Kalman Filter Learning Versus Bounded Rationality in a Heterogeneous Agent NK Model

To be presented at Jan 19 BoE-LSE Workshop

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January 16, 2024

Modelling Expectations in Macroeconomics

- Standard assumption in general equilibrium macroeconomic models is **Rational Expectations** (RE) with **perfect information** (PI) aka **full information (FI)** - hence **FIRE**.

- RE means model consistent expectations - agents know your model.

- PI (or FI) means agents observe or can infer the current and past state variables in your model.

- RE+PI (or FIRE) is a **strong assumption** .

- This paper relaxes RE or PI or both.

The Main Message of the Paper

- Information assumptions matter in our models
- In comparing Behavioural and RE models adopt a **level playing field.**
- Informational Assumptions should be the same.

Two Literatures

1. **Statistical Learning** - *Rational Expectations Versus Learning in Macroeconomics*
   - Can agents learn to be rational through econometrics (recursive least-squares learning)?
   - Is so we have **Expectational or E-Stability**

2. **Behavioural Macroeconomics**:
   - Individual rationality with heuristic rules (adaptive expectations)
   - Heterogeneous Expectations and Reinforcement Learning
   - Cognitive Discounting and Agent Inattention in otherwise rational models

Many Equilibrium Concepts "the wilderness of non-rational expectations)

Equilibria Concepts: "Small" Departures from RE

- **Minimal State Variable (MSV) OLS Learning about the RE solution**- See Evans and Honkapohja (2001), Evans and Honkapohja (2009) etc etc etc

- **Restricted Perception Stochastic Consistent Equilibria**. Perceived =Actual Law of Motion but do not ⇒ RE solution. See Hommes and Zhu (2014) and Hommes *et al.* (2023)

- **k-Level Thinking**. Iterate towards and RE equilibrium with each stage **temporary equilibrium**. See Garcia-Schmidt and Woodford (2019), Farhi and Werning (2019)

- **Inattention-Cognitive Discounting** Agents in model perceived reality with some **myopia** and **inattention** as in Gabaix (2020). They are otherwise rational. Related to **finite-time horizon** optimization as in Woodford (2018)

Equilibria Concepts: "Big" Departures from RE

- **Adaptive Expectations** (Subsuming **Heuristic Learning Rules**) - long history in macroeconomics going back to Milton Friedman. NOT Stochastic Consistent Expectations

- **Heterogenous Agents**: Some RE and some Behavioural or agents with different forcasting rules. Fixed Proportions or Endogenous Proportions with Reinforcement Learning as in Brock and Hommes (1997). See Branch and McGough (2010), Massaro (2013), De Grauwe and Ji (2019), De Grauwe (2011), De Grauwe (2012), De Grauwe and Katwasser (2012), De Grauwe and Gerba (2018).

- A further distinction in Behavioural-macro is between **Euler and Anticipated Utility** Learning

Euler Learning versus Anticipated Utility

- In the Behavioural Macro literature a division occurs between **Euler versus Anticipated Utility Learning (closely related to Internal Rationality (IR)** as in Adam and Marcet (2011))

- Under both IR and AU agents maximize utility under uncertainty, given their constraints and a consistent set of probability beliefs about payoff-relevant variables that are **beyond their control or external**.

- Then with IR beliefs take the form of a well-defined probability measure over a stochastic process (the '**fully Bayesian' plan**).

- For AU see Preston (2005) Eusepi and Preston (2011) and Woodford (2013) and Branch and McGough (2018) for EL versus AU.

- Cogley and Sargent (2008) compares the IR vs AU and encouragingly find that AU can be seen as a **good approximation** to

Imperfect Information Literature

1. A literature on **Imperfect Information in a Representative Agent (II-RA)** framework. See Minford and Peel (1983), Pearlman *et al.* (1986), Svensson and Woodford (2003), Collard *et al.* (2009), Levine *et al.* (2012), Hauk *et al.* (2021).

2. A literature on **Imperfect Information with Heterogenous Agents (II-HA)** that distinguish idiosyncratic (dispersed) and aggregate information. See Pearlman and Sargent (2005), Nimark (2008), Graham and Wright (2010), Leeper *et al.* (2013), Angeletos and Lian (2016), Rondina and Walker (2021), Angeletos and Huo (2021).

3. A **macro-econometrics** literature on **fundamentalness-invertibility**.
   - From a **macro** perspective on "**The ABC and D of VARs**" - see Fernandez-Villaverde *et al.* (2007)
   - From an **econometrics** perspective on "**Blaschke Factors**" - see Lippi and Reichlin (1994)
   - Related to II from the econometrician's perpective - Levine *et al.* (2023)

The Model, Learning Equilibrium and II Assumption

- **Workhorse NK Model**: Labour the only fop with exogenous demand (government Expenditure)
- Linearization of a Non-linear set-up with a non-zero net inflation rate
- Rational versus Bounded Rational Expectations
- Rational Expectations: Perfect versus Imperfect Information **(PI, II)**
- Bounded Rational Expectations **(BR)**
  - Anticipated Utility Learning based on Hommes *et al.* (2019)
  - Heuristic rules for expectations about variables beyond the agents' control
- Common (**imperfect**) information assumptions for RE and BR

Heuristic Rules and Information Assumptions

- **Information set for RE agents (PI)**: Full state vector (including shock processes)
- **Information set for RE agents (II)**: $I_t = [Y_{s-1}, \Pi_{s-1}, R_{n,s}]$, $s \leq t$.

- **AU learning:** agents make fully optimal decisions given their individual specification of beliefs, but have no macroeconomic model to form expectations of aggregate variables.
- We draw a clear distinction between **aggregate and local quantities** where identical agents in our model are not aware of this equilibrium.
- One-step ahead **heuristic forecasts** for households and firms are given by an adaptive expectations rule in linear (lower case) form:

$$\mathbb{E}^*_t x_{t+1} = \mathbb{E}^*_{t-1} x_t + \lambda_x (x_{t-j} - \mathbb{E}^*_{t-1} x_t)\,;\; x = w,\, r_n,\, \pi,\, \gamma - g,\, mc\,;\, j = 0, 1$$

- The real wage w, a demand term $\gamma - g$ and marginal costs, mc are **market local** variables and the nominal interest rate r_n are observed **with no lag**; inflation π is an aggregate variable obseved **with a lag**.

RE: Perfect vs Imperfect Information

The linearized form of the NK model has the following a state-space form that applies to both the RE PI and II cases:

$$\begin{bmatrix} z_{t+1} \\ \mathbb{E}_t x_{t+1} \end{bmatrix} = G \begin{bmatrix} z_t \\ x_t \end{bmatrix} + H \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \epsilon_{t+1} \quad (1)$$

$$m_t^A = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \quad (2)$$

where z_t is a $(n-m) \times 1$ vector of **predetermined variables** at time t with z_0 given; x_t x_t is a $m \times 1$ vector of **non-predetermined variables** at time t, ϵ_t is a vector of **random Gaussian zero-mean shocks** and m_t^A is a vector of **observable macro-economic variables** of the agents, which will be the data used by the econometrician.

All variables are expressed as proportional deviations about a steady state. G, H, B and $M_i, i = 1, 4$ are **fixed matrices**, ϵ_t as a vector of **random zero-mean shocks**.

PI and II Solutions: Kalman Filter

- PI saddlepath solution $x_t = -Nz_t$ is standard
- Following Pearlman *et al.* (1986), we apply the **Kalman filter** updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + K\Big[m_t^A - \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} M_3 & M_4 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix}\Big]$$

- The best estimator of (z_t, x_t) at time $t-1$ is updated by the "**Kalman gain**" K of the error in the predicted value of m_t^A given by

$$\begin{aligned} K &= \begin{bmatrix} P^A J' \\ -NP^A J' \end{bmatrix} [(M_1 - M_2 N) P^A J']^{-1} \text{ where} \\ F &\equiv G_{11} - G_{12} G_{22}^{-1} G_{21}; \qquad J \equiv M_1 - M_2 G_{22}^{-1} G_{21} \\ P^A &= Q^A P^A Q^{A'} + BB' \ \textbf{(Ricatti Equation)} \ \text{ where} \\ \mathcal{K} &\equiv P^A J' \left(J P^A J' \right)^{-1}; \quad Q^A = F[I - \mathcal{K} J] \end{aligned}$$

PI and II Solutions: Dynamics

- The **unique saddle-path stable solution under II** for the pre-determined and non-predetermined variables z_t and x_t, as shown by PCL, can then be described by processes for the **predictions** $z_{t,t-1}$ and for the **prediction errors** $\tilde{z}_t \equiv z_t - z_{t,t-1}$:

$$
\begin{aligned}
\text{Predictions of } z_t: \quad & z_{t+1,t} = A\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) \text{ where} \\
& A \equiv G_{11} + H_{11} - (G_{12} + H_{12})N \\
\text{Prediction Errors}: \quad & \tilde{z}_t = Q^A \tilde{z}_{t-1} + B\varepsilon_t \\
\text{Non-predetermined}: \quad & x_t = -N\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) \\
& \quad - G_{22}^{-1} G_{21} \left(I - \mathcal{K}J\right) \tilde{z}_t \\
\text{Measurement Equation}: \quad & m_t^A = E\left(z_{t,t-1} + \mathcal{K}J\tilde{z}_t\right) \text{ where} \\
& E \equiv M_1 + M_3 - (M_2 + M_4)N
\end{aligned}
$$

- Note the **higher order dynamics** from terms in $\tilde{z}_t$ in red

- Saddlepath condition and N matrix **as for PI**

Misperceptions about Shocks under II

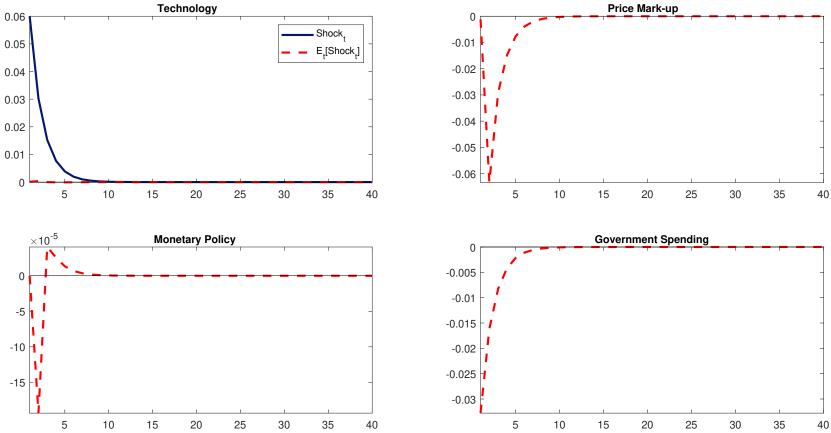

Figure: **1(a). Estimated Pure RE: Misperceptions About the Shocks under II. The graphs compare the actual structural unobserved shock process x_t with the agents belief $\mathbb{E}_t[x_t]$. Technology Shock.**

Misperceptions about Shocks under II

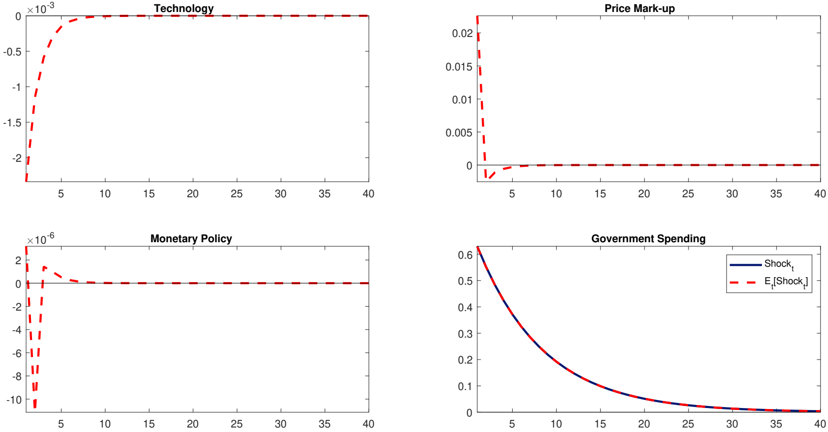

Figure: **2(b). Estimated Pure RE: Misperceptions About the Shocks under II. The graphs compare the actual structural unobserved shock process x_t with the agents belief $\mathbb{E}_t[x_t]$. Government Spending Shock.**

The Models to be Estimated

Model	Description
Pure RE(PI)	NK RE model under PI
Pure BR(AU)	NK model with AU learning
Comp RE(PI)-BR(AU)	Composite model with RE(II) and BR(AU)
Pure RE(II)	NK RE model solved and estimated under II
Comp RE(II)-BR(AU)	Composite model with RE(II) and BR(AU)

Table: **Summary of Estimated Models**

Bayesian Estimation

- **3 Observable Data Series**: Output Growth, Inflation and Nominal interest Rate

- **4 Exogenous AR(1) Shock Processes**: A_t, G_t, MS_t, $\Pi_{targ,t}$
- 4 further i.i.d. shocks including measurement errors

- Hence number of shocks $=7 >$ number of observables and the **PI solution differs from II**

- **Estimated fixed proportions** of rational households and firms with **uniform priors**

- Otherwise **standard priors** from the literature

Bayes Factor Comparison

Model	Pure RE(PI)	Pure BR(AU)	Comp RE(PI)-BR(AU)
LL	1656	1666	1672
Prob	0.0000	0.0034	0.9966

Table: **2. Log-likelihood Values and Posterior Model Odds: RE Agents with PI**

Model	Pure RE(II)	Pure BR(AU)	Comp RE(II)-BR(AU)
LL	1692	1666	1708
Prob	0.0000	0.0000	1.0000

Table: **3. Log-likelihood Values and Posterior Model Odds: RE Agents with II**

Validation

	Standard Deviation		
	Output	Inflation	Interest rate
US Data	0.58	0.24	0.61
	(0.50, 0.69)	(0.21, 0.27)	(0.55, 0.70)
Pure RE(PI)	0.80	0.88	0.86
Pure BR(AU)	0.68	0.84	0.71
Comp RE(PI)-BR(AU)	0.66	1.68	1.29
Pure RE(II)	0.74	0.66	0.76
Comp RE(II)-BR(AU)	**0.66**	**0.39**	**0.60**
	Cross-correlation with Output		
US Data	1.00	-0.12	0.22
	(-)	(-0.31, 0.10)	(0.02, 0.39)
Pure RE(PI)	1.00	0.04	-0.04
Pure BR(AU)	1.00	-0.02	0.00
Comp RE(PI)-BR(AU)	1.00	-0.02	-0.01
Pure RE(II)	1.00	-0.01	**0.01**
Comp RE(II)-BR(AU)	1.00	**-0.07**	-0.03

Table: **5. Selected Second Moments (At the Posterior Means)**: For the empirical moments computed from the data set the bootstrapped 95% confidence bounds based on the sample estimates are presented in parentheses.

Endogenous Persistence

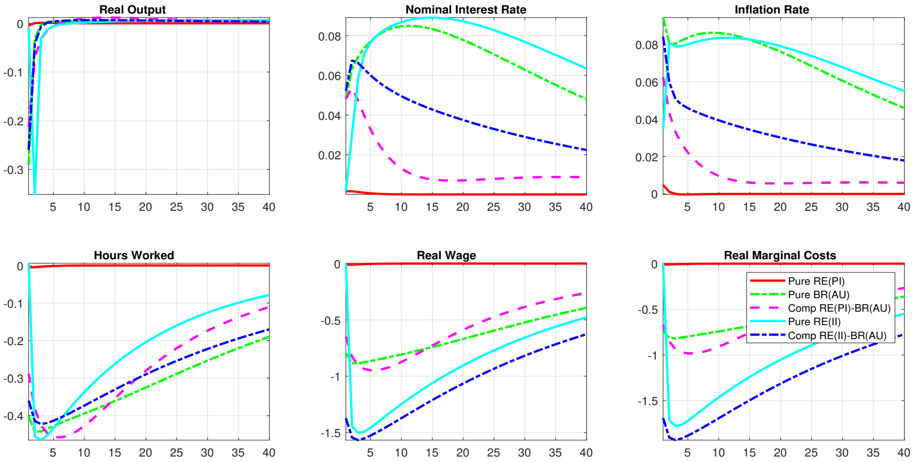

Figure: **6. Estimated Impulse Responses – Mark-up Shock**

Conclusions

- This paper studies a heterogeneous RE-BR model for which BR beliefs of economic agents are about payoff-relevant macroeconomic variables that are exogenous to their decision rules (**Anticipated Utility**).
- **Bayesian Estimation: Likelihood Race Ranking**:
  RE(II)-BR $\succ$ RE(PI)-BR $\succ$ Pure RE(II) $\succ$ Pure BR $\succ$ Pure RE(PI)
- Likelihood Ranking supported by **matching second moments** of the models to data
- These results suggest that persistence can be injected into the NK model to improve data fit in **two contrasting ways**:
  - Bounded-rationality with AE learning through heuristic rules
  - Retaining RE but with II and Kalman-filtering learning.

Adam, K. and Marcet, A. (2011). Internal Rationality, Imperfect Market Knowledge and Asset Prices. *Journal of Economic Theory*, **146**(3), 1224–1252.

Angeletos, G.-M. and Huo, Z. (2021). Myopia and anchoring. *American Economic Review*, **111**(4), 1166–1200.

Angeletos, G.-M. and Lian, C. (2016). *Incomplete Information in Macroeconomics: Accommodating Frictions on Coordination*. Elsevier. Chapter in the Handbook of Macroeconomics.

Branch, W. A. and McGough, B. (2010). Dynamic predictor election in a new keynesian model with heterogeneous agents. *Journal of Economic Dynamics and Control*, **34**(8), 1492–1508.

Branch, W. A. and McGough, B. (2018). Heterogeneous expectations and micro-foundations in macroeconomics. In *Handbook of Computational Economics*, volume 4. Elsevier Science.

Brock, W. A. and Hommes, C. H. (1997). A Rational Route to Randomness. *Econometrica*, **65**, 1059–1095.

Cogley, T. and Sargent, T. J. (2008). Anticipated utility and rational expectations as approximations of bayesian decision making. *International Economic Review*, **49**(1), 185–221.

Collard, F., Dellas, H., and Smets, F. (2009). Imperfect Information and the Business Cycle. *Journal of Monetary Economics*. Forthcoming.

De Grauwe, P. (2011). Animal spirits and monetary policy. *Economic Theory*, **47**(2-3), 423–457.

De Grauwe, P. (2012). Booms and Busts in Economic Activity: A Behavioral Explanation. *Journal of Economic Behavior and Organization*, **83**(3), 484–501.

De Grauwe, P. and Gerba, E. (2018). The role of cognitive limitations and heterogeneous expectations for aggregate production and credit cycles. *Journal of Economic Dynamics and Control*, **91**, 206–236.

De Grauwe, P. and Ji, Y. (2019). *Behavioural Macroeconomics*. Oxford University Press.

De Grauwe, P. and Katwasser, P. R. (2012). Animal Spirits in the Foreign Exchange Market. *Journal of Economic Dynamics and Control*, **36**(8), 1176–1192.

Eusepi, S. and Preston, B. (2011). Expectations, learning, and business cycle fluctuations. *American Economic Review*, **101**(6), 2844–2872.

Evans, G. W. and Honkapohja, S. (2001). *Learning and Expectations in Macroeconomics*. Princeton University Press.

Evans, G. W. and Honkapohja, S. (2009). Learning and Macroeconomics. *Annual Review of Economics*, **1**, 421–449.

Farhi, E. and Werning, I. (2019). Monetary Policy, Bounded Rationality and Incomplete Markets. *American Economic Review*, **109**(11), 3887–3928. Forthcoming.

Fernandez-Villaverde, J., Rubio-Ramirez, J., Sargent, T., and Watson, M. W. (2007). ABC (and Ds) of Understanding VARs. *American Economic Review*, **97**(3), 1021–1026.

Gabaix, X. (2020). A Behavioral New Keynesian Model. *American Economic Review*, **110**(8), 2271–2327.

Garcia-Schmidt, M. and Woodford, M. (2019). Are Low Interest Rates Deflationary? A Paradox of Perfect-Foresight Analysis. *American Economic Review* , **109**(1), 86–120.

Graham, L. and Wright, S. (2010). Information, heterogeneity and market incompleteness. *Journal of Monetary Economics*, **57**(2), 164–174.

Hauk, E., Lanteri, A., and Marcet, A. (2021). Optimal policy with general signal extraction. *Journal of Monetary Economics*, **118**(C), 54–86.

Hommes, C. and Zhu, M. (2014). Behavioural learning equilibria. *Journal of Economic Theory*, **150**, 778–814.

Hommes, C., Calvert Jump, R., and Levine, P. (2019). Learning, Heterogeneity and Complexity in the New Keynesian Model. *Journal of Economic Behaviour and Organization*, **166**(C), 446–470.

Hommes, C. H., Mavromatis, K., Ozden, T., and Zhu, M. (2023). Behavioral learning equilibria in new keynesian model. *Quantitative Economics*, **16**(6). Forthcoming.

Leeper, E. M., Walker, T., and Yang, S. S. (2013). Fiscal foresight and information flow. *Econometrica*, **81**(3), 1115–1145.

Levine, P., Pearlman, J., Perendia, G., and Yang, B. (2012). Endogenous Persistence in an Estimated DSGE Model under Imperfect Information. *Economic Journal*, **122**(565), 1287 – 1312.

Levine, P., Pearlman, J., Wright, S., and Yang, B. (2023). Imperfect Information and Hidden Dynamics. School of Economics Discussion Paper 1223, School of Economics, University of Surrey.

Lippi, M. and Reichlin, L. (1994). VAR analysis, Nonfundamental Representations, Blaschke Matrices. *Journal of Econometrics*, **63**(1), 307–325.

Massaro, D. (2013). Heterogeneous Expectations in Monetary DSGE Models. *Journal of Economic Dynamics and Control*, **37**(3), 680–692.

Minford, A. and Peel, D. (1983). Some Implications of Partial Information Sets in Macroeeconomic Models Embodying Rational Expectations. *Manchester School*, **51**, 235–249.

Nimark, K. (2008). Dynamic Pricing and Imperfect Common Knowledge. *Journal of Monetary Economics*, **55**, 365–382.

Pearlman, J. G. and Sargent, T. J. (2005). Knowing the forecasts of others. *Review of Economic Dynamics*, **8**(2), 480–497.

Pearlman, J. G., Currie, D., and Levine, P. (1986). Rational Expectations Models with Private Information. *Economic Modelling*, **3**(2), 90–105.

Preston, B. (2005). Learning about monetary policy rules when long-horizon expectations matter. *International Journal of Central Banking*, **1**, 81–126.

Rondina, G. and Walker, T. B. (2021). Confounding Dynamics. *Journal of Economic Theory*, **196**.

Svensson, L. E. O. and Woodford, M. (2003). Indicator variables for Optimal Policy. *Journal of Monetary Economics*, **50**(3), 691–720.

Woodford, M. (2013). Macroeconomic Analysis Without the Rational Expectations Hypothesis. *Annual Review of Economics, Annual Reviews*, **5**(1), 303–346.

Woodford, M. (2018). Monetary Policy Analysis when Planning Horizons are Finite. In *NBER Macroeconomics Annual 2018*, volume 33. National Bureau of Economic Research, Inc.