – see Dixit and Stiglitz (1977).

Following Calvo (1983), we assume that there is a probability of $1 - \xi$ at each period that the price of each retail good m is set optimally to $P_t^O(m)$. If the price is not re-optimized, then it is held fixed. For each retail producer m, given its real marginal cost $MC_t = \frac{P_t^W}{P_t}$, the objective is at time t to choose $\{P_t^O(m)\}$ to maximize discounted real profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^O(m) - P_{t+k} MC_{t+k} \right]$$

subject to (14), where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}}{U_{C,t}}$ is the stochastic discount factor over the interval $[t, t+k]$. The solution to this is standard and given by

$$\frac{P_t^O(m)}{P_t} = \frac{\zeta}{\zeta - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} (\Pi_{t,t+k})^\zeta Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} (\Pi_{t,t+k})^\zeta (\Pi_{t,t+k})^{-1} Y_{t+k}}$$

Denoting the numerator and denominator by J_t and JJ_t, respectively, and introducing a mark-up shock MS_t to MC_t, from Online Appendix D we write in recursive form

$$\frac{P_t^O(m)}{P_t} = \frac{J_t}{JJ_t} \tag{15}$$

$$J_t - \xi \mathbb{E}_t[\Lambda_{t,t+1}\Pi_{t+1}^{\zeta} J_{t+1}] \quad = \quad \frac{1}{1-\frac{1}{\zeta}} Y_t MC_t MS_t \tag{16}$$

$$JJ_t - \xi \mathbb{E}_t[\Lambda_{t,t+1}\Pi_{t+1}^{\zeta-1} JJ_{t+1}] \quad = \quad Y_t \tag{17}$$

Using the fact that all resetting firms will choose the same price, by the Law of Large Numbers we can find the evolution of inflation given by

$$1 = \xi \left(\Pi_{t-1,t}\right)^{\zeta-1} + (1-\xi)\left(\frac{P_t^O}{P_t}\right)^{1-\zeta} \tag{18}$$

Price dispersion lowers aggregate output as follows. Market clearing in the labour market gives

$$H_t = \sum_{m=1}^{n} H_t(m) = \sum_{m=1}^{n} \left(\frac{Y_t(m)}{A_t}\right)^{\frac{1}{\alpha}} = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{\alpha}} \sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t}\right)^{-\frac{\zeta}{\alpha}}$$

using (14). Hence equilibrium for good m gives $Y_t = \frac{Y_t^W}{\Delta_t^\alpha}$, where price dispersion is defined by

$$\Delta_t \equiv \left(\sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t}\right)^{-\frac{\zeta}{\alpha}}\right)$$

Assuming that the number of firms is large from Online Appendix E we obtain the following dynamic relationship

$$\Delta_t = \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{-\frac{\zeta}{\alpha}}$$

To close the model we first require total profits from retail and wholesale firms, Γ_t, is remitted to households. This is given in real terms by

$$\Gamma_t = \underbrace{Y_t - \frac{P_t^W}{P_t} Y_t^W}_{\text{retail}} + \underbrace{\frac{P_t^W}{P_t} Y_t^W - W_t H_t}_{\text{Wholesale}} = Y_t - \alpha \frac{P_t^W}{P_t} Y_t^W$$

using the first-order condition (12). Then to complete closure we have resource and balanced government budget constraints

$$Y_t = C_t + G_t = C_t + T_t$$

where G_t is an exogenous demand process, and a monetary policy rule for the nominal interest rate given by the following implementable Taylor-type rule

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\pi \log\left(\frac{\Pi_t}{\Pi_{targ,t}}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right) + \theta_{dy} \log\left(\frac{Y_t}{Y_{t-1}}\right)\right) + \epsilon_{MP,t} \tag{19}$$

and $\epsilon_{MP,t}$ is an i.i.d. shock to monetary policy. $\Pi_{targ,t}$ is a time-varying inflation target and together with A_t, G_t, and MS_t follows an AR(1) process. This completes the model.

2.3 Recovering the NK Workhorse Model

We now show that the linearized form of the non-linear model about the steady state reduce to the standard workhorse model in where rational expectations $\mathbb{E}_t y_{t+1}$ and $\mathbb{E}_t \pi_{t+1}$ or non-RE $\mathbb{E}_t^* y_{t+1}$ and $\mathbb{E}_t^* \pi_{t+1}$ can be treated as expectations by individual households and firms respectively of *aggregate* future output and inflation, respectively. We consider the linearized form of the above set-up about a zero inflation and growth deterministic steady state. We also ignore lending or borrowing between RE and BR agents. With RE the household j's first-order conditions take one of two forms. First, linearizing (11) we have

$$\begin{aligned} \alpha_1 c_t(j) &= \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t} \\ \omega_{1,t} &= \alpha_5 \mathbb{E}_t w_{t+1} - \alpha_6 \mathbb{E}_t r_{t+1} + \beta \mathbb{E}_t \omega_{1,t+1} \\ \omega_{2,t} &= (1-\beta)(\gamma_t - g_t) - r_t + \beta \mathbb{E}_t \omega_{2,t+1} \\ \gamma_t &= \frac{1}{\gamma_y} y_t - \frac{\alpha}{\gamma_y}(w_t + h_t) \end{aligned} \tag{20}$$

from (11) where lower case variables $x_t \equiv \log(X_t/X)$ where X is the steady state of X_t; $c_y \equiv \frac{C}{Y}$, $\gamma_y \equiv \frac{\Gamma}{Y}$, $g_y \equiv \frac{G}{Y}$ and γ_t is *exogenous* profit per household (a function of aggregate consumption and hours). Positive coefficients are given by $\alpha_1 \equiv 1 + \frac{\alpha}{\phi c_y}$, $\alpha_2 \equiv (1-\beta)(1+\frac{1}{\phi})\frac{\alpha}{c_y}$, $\alpha_3 \equiv \frac{\gamma_y}{c_y}$, $\alpha_4 \equiv \frac{\beta\alpha}{c_y}$, $\alpha_5 \equiv (1-\beta)(1+\frac{1}{\phi})$ and $\alpha_6 \equiv (1+\frac{1}{\phi})$. Alternatively from the Euler equation (4)

$$c_t = \mathbb{E}_t c_{t+1} - \mathbb{E}_t r_{t+1} \tag{21}$$

in a symmetric equilibrium. Under RE (20) or (21) lead to the same equilibrium, but under BR this is no longer the case.

Linearizing the household supply of hours decision, the resource constraint and the Fisher equation we have

$$y_t = (1-g_y)c_t + g_y g_t \tag{22}$$

$$r_t = r_{n,t-1} - \pi_t \tag{23}$$

$$h_t = \frac{1}{\phi}(w_t - c_t) \tag{24}$$

which completes the decisions of the household. Substituting out for c_t from (22)

$$y_t = \mathbb{E}_t y_{t+1} - (1-g_y)\mathbb{E}_t r_{t+1} + g_y(\mathbb{E}_t g_{t+1} - g_t) \tag{25}$$

Turning to the supply side, for the wholesale sector

$$y_t = a_t + \alpha h_t \tag{26}$$

$$mc_t = w_t - y_t + h_t \tag{27}$$

For retail firm m, linearizing the pricing dynamics (15)–(17) about a zero net equation steady state and solving forwards we have

$$\begin{aligned} p_t^o(m) - p_t &= \beta\xi \mathbb{E}_t[\pi_{t+1} + p_{t+1}^o(m) - p_{t+1}] + (1-\beta\xi)(mc_t + ms_t) \\ &= \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi)^i [\beta\xi\pi_{t+i+1} + (1-\beta\xi)(mc_{t+i} + ms_{t+i})] \end{aligned} \tag{28}$$

Then in a symmetric equilibrium we have

$$\pi_t = \frac{(1-\xi)}{\xi}\left(\mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi)^i [\beta\xi\pi_{t+i+1} + (1-\beta\xi)(mc_{t+i} + ms_{t+i})]\right) \tag{29}$$

where $\mathbb{E}_t[\pi_{t+i+1}]$ and $\mathbb{E}_t[mc_{t+i}+ms_{t+i}]$ are expectations of aggregate inflation and real marginal costs, both variables exogenous to individual price-setters. However, *if price-setters know they are identical* they know the aggregate price level over non-optimizing and optimizing firms

$$p_t(m) = \xi p_{t-1} + (1-\xi)p_t^o(m) \tag{30}$$

to obtain in a symmetric equilibrium

$$p_t^o(m) - p_t = p_t^o - p_t = \frac{\xi}{(1-\xi)}(p_t - p_{t-1}) = \frac{\xi}{(1-\xi)}\pi_t$$

Then substituting back into (28) we arrive at

$$\pi_t = \frac{(1-\xi)(1-\beta\xi)}{\xi}\mathbb{E}_t\sum_{i=0}^{\infty}\beta^i(mc_{t+i}+ms_{t+i}) \tag{31}$$

which omits learning about aggregate inflation. Under RE, (29) and (31) are equivalent.

(31) is equivalent to

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \lambda(mc_t + ms_t) \tag{32}$$

where $\lambda = \frac{(1-\xi)(1-\beta\xi)}{\xi}$ which is the familiar linearized Phillips curve expressed in terms of the real marginal cost mc_t and the mark-up shock ms_t. Substituting for the former from (26) and (27) we arrive at

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \lambda\left(\frac{1+\phi}{\alpha}(y_t - a_t) - \frac{g_y}{1-g_y}g_t + ms_t\right) \tag{33}$$

where we note that $y_t - a_t$ is the output gap. Equations (25), (33) and the Taylor rule (19) constitute the 3-equation NK RE model in output, inflation and the nominal interest rate given exogenous shock processes for g_t, ms_t and the monetary shock. A simpler form omits government spending g_t so $g_y = 0$ and replaces the aggregate demand shock in (33) with an exogenous process that can be thought of as a risk premium shock to the Fischer equation (23).

The form of Phillips curve (31) is often used in the behavioural NK literature (see, for example, De Grauwe, 2012b), but as we have shown, this assumes that firms know they are identical. In our BR model we use (20) and (29) which do not make this assumption.

3 AU Learning and Market-Consistent Information

With anticipated utility (AU) learning, our learning model is one where agents make fully optimal decisions given their individual specification of beliefs, but have no macroeconomic model to form expectations of aggregate variables. We draw a clear distinction between aggregate and internal quantities so that identical agents in our model are not aware of this equilibrium property (nor any others).

3.1 Forecasting Rules

To close the model, we need to specify the manner in which households and firms form their expectations. To do so, we assume that variables which are local to the agents, in a geographical sense, are observable within the period, whereas variables that are strictly macroeconomic are only observable with a lag. This categorization regarding information about the current state of the economy follows Nimark (2014). He distinguishes between the local information that agents acquire directly through their interactions in markets and statistics that are collected and summarized, usually by governments, and made available to the wider public.[7] The policy rate is announced by the central bank, so it is observed without a lag and it is common knowledge. Given this, we assume an adaptive expectations forecasting rule given below by (35) and (36) about variables external to agents' decisions. Let $x_t = r_t, r_{n,t}, \pi_t, w_t, \gamma_t, g_t$, then household expectations are given by

$$\mathbb{E}_t^* x_{t+i} = \mathbb{E}_t^* x_{t+1}\,; \quad i \geq 1 \tag{34}$$

Expressing $\mathbb{E}_t \omega_{1,t+1}$ and $\mathbb{E}_t \omega_{2,t+1}$ in (20) as forward-looking summations and using (34), we arrive at the *individual learning consumption equation*

$$\begin{aligned}
\alpha_1 c_t &= \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t} \\
\omega_{1,t} &= \frac{1}{1-\beta}\left[\alpha_5 \mathbb{E}_t^* w_{t+1} - \alpha_6(\beta \mathbb{E}_t^* r_{n,t+1} - \mathbb{E}_{h,t}^* \pi_{t+1})\right] - \alpha_6 r_{n,t} \\
\omega_{2,t} &= (1-\beta)(\gamma_t - g_t) - r_t + \frac{\beta}{1-\beta}((1-\beta)(\mathbb{E}_t^* \gamma_{t+1} - \mathbb{E}_t^* g_{t+1}) - \mathbb{E}_t^* r_{t+1})
\end{aligned}$$

which is now expressed in terms of one-step ahead forecasts by the standard adaptive expectations rule:[8]

$$\mathbb{E}_t^* x_{t+1} = \mathbb{E}_{t-1}^* x_t + \lambda_x(x_{t-j} - \mathbb{E}_{t-1}^* x_t)\,; \quad x = w,\, r_n,\, \pi,\, \gamma - g\,; \quad j = 0, 1 \tag{35}$$

Households make inter-temporal decisions for their consumption and hours supplied given adaptive expectations of the wage rate, the nominal interest rate, inflation and profits. These macro-variables may in principle be observed with or without a one-period lag ($j = 1, 0$), but as stated earlier we assume $j = 0$ for market-specific variables w_t, $\gamma_t - g_t$, and $j = 1$ for aggregate inflation π_t. However we assume the current nominal interest rate, $r_{n,t}$, is announced and therefore also observed without a lag.

We distinguish household and firm expectations $\mathbb{E}_{h,t}^* \pi_{t+1}$, $\mathbb{E}_{f,t}^* \pi_{t+1}$. Then for retail firm m

$$\begin{aligned}
\mathbb{E}_t^* \pi_{t+i+1} &= \mathbb{E}_t^* \pi_{t+1}\,; \quad i \geq 0 \\
\mathbb{E}_t^*(mc_{t+i} + ms_{t+i}) &= \mathbb{E}_t^*(mc_{t+1} + ms_{t+1})\,; \quad i \geq 1 \\
p_t^o(m) - p_t &= \frac{\beta\xi}{1-\beta}\mathbb{E}_{f,t}^* \pi_{t+1} + (1-\beta\xi)(mc_t + ms_t) + \frac{\beta}{1-\beta}\mathbb{E}_t^*(mc_{t+1} + ms_{t+1})
\end{aligned}$$

where again one-step ahead forecasts are given by the adaptive expectations rule:

$$\mathbb{E}_t^* x_{t+1} = \mathbb{E}_{t-1}^* x_t + \lambda_x(x_{t-j} - \mathbb{E}_{t-1}^* x_t)\,; \quad x = \pi,\, (mc + ms); \quad j = 0, 1 \tag{36}$$

Retail firms make inter-temporal decisions for their price and output given adaptive expectations of the aggregate inflation rate and their post-shock real marginal shock wage rate. As before these variables may be observed with or without a one-period lag ($j = 1, 0$), but for aggregate inflation we assume $j = 1$ as for households, but $j = 0$ for the market-specific variable mc_t. Note that we can in principle distinguish between households' and firms' expectations of inflation.

[7] His paper actually focuses on a third category, information provided by the news media, and allows for II in the form of noisy signals, issues which go beyond the scope of our paper.

[8] We construct a local variable $\gamma - g_t$ assumed to be observed at the local level. An alternative set-up would be to assume $g_t = 0$.

3.2 Alternative Beliefs with Credibility

An extension to our behavioural model in this section is to generalize the adaptive expectations forecasting rule to allow for partial credibility. The aim is to introduce more rationality in beliefs with various degrees of credibility which should help improve the empirical performance of the BR models. To set out a simplified version of the model, instead of assuming one-period ahead adaptive expectations for $R_{n,t}$, we now allow agents to know the rule

$$
\begin{aligned}
R_{n,t} &= R_{n,t-1}^{\rho_r} X_t^{1-\rho_r} && (37)\\
X_t &\equiv \Pi_t^{\theta_\pi} Y_t^{\theta_y} (Y_t/Y_{t-1})^{\theta_{dy}} && (38)
\end{aligned}
$$

but still lack knowledge of the model that generates Π_t and Y_t.[9] Thus, $\mathbb{E}_t^*\Pi_{t+i} = \mathbb{E}_t^*\Pi_{t+1}$ and $\mathbb{E}_t^* Y_{t+i} = \mathbb{E}_t^* Y_{t+1}$ as before and therefore $\mathbb{E}_t^* X_{t+i} = \mathbb{E}_t^* X_{t+1}$, but now agents' beliefs *perceive nominal interest rate persistence* with one-period ahead forecasts of $R_{n,t}$ with

$$
\mathbb{E}_t^* R_{n,t+1} = R_{n,t}^{\rho_r} \mathbb{E}_t^* X_{t+1}^{1-\rho_r} \tag{39}
$$

which with complete credibility now replaces the adaptive expectations rule

$$
\mathbb{E}_t^* R_{n,t+j} = \mathbb{E}_t^* R_{n,t+1} = \mathbb{E}_{t-1}^* R_{n,t} + \lambda_1 (R_{n,t} - \mathbb{E}_{t-1}^* R_{n,t}); \quad j \geq 1 \tag{40}
$$

Partial credibility can be modelled by the following rule which is set up relative to a given steady state

$$
\begin{aligned}
\mathbb{E}_t^* R_{n,t+j}/R_n &= \omega \mathbb{E}_t^* R_{n,t+1}/R_n = (R_{n,t}/R_n)^{\rho_r} \mathbb{E}_t^* (X_{t+1}/X)^{1-\rho_r} \\
&+ (1-\omega)(\mathbb{E}_{t-1}^* R_{n,t} + \lambda_1 \left(R_{n,t} - \mathbb{E}_{t-1}^* R_{n,t} \right)) / R_n \,; \quad j \geq 1 && (41)
\end{aligned}
$$

where $\omega_x = 0$ reduces to the previous model, $\omega_x = 1$ assumes full credibility and $\omega_x \in [0, 1]$, where $x = h, f$, measures various degrees of credibility.

4 Rational Expectations: Perfect vs Imperfect Information

We now examine the information assumptions that are made explicitly or implicitly in the RE and boundedly rational forms of the NK model. The linearized form of the NK model has the following a state-space form that applies to both the PI and II cases:[10]

$$
\begin{aligned}
\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathbb{E}_t \mathsf{x}_{t+1} \end{bmatrix} &= G \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + H \begin{bmatrix} \mathbb{E}_t \mathsf{z}_t \\ \mathbb{E}_t \mathsf{x}_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \epsilon_{t+1} && (42)\\
\mathsf{m}_t^A &= \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t \mathsf{z}_t \\ \mathbb{E}_t \mathsf{x}_t \end{bmatrix} && (43)
\end{aligned}
$$

where z_t is a $(n-m) \times 1$ vector of predetermined variables at time t with z_0 given, x_t **is a** $m \times 1$ **vector of non-predetermined variables at time** t and m_t^A is a vector of observable macro-economic variables of the agents, which when we come to estimation will be the data used by the econometrician. All variables are expressed as proportional deviations about a

[9] We can conceive this as a *credibility assumption* where the policymaker announces and commits to the rule.

[10] Levine *et al.* (2023) derive this result that applies to a general linear RE model such as the linearized form of the NK model of Section 2.

steady state. G**,** H**,** B **and** $M_i, i = 1, 4$ **are fixed matrices**, ϵ_t is a vector of random Gaussian zero-mean shocks. RE under PI are formed assuming a full information set $\{z_s, x_s, \epsilon_s\}$, $s \leq t$ so that we can put $H = M_3 = M_4 = 0$ and $[M_1\ M_2]$ is the identity matrix. Note that the expressions involving $\mathbb{E}_t \mathsf{z}_t, \mathbb{E}_t \mathsf{x}_t$ arise from writing the original model in Blanchard-Kahn form (42).

If the number of eigenvalues outside the unit circle is equal to the number of non-predetermined variables, the system has a unique equilibrium which is also stable with saddle-path $x_t = -N z_t$ where N will depend on the instrument rules incorporated into the set-up (42) (see Blanchard and Kahn, 1980; Currie and Levine, 1993). Instability (indeterminacy) occurs when the number of eigenvalues of $G + H$ outside the unit circle is larger (smaller) than the number of non-predetermined variables. We proceed on the assumption that the determinacy-stability condition holds.

For ease of notation we assume that if any variables are observed with measurement error, then these variables are included in the state space, and the measurement errors are then part of the vector ϵ_t. Given the fact that expectations of forward-looking variables depend on the information set, it is hardly surprising that the absence of PI will impact on the path of the system.

4.1 The RE Solution Under Perfect and Imperfect Information (PI and II)

A full derivation of the II solution for the general linear setup above is provided in Pearlman *et al.* (1986). We now provide an outline solution starting with the PI solution. For this case we assume (without seeking to justify this assumption) that the single agent directly observes all elements of Y_t, hence of (z_t, x_t) as an endowment. Hence $z_{t,t} = z_t,\ x_{t,t} = x_t$, and using standard solution methods, there is a saddle path satisfying

$$x_t + N z_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix} (G + H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{44}$$

where Λ^U is a matrix with unstable eigenvalues. The saddlepath matrix N can be calculated by standard techniques. If the number of unstable eigenvalues of $(G + H)$ is the same as the dimension of x_t, then the system will be determinate.

Given this determinacy condition, after substituting for x_t, a unique saddle-path stable RE solution exists for the states under PI of the following form

$$z_t = A z_{t-1} + B \varepsilon_t \tag{45}$$

where

$$A \equiv G_{11} + H_{11} - (G_{12} + H_{12}) N \tag{46}$$

Under II, the transformation of the non-linear model into the form (42) and (43) allows us to apply the solution techniques originally derived in PCL. We briefly outline this solution method below.

We first define matrices G and H in (42) conformably with z_t and x_t, and define two more structural matrices F, J

$$G \equiv \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad H \equiv \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{47}$$

$$F \equiv G_{11} - G_{12}G_{22}^{-1}G_{21} \qquad J \equiv M_1 - M_2G_{22}^{-1}G_{21} \tag{48}$$

where F and J capture intrinsic dynamics in the system, that are invariant to expectations formation. Both PCL and BGW show that the filtering problem is unaffected by these additional terms.[11]

Following PCL, we apply the Kalman filter updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + K \left[m_t^A - \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} \right]$$

The single agent's best estimate of (z_t, x_t) based on current information is a weighted average of their best estimate using last period's information and the new information m_t^A. Thus the best estimator of (z_t, x_t) at time $t-1$ is updated by the "Kalman gain" K of the error in the predicted value of the measurement. PCL show that K is solved endogenously as $K = \begin{bmatrix} P^AJ' \\ -NP^AJ' \end{bmatrix} [(M_1 - M_2N)P^AJ']^{-1}$, where P^A is defined below in (55), but this version of the Kalman gain is not directly incorporated into the solution for (z_t, x_t).

The unique saddle-path stable solution under II, as derived by Pearlman *et al.* (1986) for the pre-determined and non-predetermined variables z_t and x_t, can then be described by processes for the predictions $z_{t,t-1}$ and for the prediction errors $\tilde{z}_t \equiv z_t - z_{t,t-1}$:

$$\begin{aligned} \text{Predictions}: \quad & z_{t+1,t} = A\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) && (49) \\ \text{Prediction Errors}: \quad & \tilde{z}_t = Q^A\tilde{z}_{t-1} + B\varepsilon_t && (50) \\ \text{Non-predetermined}: \quad & x_t = -N\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) - G_{22}^{-1}G_{21}\left(I - \mathcal{K}J\right)\tilde{z}_t && (51) \\ \text{Measurement Equation}: \quad & m_t^A = E\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) && (52) \end{aligned}$$

where

$$\mathcal{K} = P^AJ'\left(JP^AJ'\right)^{-1}; \quad Q^A = F\left[I - \mathcal{K}J\right] \tag{53}$$

F and J are as defined above in (48), $\mathcal{K}$ is an alternative Kalman gain matrix after stripping out the predictable variation in the state variables z_{t+1} arising from dependence on x_t. The matrix A, defined in (46), is the autoregressive matrix of the states z_t in the solution under PI. We have introduced another non-structural matrix E defined by

$$E \equiv M_1 + M_3 - (M_2 + M_4)N \tag{54}$$

which captures the impact of predictions and prediction errors for z_t on observable variables. B captures the direct (but unobservable) impact of the structural shocks ε_t and $P^A = \mathbb{E}[\tilde{z}_t\tilde{z}_t']$ is the solution of a Riccati equation given by

$$P^A = Q^AP^AQ^{A'} + BB' \tag{55}$$

To ensure stability of the solution P^A, we also need to satisfy the convergence condition, that Q^A has all eigenvalues in the unit circle. Since the matrix Q^A is also the autoregressive matrix of the prediction errors $\tilde{z}_t$ in (50), this is equivalent to requiring that prediction errors

[11] By substituting from the second block of equations in (42), we can write $z_t = Fz_{t-1} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varepsilon_{t+1}$ plus additional terms involving expectations formed at time t; and $m_t^A = Jz_t +$ additional terms likewise. Since all expectational terms are known at time t, they do not affect the solution to the filtering problem.

are stable. Since there is a unique solution of the Riccati equation under mild conditions that satisfies this condition, it follows that the solution (49)–(52) is also unique thereby extending this property of the PI BK solution to the II case.

We can thus see that the solution procedure above is a generalization of the BK solution for PI and that the determinacy of the system is independent of the information set.

We finally note that the II solution can be transformed into the PI solution when the agent's information set is (z_t, x_t). Choose just a subset of the information, $m_t = Jz_t$, such that JB is invertible. We then deduce from (55) that $P^A = BB'$ and hence $\tilde{z}_t = B\varepsilon_t$. Substituting into (49) yields $z_{t+1,t} = Az_{t,t-1} + AB\varepsilon_t = A(z_{t,t-1} + \tilde{z}_t) = Az_t$. Adding this to $\tilde{z}_{t+1} = B\varepsilon_{t+1}$ yields $z_{t+1} = Az_t + B\varepsilon_{t+1}$, the PI solution.

4.2 Misperceptions about Shocks under II

Before turning to the heterogeneous composite expectations RE-BR model we first examine the pure RE model under the two information assumptions PI and II. The model has 4 exogenous AR(1) shocks for technology, the price mark-up, government spending and the inflation target. We focus on the first three of these and in addition the i.i.d monetary shock. There is in addition i.i.d measurements errors for output and inflation and an i.i.d shock to the trend. this makes 8 shocks in total with only three observable variables. It follows from Section 4.3 that RE solutions under PI and II must differ. A full examination of these differences is deferred to Section 6.8; here we focus on the misperceptions of our selection of shocks under II.

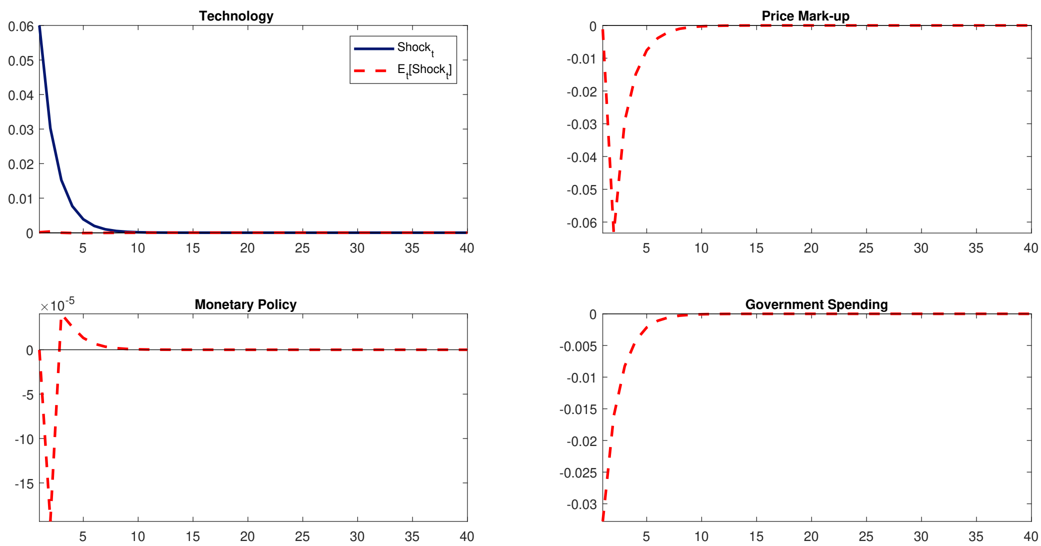
Figures 1 and 2 compare the actual structural unobserved shock process x_t with the agents belief $\mathbb{E}_t[x_t]$ for each of the four processes in turn. In Figure 1 (a) under II the rational agents mistake a technology shock for a combination of negative mark-up, monetary policy and government spending shocks. Given their observations of lagged output and inflation and the current nominal interest rate these beliefs obtained by Kalman Filter learning constitute a RE saddle-path stable equilibrium. In Figure 1 (b) A far less extreme confusion takes place with a price mark-up shock that is perceived only temporarily as a negative technology and monetary policy shocks alongside a boost to government spending.

The i.i.d monetary policy shock in Figure 2 (a) is barely picked up at all and perceived of a combination of small shocks to technology, the price mark-up and government spending. Finally in Figure 2 (b), the government spending shock is almost completely picked up but there remains a misperception of a small expansionary monetary policy that cancels out a temporary negative technology and positive mark-up shocks.

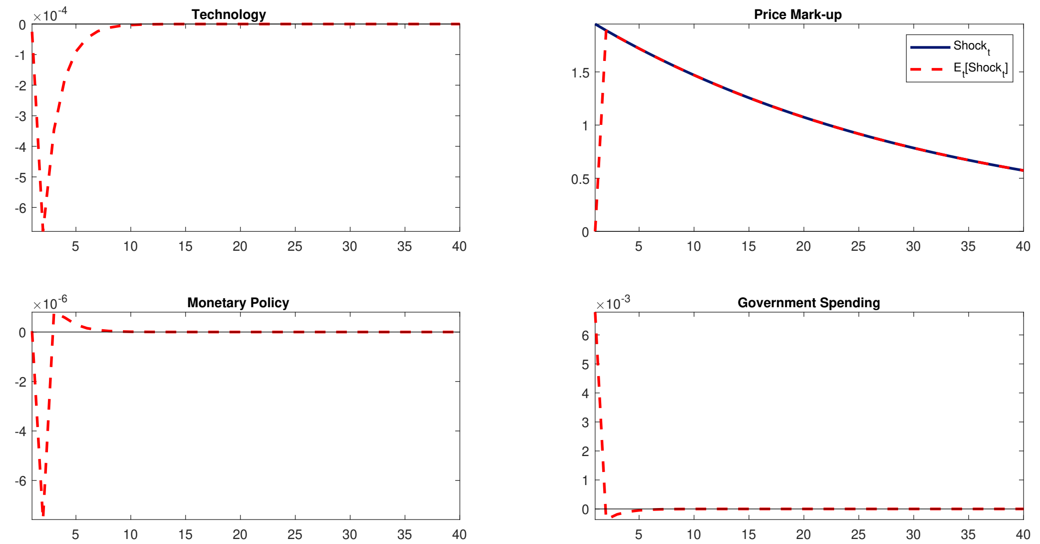
4.3 When PI and II RE Solutions Differ and Heuristic Forecasting Rules

We now pose the question: given the agents' and econometrician's information set, under what conditions do the RE solutions under agents' different the information sets PI and II actually differ? When can the both agents and econometrician infer the full state vector, including shocks?

To address this question we consider the data series $\mathsf{m}_t = [y_t, \pi_t, r_{n,t}]'$. We assume these are also the observations of the agents so $m_t^A = m_t$ in (43). Observing these three time-series under RE *may* enable agents (and the econometrician) to back out the shocks and by expressing m_t as an infinite VAR (Fernandez-Villaverde *et al.*, 2007 and Levine *et al.*, 2012). This is the case of invertibility. A necessary condition is that the number of shocks equals the number of observables. In the NK model there are 3 variables in the observations and 4 exogenous AR(1)

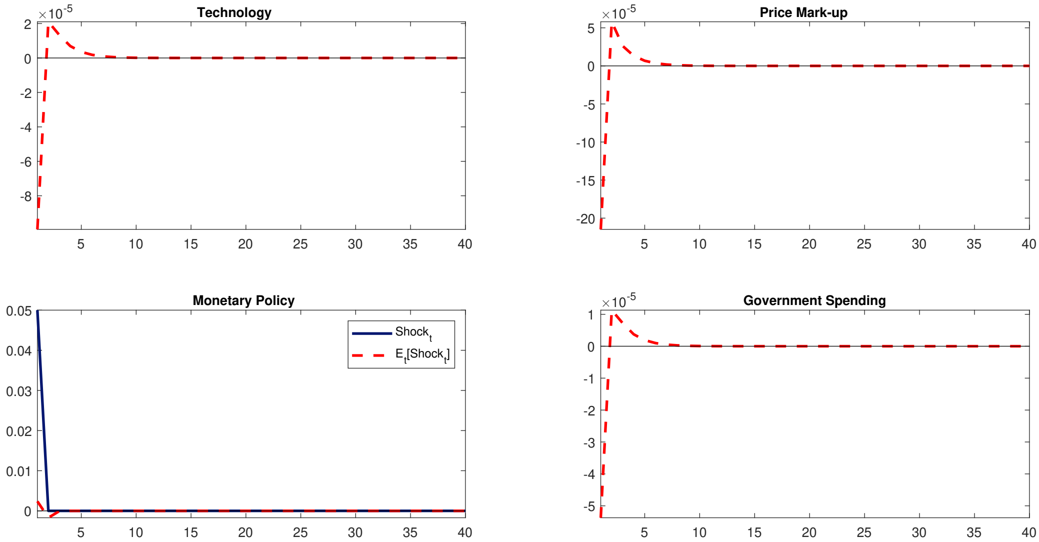


(a) Technology

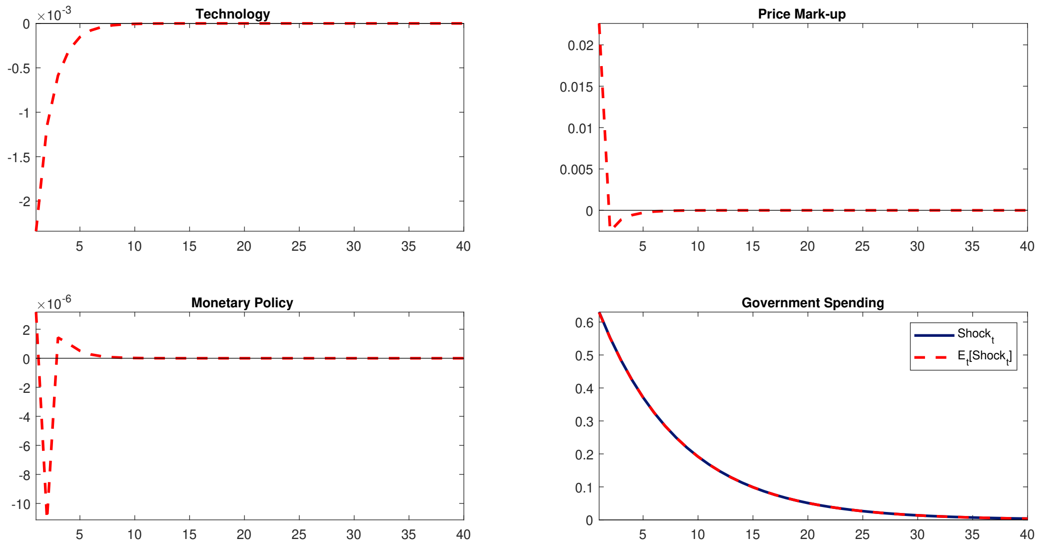


(b) Price Mark-up

Figure 1: **Estimated Pure RE: Misperceptions About the Shocks under II. The graphs compare the actual structural unobserved shock process x_t with the agents belief $\mathbb{E}_t[x_t]$. Technology and Price mark-up Shocks.**



(a) Monetary Policy



(b) Government Spending

Figure 2: **Estimated Pure RE: Misperceptions About the Shocks under II. The graphs compare the actual structural unobserved shock process x_t with the agents belief $\mathbb{E}_t[x_t]$. Monetary Policy and Government Policy Shocks.**

processes (A_t, G_t, MS_t, $\Pi_{targ,t}$). For now let us drop one of these shocks to give a square system which may be invertible.

To derive the necessary and sufficient conditions for invertibility, write the RE solution for both PI and II set out in the previous sub-section as the following VARMA process

$$\mathsf{s}_t = \tilde{A}\mathsf{s}_{t-1} + \tilde{B}\epsilon_t \tag{56}$$

$$\mathsf{m}_t = \tilde{E}\mathsf{s}_{t-1} = \tilde{C}\mathsf{s}_{t-1} + \tilde{D}\epsilon_t \tag{57}$$

where $\tilde{C} \equiv \tilde{E}\tilde{A}$ and $\tilde{D} \equiv \tilde{E}\tilde{B}$.

For the PI case, given the informational assumptions set out above, we have, straightforwardly, $s_t = z_t, \tilde{A} = A, \tilde{B} = B, \tilde{E} = E$. For the II case, we have

$$s_t = \begin{bmatrix} z_{t,t-1} \\ \tilde{z}_t \end{bmatrix} \tag{58}$$

$$\tilde{A} \equiv \begin{bmatrix} A & A\mathcal{K}J \\ 0 & Q^A \end{bmatrix} \tag{59}$$

$$\tilde{B} \equiv \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{60}$$

$$\tilde{E} \equiv \begin{bmatrix} E & E\mathcal{K}J \end{bmatrix} \tag{61}$$

where $\mathcal{K}$, Q^A and E are as defined in (53) to (55).

Because we have three shocks and three observables, the matrix $\tilde{D}$ is square. Assume now it is also non-singular which is only possible if m_t are *observations without lags*. Then $\epsilon_t = \tilde{D}^{-1}(\mathsf{m}_t - \tilde{C}\mathsf{z}_{t-1})$ and substituting into (56) and denoting the lag operator by L, we have

$$[(I - (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})L]\mathsf{z}_t = \tilde{B}\tilde{D}^{-1}\mathsf{m}_t \tag{62}$$

Hence combining (56)–(62) we have

$$\mathsf{z}_t = \sum_{i=0}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^i\tilde{B}\tilde{D}^{-1}\mathsf{m}_{t-i} \tag{63}$$

$$\mathsf{m}_t = \tilde{C}\sum_{i=1}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^i\tilde{B}\tilde{D}^{-1}\mathsf{m}_{t-i} + \tilde{D}\epsilon_t \tag{64}$$

Convergence of the summations in (63) and (64) requires that the matrix $(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})$ has all eigenvalues within the unit circle. Then equation (64) is an infinite VAR for the three observables $\mathsf{m}_t = [y_t, \pi_t, r_{n,t}]'$ which is estimatable from output, inflation and interest rate data.[12] It follows that the RE forecast is

$$\mathbb{E}_t\mathsf{m}_{t+1} = \tilde{C}\sum_{i=0}^{\infty}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^i\tilde{D}^{-1}\mathsf{m}_{t-i} \tag{65}$$

[12] See Levine *et al.* (2023) for a full treatment of invertibility under PI or II.

whereas the adaptive heuristic rules (66) and (67)

$$\mathbb{E}_t^* y_{t+1} = \sum_{i=0}^{\infty} \lambda_y^i y_{t-i}\,; \quad \text{or} \quad \mathbb{E}_t^* y_{t+1} = \sum_{i=1}^{\infty} \lambda_y^i y_{t-i} \tag{66}$$

$$\mathbb{E}_t^* \pi_{t+1} = \sum_{i=0}^{\infty} \lambda_\pi^i \pi_{t-i}\,; \quad \text{or} \quad \mathbb{E}_t^* \pi_{t+1} = \sum_{i=1}^{\infty} \lambda_\pi^i \pi_{t-i} \tag{67}$$

for the two cases of non-lagged and lagged data for output and inflation. These we can see are parsimonious representations of (65) so we can interpret the heuristic rules as parsimonious forecasting models in which non-rational agents choose under-parameterized predictors (see Branch and Evans, 2011).

We conclude that unless shock processes are either known or observed then at best with the number of shocks equal to the number of observables and no lags in the latter, a well-specified forecasting rule in the form of an infinite VAR is available.[13] Otherwise the VARMA solution (56)–(57) is not invertible. In fact none of these conditions are satisfied in the set-up we consider when we come to estimation: we have *more* shocks than observables, our heuristic rules assume aggregate variables are observed with a lag[14] and there are extra i.i.d. shocks that to consider in the form of measurement errors and a shock to trend. Neither the econometricians nor the agents can back out the shock processes from their information set and it is this feature that drives the wedge between PI and II.

5 Heterogeneous Expectations: Persistence through Bounded Rationality

Now we turn to the heterogeneous expectations model with BR(AU) agents alongside RE agents with fixed proportions of each type. We assume all RE agents *know the composite model*. In addition we impose informational inconsistency by assuming they have the *same II set* as the BR(AU) agents. The latter do not know the model, but do make individually optimal decisions given individual observations of the states and belief formations. The composite RE-BR model then has an equilibrium (in non-linear form)

$$\begin{aligned}
H_t^d &= n_{h,t}\,(H_t^s)^{RE} + (1-n_{h,t})\,(H_t^s)^{BR} \\
C_t &= n_{h,t}\,(C_t)^{RE} + (1-n_{h,t})\,(C_t)^{BR} = Y_t - G_t \\
\frac{P_t^o}{P_t} &= n_{f,t}\left(\frac{P_t^o}{P_t}\right)^{RE} + (1-n_{f,t})\left(\frac{P_t^o}{P_t}\right)^{BR}
\end{aligned}$$

Zero net wealth *in aggregate* implies that $n_{h,t}B_t^{RE} = -(1-n_{h,t})B_t^{BR}$. We consider the properties of the model with fixed exogenous proportions of RE and BR agents.

For our model of BR with AU, Figure 3 plots the impulse response functions (IRFs) with standard parameters for the rule for a shock to monetary policy under fast and slow learning. Figures 1 and 2 in Online Appendix E show IRFs for the technology and mark-up shocks. Not surprisingly fast learning sees an IRF converge faster to the RE case, but in either case BR introduces *more persistence* compared with RE. This suggests that this feature should lead to

[13] Approximating the infinite lag with a finite one introduces a further degree of misspecification.

[14] Bullard and Eusepi (2014) also examine learning with lagged observations which is distinct from rational expectations with II considered in this paper.

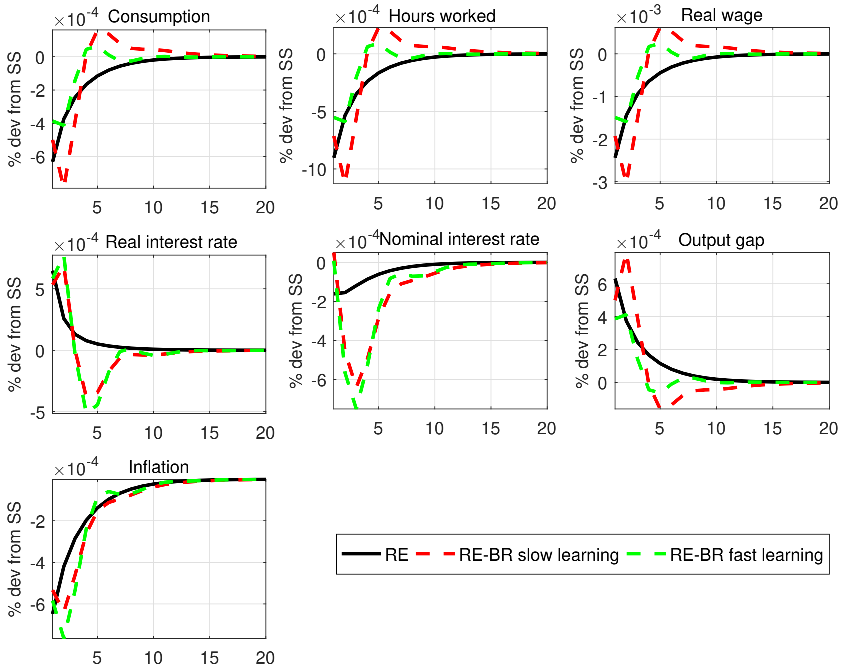


Figure 3: **RE vs RE-BR Composite Expectations with $n_h = n_f = 0.5$, $\lambda_x = 0.25, 1.0$; Taylor rule with $\rho_r = 0.7$, $\theta_\pi = 1.5$ and $\theta_y = 0.3$, $\theta_{dy} = 0$; Monetary Policy Shock**

a better fit of the data without relying on other persistence mechanisms (shocks, habit or price indexing). This we examine in the estimation of our model.[15]

6 Bayesian Estimation

We now turn to the estimation of an empirical NK behavioural model which differs from the linearized form used up to now in two respects. First, we retain the G_t shock process but drop the risk premium RS_t. Second, we assume that the steady state about which the perturbation solution is computed has a non-zero net growth and inflation. The former is stochastic and given by $g_t = (1 + g)\exp(\epsilon_{Atrend}) - 1$ where ϵ_{Atrend} is a shock to technology trend. The estimation then is conducted to be consistent with the long-term trend of output and inflation in the data used in the estimation.

6.1 The Models and Observables

We estimate five models with wealth distribution (Table 1). For the RE agents in either the 'pure' or composite RE model, we assume and compare the PI or II sets as discussed in Section 4. composite model with RE(II) and BR(AU) learning

6.2 The Models and Observables

We estimate five models with wealth distribution (Table 1). For the RE agents in either the 'pure' or composite RE model, we assume and compare the PI or II sets as discussed in Section 4.

[15]The stability properties of the model are examined in the WP version of the paper.

Model	Description
Pure RE(PI)	NK RE model under PI
Pure BR(AU)	NK model with AU learning
Comp RE(PI)-BR(AU)	Composite model with RE(II) and BR(AU) learning
Pure RE(II)	NK RE model solved and estimated under II
Comp RE(II)-BR(AU)	Composite model with RE(II) and BR(AU) learning

Table 1: **Summary of Estimated Models**

Bayesian methods are employed using Dynare adapted to handle II.[16] The sample period is 1984:1-2008:2, a subset of that used in Smets and Wouters (2007), which is also used extensively in the empirical and RBC literature. These observable variables are the log differences of real GDP (GDP_t) and the GDP deflator (DEF_t), and the federal funds rate ($FEDFUNDS_t$). All series are seasonally adjusted and taken from the FRED Database available through the Federal Reserve Bank of St.Louis and the US Bureau of Labour Statistics.

6.3 The Measurement Equations and Priors

The corresponding measurement equations for the 3 observables are[17]

$$\begin{bmatrix} D(\log GDP_t) * 100 \\ \log(DEF_t/DEF_{t-1}) * 100 \\ FEDFUNDS_t/4 * 100 \end{bmatrix} = \begin{bmatrix} \log\left(\frac{Y_t}{\overline{Y}_t}\right) - \log\left(\frac{Y_{t-1}}{\overline{Y}_{t-1}}\right) + \text{trend} + \epsilon_{y,t} - \epsilon_{y,t-1} + \epsilon_{A,t} \\ \log\left(\frac{\Pi_t}{\Pi}\right) + \text{cons}_\pi + \epsilon_{\pi,t} \\ \log\left(\frac{R_{n,t}}{R_n}\right) + \text{cons}_r \end{bmatrix}$$

where constants $trend$, $cons_\pi$ and $cons_r$ are related to the steady state of our model by $\log(1+g) = \text{trend}/100$, $\Pi = \text{cons}_\pi/100 + 1$ and $R_n = \frac{\Pi}{\beta_g} = \frac{\Pi(1+g)}{\beta} = \text{cons}_r/100 + 1$, respectively.[18] We introduce measurement errors on two observables, output and inflation ($\epsilon_{y,t}$ and $\epsilon_{\pi,t}$) so in total there are 3 variables in the observations, 4 exogenous AR(1) processes (A_t, G_t, MS_t, $\Pi_{targ,t}$) and 4 further i.i.d. shocks including measurement errors, $\epsilon_{MP,t}$, $\epsilon_{Atrend,t}$ and $\epsilon_{y,t}$, $\epsilon_{\pi,t}$.[19] Thus there are 8 shocks and 3 observables meaning that the invertibility condition discussed in Section 4 is not satisfied. Structural parameters $[\zeta, \alpha]$ are fixed with standard choices from the DSGE literature. These parameters are necessary to solve and linearize the models but are problematic for identification.

For the remainder of parameters gamma and inverse gamma distributions are used as priors when non-negativity constraints are necessary, and beta distributions for fractions or probabilities. Normal distributions are used when more informative priors seem to be necessary. The values of priors are in line with those in Smets and Wouters (2007). For the Taylor rule parameter on inflation the prior is set to obey the Taylor principle is centred at the value suggested by Taylor. The beta distribution we use on the adaptive expectations learning parameters λ_x and ω_x also restricts it to the open unit interval with support of 0.28 standard deviation, which is the highest value for a beta distribution, to impose more prior uncertainty. For all these

[16] Levine *et al.* (2020) provides full details of this addition to Dynare.

[17] $Y_t = GDP_t$, $\overline{Y}_t$ =trend and trend growth $=\log \overline{Y}_t - \log \overline{Y}_{t-1} = \log(1+g) + \epsilon_{A,t}$. $\epsilon_{y,t}$ and $\epsilon_{\pi,t}$ are measurement equations for output and inflation, respectively.

[18] This implies that β is determined empirically as $\beta = \left(\frac{\text{cons}_\pi + 100}{\text{cons}_r + 100}\right)(1+g)$.

[19] The monetary policy rules for the nominal interest rate and the AR(1) processes we use in the estimation are specified in Online Appendix B. The AR(1) models introduce 4 AR coefficients $[\rho_A, \rho_G, \rho_{MS}, \rho_\pi]$ and 4 i.i.d. shocks $[\epsilon_{A,t}, \epsilon_{G,t}, \epsilon_{MS,t}, \epsilon_{\pi,t}]$.

beta distribution parameters, we centre the prior density in the middle of the unit interval. For the parameters for the proportions of rationality n_h and n_f, we draw our priors from an interval $n_h, n_f \in [0,1]$. Our prior mean $n_h, n_f = 0.5$ lies at the mid-point of this range which is uniformly distributed. A common theme in papers that study empirical RBC/DSGE models is the difficulty in pinning down the parameter of labour supply elasticity ϕ. Inference on the inverse Frisch elasticity of labour supply has been found susceptible to model specifications, and exhibiting wide posterior probability intervals. So we assume a normal distribution with mean 2.0 and standard deviation of 0.5 for the parameter which is well within the range of point estimates reported in the RBC and labour literature.

6.4 Posterior Simulations, Identification

The posterior mode and the Hessian matrix are obtained via standard numerical optimization routines. The latter is then used in the Metropolis-Hastings (MH) algorithm to generate a sample from the posterior distribution. For each estimated model, two parallel chains of 1,000,000 random draws are used in the Monte-Carlo Markov Chain Metropolis-Hastings (MCMC-MH) algorithm. We run an iterative process of MCMC simulations in order to calibrate the scaling factor to achieve the desired rate of acceptance which is key for the speed of convergence of the MCMC-MH chains, which are also sensitive to the number of MCMC iterations. The former ensures that more of the parameter region is searched more regularly, but at the expense of reducing the acceptance ratio. In this estimation the number of draws we choose is sufficient to allow for convergence.[20]

Based on the prior information, we first conduct some pre-estimation identification diagnostics and report them in detail in Online Appendix F. The aim of this exercise is to scan the parameters we choose to estimate in terms of their identification in our models. We focus this exercise on the most general Comp RE-BR(AU) model and on the identification evaluation at the point values of the prior means and from a Monte Carlo sample drawn from the prior space. Our checks are also performed for the identification evaluation at the point values of the posterior means. To focus on the case when weak identification arises, Figure 9 in Online Appendix F also shows the identification strength and sensitivity component in the moments using the composite RE-BR priors and estimation results. Although, no parameter identification difficulties are detected from both the prior space and across the estimated parameters, the sensitivity strength in the moments of a few parameters at their posterior mean point is relatively weak. In light of the findings, we perform a number of robustness checks.

6.5 Bayes Factor Comparison

We first focus on Pure RE, Pure BR(AU) and Comp RE(PI)-BR(AU) when RE agents have a PI set. We employ the Bayes Factor (BF) from the model marginal likelihoods to gauge the relative merits across the three models in Table 2.

The BR models - Pure BR(AU) and Comp RE(PI)-BR(AU) - all substantially outperform their RE counterpart which is firmly rejected by the data. Formally, using the Bayesian statistical language of Kass and Raftery (1995), a BF, the quotient of the probabilities reported, greater than 100 (marginal log-likelihood difference over 4.61) offers "decisive evidence". Thus, we have decisive support for the pure BR and some composite behaviour from the US data we observe. The BF difference between the non-RE models is also strong.

[20] To formally test and to check the convergence, besides calibrating the acceptance rate, we use the convergence indicators recommended by Brooks and Gelman (1998) and Gelman *et al.* (2004).

Model	Pure RE(PI)	Pure BR(AU)	Comp RE(PI)-BR(AU)
LL	1656	1666	1672
Prob	0.0000	0.0034	0.9966

Table 2: **Log-likelihood Values and Posterior Model Odds: RE Agents with PI**

Next we assume an II set for the RE agents: $I_t = [Y_{s-1}, \Pi_{s-1}, R_{n,s}]$, $s \leq t$. An important point to stress is that this is the *same information set* we assume for BR agents when they come to update their heuristic rule. In this sense, we now have *informational consistency* across BR and RE agents, and also with the econometrician estimating the model. This feature we believe is new for the heterogeneous behavioural NK model literature. The results for the likelihood race are reported in Table 3.

Model	Pure RE(II)	Pure BR(AU)	Comp RE(II)-BR(AU)
LL	1692	1666	1708
Prob	0.0000	0.0000	1.0000

Table 3: **Log-likelihood Values and Posterior Model Odds: RE Agents with II**

A very similar picture emerges when comparing the RE model with the behavioural alternatives. Two results are worth noting. First, RE with imperfect information (Pure RE(II)) wins the likelihood race against both Pure BR(AU) and Pure RE(PI). Again, in formal Bayesian language, the RE(II) model decisively dominates the pure BR-AU-learning model and, not surprisingly, decisively dominates RE(PI) - a finding that is consistent with that in Levine *et al.* (2012). The second interesting result is that, when the composite heterogenous expectations model is estimated assuming the same II information set for everyone (Comp RE(II)-BR(AU)), it generates the highest log-likelihood value and outperforms all the completing models in fitting the data. In Section 6.7, we examine whether the ability to match second moments of the data is able to provide more evidence.

6.6 Parameter Estimation Results

Table 4 contains summary statistics of the posterior distributions of the NK models. We report posterior means of the parameters of interest and 95% probability intervals alongside the posterior marginal likelihoods for all 7 models so far: Pure RE(PI), Pure BR(AU), Comp RE(PI)-BR with $n = 0.5, 0.1$, Comp RE(PI)-BR(AU), Pure RE(II) and Comp RE(II)-BR(AU). As expected, the RE solutions yield the estimates that are in the range often found in the existing literature. The BR solution equilibrium we propose departs from the standard RE solutions and allows a process of adaptive learning driven by the speed of learning parameter $\lambda_x \in [0, 1]$ for the household and firms, respectively.

Focusing on the parameter characterizing the degree of price stickiness, ξ, the mean estimates report an average price contract duration of around 2.70 and 3.45 quarters for BR and RE(PI)-BR. Their estimated 95% intervals imply that price contracts change in the ranges of $\in (2.33, 3.23)$ and $\in (2.94, 4.00)$ suggesting that the firms of BR and RE-BR economies change prices as frequently as once every 2.33 quarters. The estimated contract length is shorter in the non-pure-RE models.

Interesting to note that, assuming very diffuse priors on the learning parameters λ_x, we find

Parameter	Notation	Prior distribution			Pure RE(PI)	Pure BR(AU)	RE(PI)-BR(n=0.1)	RE(PI)-BR(n=0.5)	RE(PI)-BR(AU)	Pure RE(II)	RE(II)-BR(AU)
		Density	Mean	S.D/df							
Calvo prices	ξ	$\mathcal{B}$	0.50	0.10	0.77 [0.73:0.81]	0.63 [0.57:0.69]	0.66 [0.61:0.71]	0.70 [0.64:0.76]	0.71 [0.66:0.75]	0.84 [0.80:0.87]	0.67 [0.59:0.74]
Labour supply elasticity	ϕ	$\mathcal{N}$	2.00	0.50	0.54 [0.26:0.85]	0.70 [0.16:1.20]	0.52 [0.12:0.90]	0.65 [0.18:1.11]	1.24 [0.53:1.85]	2.88 [1.64:4.14]	3.86 [2.77:4.97]
Adaptive learning											
Learning $\in [0,1]$	λ_h	$\mathcal{B}$	0.50	0.28	-	0.20 [0.13:0.26]	0.21 [0.14:0.27]	0.23 [0.17:0.30]	0.27 [0.12:0.35]	-	0.04 [0.01:0.07]
Learning $\in [0,1]$	λ_f	$\mathcal{B}$	0.50	0.28	-	0.05 [0.01:0.09]	0.05 [0.01:0.08]	0.17 [0.04:0.29]	0.03 [0.01:0.05]	-	0.08 [0.00:0.16]
Credibility $\in [0,1]$	ω_h	$\mathcal{B}$	0.50	0.28	-	0.95 [0.90:1.00]	0.95 [0.88:1.00]	0.94 [0.87:1.00]	0.93 [0.86:1.00]	-	0.96 [0.91:1.00]
Credibility $\in [0,1]$	ω_f	$\mathcal{B}$	0.50	0.28	-	0.52 [0.12:0.99]	0.52 [0.11:0.99]	0.50 [0.03:0.92]	0.43 [0.00:0.84]	-	0.53 [0.13:0.99]
Proportion of rationality											
Rational households	n_h	$\mathcal{U}$	0.5	0.2887	1.00	0.00	0.10	0.50	0.45 [0.42:0.50]	1.00	0.95 [0.92:0.97]
Rational firms	n_f	$\mathcal{U}$	0.5	0.2887	1.00	0.00	0.10	0.50	0.04 [0.00:0.10]	1.00	0.16 [0.00:0.51]
Interest rate rule											
Inflation	θ_π	$\mathcal{N}$	1.50	0.25	1.58 [1.38:1.79]	1.42 [1.10:1.75]	1.36 [1.10:1.63]	1.91 [1.65:2.16]	1.57 [1.33:1.80]	1.97 [1.68:2.27]	2.14 [1.83:2.43]
Output	θ_y	$\mathcal{N}$	0.12	0.05	0.04 [-0.00:0.08]	0.13 [0.07:0.18]	0.09 [0.04:0.14]	0.10 [0.05:0.14]	0.01 [-0.05:0.07]	0.19 [0.12:0.24]	0.14 [0.08:0.19]
Output growth	θ_{dy}	$\mathcal{N}$	0.12	0.05	0.22 [0.18:0.27]	0.08 [0.03:0.13]	0.09 [0.04:0.13]	0.09 [0.04:0.14]	0.11 [0.07:0.15]	0.12 [0.04:0.19]	0.22 [0.15:0.29]
Interest rate smoothing	ρ_r	$\mathcal{B}$	0.75	0.10	0.76 [0.71:0.80]	0.35 [0.26:0.44]	0.39 [0.33:0.45]	0.35 [0.28:0.43]	0.32 [0.25:0.40]	0.41 [0.31:0.51]	0.41 [0.31:0.50]
AR(1) coefficients											
Technology	ρ_A	$\mathcal{B}$	0.50	0.20	0.62 [0.40:0.84]	0.95 [0.90:0.99]	0.95 [0.91:0.99]	0.96 [0.92:0.99]	0.92 [0.87:0.99]	0.50 [0.18:0.83]	0.51 [0.17:0.85]
Government spending	ρ_G	$\mathcal{B}$	0.50	0.20	0.94 [0.92:0.97]	0.50 [0.17:0.83]	0.50 [0.16:0.82]	0.51 [0.14:0.80]	0.45 [0.14:0.80]	0.88 [0.84:0.91]	0.85 [0.80:0.90]
Price mark-up	ρ_{MS}	$\mathcal{B}$	0.50	0.20	0.39 [0.30:0.47]	0.97 [0.94:0.99]	0.97 [0.95:0.99]	0.97 [0.95:0.99]	0.96 [0.94:0.99]	0.97 [0.95:0.99]	0.97 [0.96:0.99]
Inflation target	ρ_π	$\mathcal{B}$	0.50	0.20	0.99 [0.98:1.00]	0.92 [0.87:0.98]	0.96 [0.94:0.99]	0.96 [0.94:0.99]	0.98 [0.96:0.99]	0.76 [0.68:0.83]	0.76 [0.61:0.95]
Standard deviation of shocks											
Technology trend	ϵ_{Atrend}	$\mathcal{IG}$	0.10	2.00	0.47 [0.41:0.52]	0.37 [0.29:0.45]	0.34 [0.26:0.42]	0.34 [0.26:0.42]	0.29 [0.21:0.37]	0.48 [0.42:0.54]	0.48 [0.42:0.54]
Technology	ϵ_A	$\mathcal{IG}$	0.10	2.00	0.07 [0.02:0.12]	0.54 [0.46:0.61]	0.53 [0.45:0.60]	0.53 [0.45:0.60]	0.52 [0.45:0.58]	0.06 [0.02:0.10]	0.07 [0.03:0.11]
Government spending	ϵ_G	$\mathcal{IG}$	0.10	2.00	2.34 [1.93:2.74]	0.10 [0.02:0.18]	0.17 [0.02:0.44]	0.17 [0.02:0.44]	0.08 [0.02:0.14]	3.18 [2.43:3.91]	4.57 [3.83:5.79]
Price mark-up	ϵ_{MS}	$\mathcal{IG}$	0.10	2.00	0.09 [0.02:0.17]	1.35 [0.95:1.75]	1.05 [0.75:1.35]	1.05 [0.75:1.35]	1.42 [0.96:1.87]	1.95 [1.18:2.74]	2.50 [1.89:3.10]
Inflation target	ϵ_π	$\mathcal{IG}$	0.10	2.00	0.05 [0.03:0.06]	0.19 [0.14:0.25]	0.20 [0.17:0.24]	0.20 [0.17:0.24]	0.18 [0.15:0.22]	0.19 [0.15:0.22]	0.13 [0.06:0.19]
Monetary policy	ϵ_{MP}	$\mathcal{IG}$	0.10	2.00	0.22 [0.18:0.25]	0.07 [0.03:0.11]	0.05 [0.03:0.07]	0.05 [0.03:0.07]	0.06 [0.03:0.08]	0.05 [0.03:0.08]	0.10 [0.03:0.18]
Price contract length	$\frac{1}{1-\xi}$	-	-	-	4.35	2.70	2.94	3.33	3.45	6.25	3.03
Marginal likelihood					1655.838984	1666.473189	1673.903923	1671.310091	1672.140275	1691.799452	1708.046087

Table 4: **Bayesian Prior and Posterior Distributions for RE, BR and Composite RE-BR Models**: Perfect Information (PI) and Imperfect Information (II) Assumptions for RE Agents. For all estimated models we use observations with a lag and the information set for lag 1 case at time t is $I_t = \{Y_{t-1}, \Pi_{t-1}, R_{n,t}\}$. $n = n_h = n_f = 0.1, 0.5$ are also imposed in this estimation. The trend or mean of the data variables are calculated directly from the data and not estimated with the rest of the model. The steady state is consistent with these values.

that the data is very informative about these parameters, strengthening the strong empirical support from the observations for the learning processes in the BR and RE-BR economies which improve on RE without consistent information assumptions. This observation also applies to the estimated proportions of rational or BR agents. In addition, none of the estimated λ_x is very close to 0, suggesting that some form of information without RE is relevant for updating and learning in the 5 BR models. For example, our estimates suggest that some fast learning takes place in the household sector (λ_h is around 0.20-0.27), although when agents have the same II set λ_h has become much smaller (0.04). The explanation is clear: with RE(II), learning from lagged information becomes less relevant.

For the policy rule, we find that for the behavioural models there is a low degree of persistence in the nominal interest rate which is much lower than observed in the literature. The responses to output (θ_y, θ_{dy}) are very low, in some cases, nearly non-existent, while the feedback to inflation (θ_π) is strong, implying a stronger concern from the monetary authorities about inflation variability, relative to the moments in output, which is caused by the varying forecast behaviours from agents' heterogenous expectations.

The estimates of the AR(1) coefficients show that the inflation target and price mark-up shocks are significantly inertial. With BR in the model, the exogenous price mark-up shock volatility contributes the most to the variation in the data and the monetary/fiscal policy volatility matters much less for this aspect of the fit. Assuming II in the model, the government spending volatility has increased significantly. The price mark-up shock (the uncertainty interval) is sightly larger than that of the RE model because the expectation heterogeneity in the model increases inflation volatility (uncertainty) and acts as a persistent force in this behavioural economy in the inflation fluctuations (this is also evident in Section 6.7 when we examine the implied model moments).

Overall, the parameter estimates are reasonably robust across information specifications, despite the fact that the II alternative leads to a much better model fit based on the corresponding posterior marginal likelihood. It is interesting to note that the point estimates of almost every single parameter under II are tighter and more strongly determined compared with the case under PI which this helps to explain its superior performance in the likelihood race.

6.7 Matching Second Moments

In this sub-section, we examine the model second moments, which has been a standard practice for researchers in the RBC tradition. We consider second moments and autocorrelations in turn. We mainly focus our analysis on the baseline RE model with its II variant, the behavioural BR and the outperforming composite models.

In terms of matching volatility the behavioural composite RE(II)-BR(AU) is able to match precisely the interest rate standard deviation in the data and performs very well at matching the data, whereas the pure RE model (including Comp RE(PI)-BR(AU)) performs rather poorly at capturing inflation and interest rate volatility, lying well-outside the 95% confidence bands. However, for the behavioural composite, there is room for improvement in matching inflation volatility. The model's ability of matching inflation moments is distorted, generating more volatility in inflation than the data and as noted this can be explained by the role played by the more volatile pricing shock (ϵ_{MS}) found in the estimated models which gives rise to the amplification effects on inflation dynamics caused by the expectation heterogeneity in the behavioural economy. The pure BR model is able to reduce this volatility while still matching output well.

	Standard Deviation		
	Output	Inflation	Interest rate
US Data	0.58	0.24	0.61
	(0.50, 0.69)	(0.21, 0.27)	(0.55, 0.70)
Pure RE(PI)	0.80	0.88	0.86
Pure BR(AU)	0.68	0.84	0.71
Comp RE(PI)-BR(AU)	0.66	1.68	1.29
Pure RE(II)	0.74	0.66	0.76
Comp RE(II)-BR(AU)	**0.66**	**0.39**	**0.60**
	Cross-correlation with Output		
US Data	1.00	-0.12	0.22
	(-)	(-0.31, 0.10)	(0.02, 0.39)
Pure RE(PI)	1.00	0.04	-0.04
Pure BR(AU)	1.00	-0.02	0.00
Comp RE(PI)-BR(AU)	1.00	-0.02	-0.01
Pure RE(II)	1.00	-0.01	**0.01**
Comp RE(II)-BR(AU)	1.00	**-0.07**	-0.03

Table 5: **Selected Second Moments (At the Posterior Means)**: For the empirical moments computed from the data set the bootstrapped 95% confidence bounds based on the sample estimates are presented in parentheses.

Table 5 also reports the cross-correlations of the 3 observable variables vis-a-vis output. All the estimated models except Pure RE(PI) do well and predict the correct sign for the output-inflation cross-correlation and the best performing Comp RE(II)-BR(AU) is also highly successful in reproducing the co-movement in the data. However, in terms of the output-interest rate correlation, all models perform poorly and most have the wrong sign although the RE(II) assumption improves in this dimension, getting the correct sign and closer to the bootstrapped 95% lower bound. Overall, the strength of the composite-II behaviour in reproducing business cycles lies in the output and interest rate moments as the estimated model matches most of the US data and the empirical moments are captured well-within the 95% uncertainty bands.

If we look at the autocorrelations up to 10 lags in Figure 4, the picture is also somewhat mixed. Overall, it shows very good goodness-of-fit of the RE-BR composite under II to data in terms of successfully capturing the autocorrelations up to many lags. Almost all of the moments are inside the 95% confidence intervals of the empirical moments of autocorrelations, which leads to some confidence in the estimated models. Model RE with PI and II is the most problematic one in reproducing the output autocorrelations at the first two and three orders, ACF lying outside of the lower interval and having the wrong sign. In addition, the implies moments from Pure RE(PI) and Comp RE(PI)-BR(AU) are well-outside the 95% confidence intervals over the entire ACF horizon.

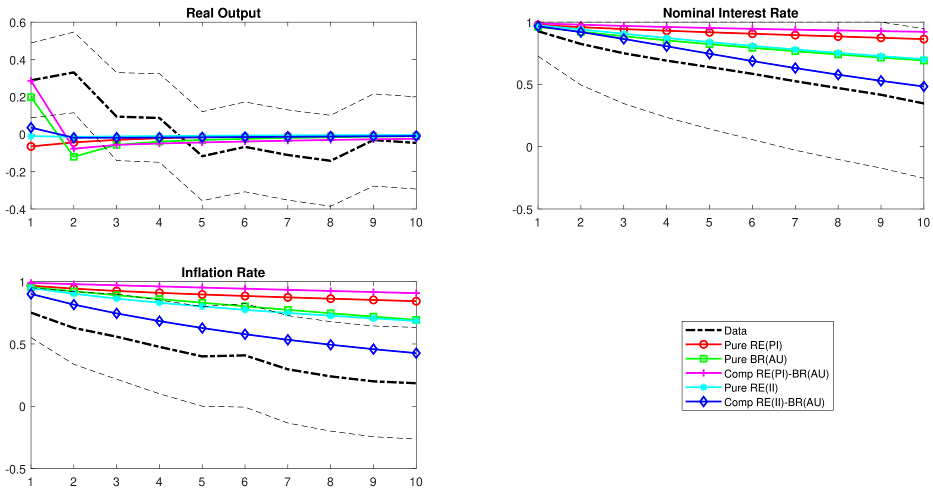


Figure 4: **Autocorrelations of Observables in the Actual Data and in the Estimated Models**: The approximate 95% confidence bands are constructed using the large-lag standard errors (see Anderson, 1976).

On the other hand, the two models estimated under II, namely, Comp RE(II)-BR(AU) and Pure RE(II), are capable of generating the persistence seen in inflation and interest rate than the BR special case and the reason for this lies in the estimated learning mechanism of the adaptive expectations scheme in, for example, forecasting inflation movements from their RE(PI) counterparts. These autocorrelations are able to reproduce an important stylized fact, namely the persistence of aggregate inflation usually observed in empirical data, generating much inertia in the time path to match the actual inflation (also shown in the IRF predictions below). This is more effective than the pure RE case with II learning and/or exogenous shock dynamics which generates too much inertia. Finally, switching the information set from PI to II for the RE model produces a little more persistence, captured by the implied correlograms of inflation.[21] The analysis shows an improved ability of the DSGE model with II and BR

[21] The findings are generally in line with those in Jang and Sacht (2016), who conduct an empirical investigation on moment matching using a bounded rationality behavioural model à la De Grauwe (2011) estimated by the Simulated Method of Moments for the Euro Area. They find that their results can mimic the real data well,

behaviour to generate endogenous propagation mechanisms, in particular, how they capture the autocorrelation dynamics and output volatility. This explains the improved overall model fit in the comparison section.

6.8 Posterior Impulse Responses and Endogenous Persistence

As shown above from the estimated models and the moment analysis, both the heuristic rules and RE-II learning mechanisms introduces more dynamics (persistence) into the model solutions. As a result, the empirical models incorporating either form of endogenous learning can significantly outperform the standard RE-PI model in the likelihood comparison. The empirical IRFs from the estimated models in this section support these conclusions. In Figures 5 and 7, relaxing PI in particular introduces more persistence compared with RE-PI, generating more hump-shaped trajectories after the system is shocked. This suggests this feature should lead to a better fit of the data without relying on other model internal inertia mechanisms.

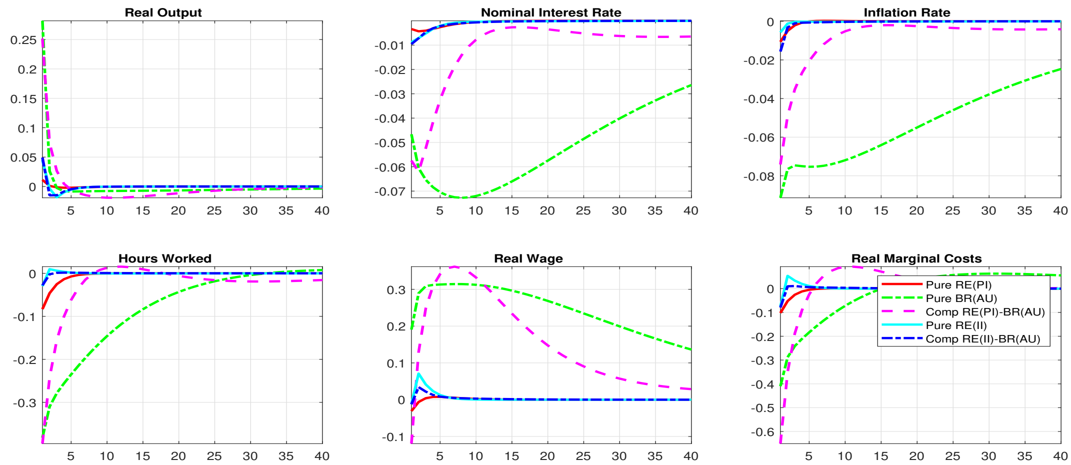


Figure 5: **Estimated Impulse Responses – Technology Shock**

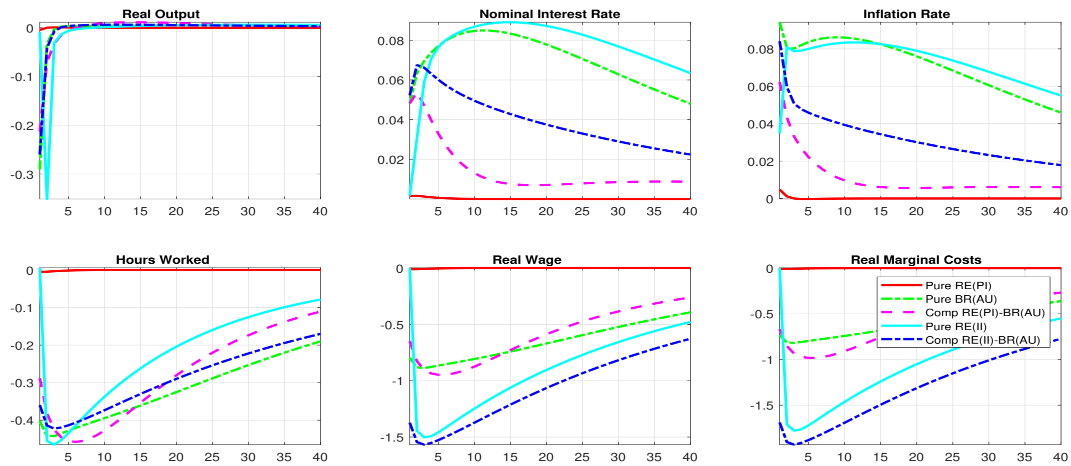


Figure 6: **Estimated Impulse Responses – Mark-up Shock**

slightly outperforming the linear RE counterpart in some of the moments, or are at least as good as the RE model in terms of providing fits for auto- and cross-covariances of the data.

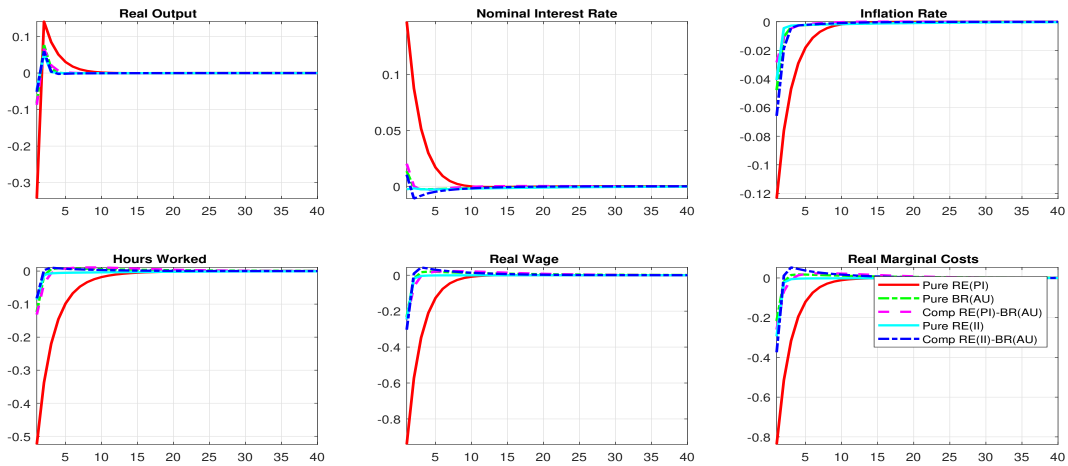


Figure 7: **Estimated Impulse Responses – Monetary Policy Shock**

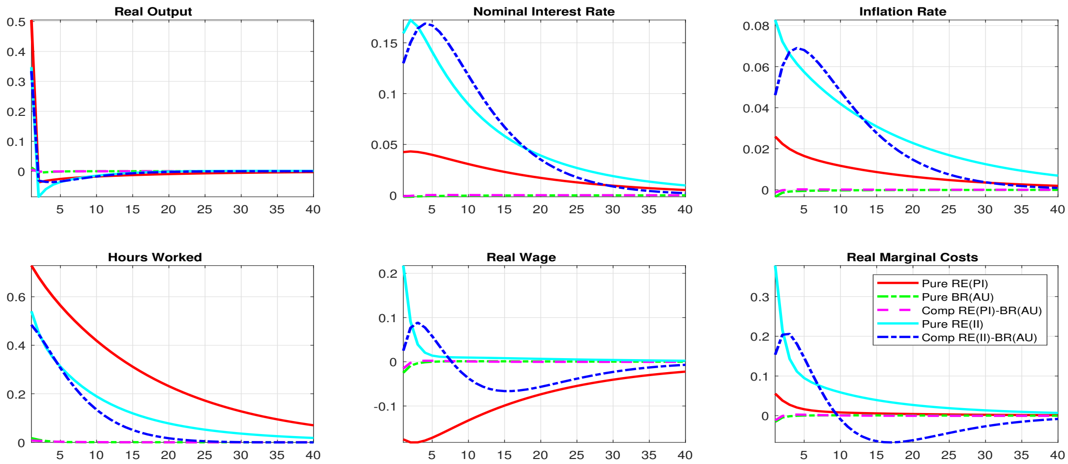


Figure 8: **Estimated Impulse Responses – Government Spending Shock**

The IRFs also attempt to address the difficulty of generating reasonable endogenous persistence in DSGE frameworks and replicating the observed business cycle stylized facts. As already seen in Table 4, our baseline RE model with II learning statistically dominates all other models. Relaxing 50% pure rationality in the baseline model with the general heuristic learning rule also performs well. Model fit can be much improved without resorting to building a large number of frictions and shocks, offering a parsimonious approach while relaxing the extreme RE and PI. Of particular interest for the evaluation of using internal propagation mechanisms, relaxing full rationality leads to a reduction in the estimated degree of price stickiness ξ. In addition, relaxing the RE and PI restrictions generally leads to a reduction in the estimated persistence of the shock processes (e.g. ρ_A or ρ_π in particular).

We also find that the lagged interest rate is highly significant in the estimated policy rule, but the estimated inertia is much reduced when BR and II are introduced, suggesting a reduction in the persistence needed in the rule. The monetary policy volatility matters much less for explaining the data variation aspect of the fit when the model is no longer pure RE. Overall taking the results reported in Sections 6, 6.7 and 6.8, we can capture business cycle movements without having to assume either highly autocorrelated shocks, high policy rule persistence and/or the presence of endogenous inertia in the model due to, for example, habit formation in consumption and lengthy price-setting contracts. This contrasts with standard DSGE models in a RE-PI environment.

6.9 Robustness Checks

In order to ensure robustness of our main results, we further estimate and evaluate a number of additional models and subject our data to a wide array of tests. Online Appendix F conducts additional model estimations to check our results (i.e. the likelihood comparisons) across different BR specifications with alternative prior specifications on the speed of learning parameter λ_x (e.g. priors with a looser precision) and parameters values (with λ_x centered at the prior mean and with the proportions $n_h = n_f = n = 0.5, 0.1$ being fixed to the values we used in the simulation analysis[22]).

7 Endogenous Proportions of Rational and Non-Rational Agents

Up to now we assume that proportions of rational and non-rational agents $n_{y,t}$ and $n_{\pi,t}$ are exogenous. As in Massaro (2013), in the estimation and main conclusions that follow, we retain this assumption, but in this sub-section we explore the extension that endogenizes these decision by agents. Following Brock and Hommes (1997) and the reinforcement learning literature in generalize these can be chosen as follows:

$$n_{x,t} = \frac{\exp(-\gamma\Phi^{RE}_{x,t}(\{x_t\}))}{\exp(-\gamma\Phi^{RE}_{x,t}(\{x_t\})) + \exp(-\gamma\Phi^{AE}_{x,t}(\{x_t\}))} \tag{68}$$

where $-\Phi^{RE}_{x,t}(\{x_t)\})$ and $-\Phi^{AE}_{x,t}(\{x_t)\})$ are 'fitness' measures respectively of the forecast performance of the rational and non-rational predictor of outcome $\{x_t\} = \{y_t\}, \{\pi_t\}$ given by a discounted least squares error predictor

$$\Phi^{RE}_{x,t}(\{x_t\}) = \mu_{RE}\Phi^{RE}_{x,t-1}(\{x_t\}) + (1-\mu_{RE})([x_t - \mathbb{E}_{t-1}\, x_t]^2 + C_x) \tag{69}$$
$$\Phi^{AE}_{x,t}(\{x_t\}) = \mu_{AE}\Phi^{AE}_{x,t-1}(\{x_t\}) + (1-\mu_{AE})[x_{t-j} - \mathbb{E}^*_{t-1-j}\, x_{t-1}]^2\,; j = 0, 1 \tag{70}$$

where ρ_{RE} and ρ_{AE} capture the memory of the agents forming RE and AE (a measure of forgetfulness of past observations). C_x represents the relative costs of being rational in learning about variable x_t. Thus the proportion of rational agents in the steady state is given by

$$n_x = \frac{\exp(-\gamma C_x)}{\exp(-\gamma C_x) + 1}$$

which is pinned down by the γC_x.

A complete treatment of the model would require a departure from the linear Kalman Filter solution for the II case for which we exploit the closed-form saddlepath solution that Pearlman *et al.* (1986) shows both exists and is unique. We have also exploited the convenience of linear Bayesian estimation. In what follows we confine ourselves to the RE PI case and use the linear estimates obtained up to now.

[22] As part of the robustness check, we also compare different values of $n = n_f = n_h \in [0, 1]$, and as $n \to 0$ we gradually reduce rationality in the model. We find that, as expected, as n decreases, the empirical support of the model improves gradually from the pure RE to the pure behavioural model. In this analysis, we systematically estimate and compare the different models with a grid of values for $n \in [0, 1]$ and find robust results and only report the fixed $n = 0.1, 0.5$ in Online Appendix F.

Agents with reinforcement learning now have proportions of rational households ($n_{h,t}$) and firms ($n_{f,t}$) are given by (68). Table 6 provides a third order perturbation solution of non-linear NK RE(PI) -BR Model. We use the Bayesian estimation of the linear model in Section 6 where the model is linearized and the proportions $n_{h,t}$ and $n_{f,t}$ are fixed. Non-linear estimation would be required to pin down the parameters n_h, n_f in the steady state, and $\mu_h^{RE,BR}$, $\mu_f^{RE,BR}$ and γ in the reinforcement learning process and goes beyond the scope of this paper. So here we impose them as reported in the table. We also scale the estimated standard deviations of the shocks using a parameter $\sigma = 1, 2$.

Variable	Stochastic Mean	Standard Deviation (%)	Skewness	Kurtosis
$\frac{C_t}{C}$	0.9993	2.47	0.2792	0.0371
$\frac{H_t}{H}$	1.0002	0.19	0.0192	0.0327
$\frac{W_t}{W}$	0.9996	2.15	0.2771	0.0215
$\frac{\Pi_t}{\Pi}$	0.9999	0.46	0.0159	0.0645
$\frac{R_{n,t}}{R_n}$	0.9999	0.46	0.0070	0.0651
$\Phi_{h,t}^{RE} - C_h$	-0.000065	0.000020	-0.7589	0.9487
$\Phi_{h,t}^{AE}$	-0.000084	0.000054	-1.8238	5.7852
$\Phi_{f,t}^{RE} - C_f$	-0.000011	0.000009	-0.7203	0.7834
$\Phi_{f,t}^{AE}$	-0.000069	0.000053	-2.2156	8.8686
$n_{h,t}(\gamma = 1; \sigma = 1)$	0.093301	0.000004	1.8039	6.0897
$n_{f,t}(\gamma = 1; \sigma = 1)$	0.098603	0.000004	2.2688	9.2725
$n_{h,t}(\gamma = 100; \sigma = 1)$	0.094221	0.003634	1.8039	6.0897
$n_{f,t}(\gamma = 100; \sigma = 1)$	0.101751	0.004303	2.2688	9.2725
$n_{h,t}(\gamma = 1000; \sigma = 1)$	0.102506	0.036343	1.8039	6.0897
$n_{f,t}(\gamma = 1000; \sigma = 1)$	0.130105	0.043030	2.2688	9.2725
$n_{h,t}(\gamma = 1000; \sigma = 2)$	0.129993	0.146939	1.8403	6.6096
$n_{f,t}(\gamma = 1000; \sigma = 2)$	0.224367	0.174046	2.3668	10.5098

Table 6: **Third Order Solution of the Estimated NK RE(PI)-BR Model**; $\mu_h^{RE} = \mu_h^{BR} = \mu_f^{RE} = \mu_f^{BR} = 0$; $\gamma = 1, 100, 1000$

The main results from these simulations are as follows. First, that reinforcement learning introduces *high kurtosis and skewness*[23] in macro variables. Second, reinforcement learning coupled with higher volatility of exogenous shocks results in the numbers of rational agents increasing from the estimated deterministic steady state value of 0.093 and 0.099 to 0.13 and 0.22 for households and firms respectively in the stochastic steady state. Third, given that bounded rationality is a welfare-reducing friction in these models it follows that volatility can actually be welfare-increasing in our homogeneous expectations setting.

[23] The absence of kurtosis in the standard NK model, often highlighted in the literature (see, for example, De Grauwe (2012a)) is in part simply the consequence of linearization and non-normality is a feature of higher order approximations.

8 Conclusions

This paper studies an NK behavioural model for which boundedly rational beliefs of economic agents are about payoff-relevant macroeconomic variables that are exogenous to their decision rules. In a Bayesian estimation of the RE-BR composite model with exogenous proportions of RE and BR agents, informational assumptions are central to the paper. In comparisons of different composites including the pure RE and BR cases, we impose what we term *informational consistency* where RE and BR agents in the model share the same II as the econometrician estimating the model. We contrast this with the standard assumption that RE agents have PI of the current state variables.

We find in a likelihood race that the pure RE model with II outperforms the pure BR model which in turn outperforms the pure RE with PI. When we examine the behavioural composite model with a general heuristic forecasting rule, we find that the composite RE(II)-BR estimated model, with estimated proportions of RE and BR agents, outperforms its RE(PI)-BR counterpart in terms of both a likelihood race and the fit of model second moments with those of the data. These results suggest that persistence can be injected into the NK model to improve data fit in two contrasting ways: bounded-rationality with learning through heuristic rules, or retaining RE but with II and Kalman-filtering learning.

Our results for the workhorse NK model suggest a new perspective for the macro/NK/learning literature. Avenues for future work could embed the RE-BR composite model into a richer NK model along the lines of Smets and Wouters (2007), extend the linear Kalman Filter to accommodate the non-linearity in reinforcement learning and use non-linear estimation methods to identify a number of parameters that cannot be identified using linear Bayesian estimation. The latter two non-linear extensions are major challenges. Future work could also examine optimal monetary policy and follow Geweke and Amisano (2012) and Deak *et al.* (2019) and estimate an optimal pool of RE(II) and RE-BR composites to be used to design a robust rule across BR model variants discussed in the Introduction.

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