Learning about Bond Prices

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Learning and Bond price volatility:

- Many studies on stock price volatility, fewer on bond price volatility
- We learnt a lot abut stock prices using models where agents learn about prices (internal rationality).
- Recent literature studies the role of expectations on many aspects of the economy.
- How much can we learn about bond yields using models of internally rational learning?

Roadmap

Data

- Sample 1983:12 to 2019:12.
- Monthly data, US government bonds
- Nominal zero-coupon yields
- Surveys: Reuters' Blue Chip Financial Forecasters survey on expected bond yields.
- One-year maturities, leads, forecasting horizons.

Flat Term structure of Yield Volatilities

Table 1: Yields 1st and 2d moments

	$1y$	$2y$	$3y$	$5y$	$7y$	$10y$
$\widehat{Y}^n$	388.83	417.23	441.39	481.19	510.89	541.69
$\widehat{\sigma}_{Y^n}$	289.15	294.34	291.41	281.62	272.50	258.74
$\widehat{corr}(Y^n)_{-1y}$	0.89	0.90	0.91	0.93	0.93	0.93

Hard to reconcile with RE and reasonable serial correlation of expected inflation.

Failure of Expectations Hypothesis

$$Y_{t+1y}^{n-1} - Y_t^n = \alpha^n + \beta^n \frac{Y_t^n - Y_t^1}{n-1} + U_t^n \tag{1}$$

EH says β's ≈ 1

Table 2: Failure of Expectations Hypothesis

	$2y$	$3y$	$5y$	$7y$	$10y$
$\widehat{\beta}^n$	-0.31	-0.50	-0.99	-1.28	-1.67
t-stat	-0.43	-0.60	-1.21	-1.56	-2.16

horizon$=$ 1y here and in all tables below
n means "n years"

Failure of EH: "Modern" version

Let Q_t^n be the price of a zero-coupon bond of maturity n.

Define excess bond returns

$$xr_{t+1y}^n = log\left(\frac{Q_{t+1y}^{n-1}}{Q_t^n}\right) - log\left(\frac{1}{Q_t^1}\right)$$

Or, in terms of yields $Y_t^n \equiv -\frac{log(Q_t^n)}{n}$

$$xr_{t+1y}^n = -(n-1)Y_{t+1y}^{n-1} + nY_t^n - Y_t^1$$

Failure of EH: "Modern" version

EH can also be expressed in terms of *xr* forecasts:

$$xr^n_{t+1y} = \alpha^{xr}_n + b^{xr}_n \frac{Y^n_t - Y^1_t}{n-1} + U^n_t$$

EH implies b^{xr}'s ≈ 0.
LHS are marginal profits from investing in long bonds.

Table 3: Failure of EH "modern" version

	$2y$	$3y$	$6y$	$8y$	$11y$
$\widehat{b^{xr}_n}$	0.61	2.54	10.69	15.87	24.17
$t-stat$	0.52	1.14	2.40	2.87	3.54
R^2	0.32	1.59	6.00	7.88	10.41

High volatility of excess bond returns

$$xr^n_{t+1y} = log\left(\frac{Q^{n-1}_{t+1y}}{Q^n_t}\right) - log\left(\frac{1}{Q^1_t}\right)$$

Table 4: Excess Returns volatility

maturity n	$2y$	$3y$	$5y$	$7y$	$10y$
$\widehat{\sigma}_{xr^n}$	136.60	256.20	455.74	636.45	880.10

Excess volatility of yields, Intuition

$$\begin{aligned} Q_t^n &= Q_t^1 \; E_t^{\mathbb{Q}}\left(Q_{t+1}^{n-1}\right) \\ &= E_t^{\mathbb{Q}}\left(\prod_{i=0}^{n-1} Q_{t+i}^1\right) \end{aligned}$$

where $\mathbb{Q}$ refers to "risk-neutral" distribution.
Taking log-linear approximation

$$Y_t^n \approx E_t^{\mathbb{Q}}\left(\frac{\sum_{i=0}^{n-1} Y_{t+i}^1}{n}\right)$$

therefore we expect $\sigma_{Y_t^n} < \sigma_{Y_t^1}$

Excess volatility of excess bond returns, Intuition

Furthermore

$$\begin{aligned} xr_{t+1}^n &= nY_t^n - (n-1)Y_{t+1}^{n-1} - Y_t^1 \\ &\approx (E_t^{\mathbb{Q}} - E_{t+1}^{\mathbb{Q}}) \left(\sum_{i=1}^{n-1} Y_{t+i}^1 \right) \end{aligned}$$

so $\sigma^2_{xr_t^n}$ variance of a change in expectations, so it should be small.

Bond survey expectations

- Much recent work on surveys about asset prices
- Giacoletti, Laursen and Singleton (2020) and Singleton (2021) AFA Presidential Address (GLS) :
  - Describe investor bond yield expectations
  - Investors (consensus) bond yield expectations are on average similar to those arising from an affine model with principal components

RE Tests

Adam, Marcet, Beutel (2017)
Let $\mathcal{E}$ denote survey expectations.
Consider a regression like EH but with expectations in left hand side

$$\mathcal{E}_t(xr^n_{t+1y}) \;=\; \alpha^{\mathcal{E}}_n + b^{\mathcal{E}}_n \frac{Y^n_t - Y^1_t}{n-1} + U^{n,\mathcal{E}}_t$$

Under RE $b^{xr}_n = b^{\mathcal{E}}_n$.

Related to Coibion and Gorodnichenko approach, two differences:
-CG work with differenced regression
-CG introduce forecast errors or forecast updates on the RHS

RE Regressions

$$xr^n_{t+1y} = \alpha^{xr}_n + b^{xr}_n \frac{Y^n_t - Y^1_t}{n-1} + U^{n,xr}_t$$

$$\mathcal{E}_t(xr^n_{t+1y}) = \alpha^{\mathcal{E}}_n + b^{\mathcal{E}}_n \frac{Y^n_t - Y^1_t}{n-1} + U^{n,\mathcal{E}}_t$$

Table 5: RE Regressions

	$2y$	$3y$	$6y$	$8y$	$11y$
$\widehat{b^{xr}_n}$	0.61	2.54	10.69	15.87	24.17
$\widehat{b^{\mathcal{E}}_n}$	0.16	0.80	2.50	1.27	3.69
t-stat $b^{xr}_n = b^{\mathcal{E}}_n$	0.37	0.76	1.75	2.47	2.70

Surveys underestimate the role of the slope in predicting xr^n_{t+1y}

$t - stat$ rejects RE, more strongly for long maturities

Test would be valid for any regressor in the information set at t

RE Regressions, additional versions

Alternative regressors:

-PC_t^2: 2d Estimated Principal Component

-$\widehat{\mathbf{s}}_t$: Investor's estimate of underlying slope according to our model

Table 6: RE Regressions

regressor		$2y$	$3y$	$6y$	$8y$	$11y$
PC_t^2	$\widehat{b_n^{xr}}$	0.06	0.13	0.43	0.61	0.89
	$\widehat{b_n^{\mathcal{E}}}$	0.02	0.05	0.14	0.13	0.19
	t-stat $b_n^{xr} = b_n^{\mathcal{E}}$	0.85	1.11	1.98	2.57	2.87
$\widehat{\mathbf{s}}_{t-3}$	$\widehat{b_n^{xr}}$	1.04	2.16	9.14	13.90	22.67
	$\widehat{b_n^{\mathcal{E}}}$	0.41	0.57	0.28	−2.28	−2.86
	t-stat $b_n^{xr} = b_n^{\mathcal{E}}$	0.77	0.95	2.12	2.79	3.08
$\widehat{\mathbf{s}}_t, \widehat{\mathbf{s}}_{t-3}$	p-value $b_n^{xr} = b_n^{\mathcal{E}}$	0.70	0.59	0.09	0.01	0.00

Roadmap

Literature

- Difficult to match EH-failure and volatilities with standard equilibrium asset pricing models.
  High volatility of *xr* documented since Shiller (1979) but very rarely addressed.
- Much progress has been made:
  - Time-varying reward for risk:
    Dai and Singleton (2002), Wachter (2006), Gabaix (2008), Bansal and Shaliastovich (2013)
  - Bayesian/RE: agents disagree about forecast of inflation but they know the pricing function:
    Xiong and Yan (2010), Buraschi and Whelan (2022), Buraschi and Jiltsov (2007), Ehling Gallmeyer, Heyerdahl-Larsen and Illeditsch (2018).

Literature

- Tables 1-4 remain hard to explain. Giglio et al (2018) argue standard models can't explain observations.
- Tables 5-6 are a new puzzle
- It matters whether the model is RE with time-varying risk or learning!!!

Literature, Adaptive Learning, partial information and bond yields

- Sinha (2009), Piazzesi and Schneider (2006), Laubach, Tetlow, and Williams (2007) models of adaptive learning about how macro variables or inflation influence bond prices.
- Ellison Tischbirekz (2017) beauty contest.

This Project

Improve match of the data under *Internal Rationality*.

- investors have a *consistent system of beliefs* about bond prices
- investors are *fully rational* given their beliefs.

A "minor" deviation from standard RE paradigm

Beliefs about bond prices are not RE, but the are quite good in equilibrium:

- close to RE beliefs
- model converges to RE
- "close" to the true distribution, (hard to reject using observed data)
- investors' beliefs match some features of expectations surveys

Roadmap

A model of bond prices

N zero-coupon nominal bonds of maturity $n = 1, ..., N$

Bond prices $Q_t = \left[Q_t^1, ..., Q_t^N\right]'$.

Homogeneous investors with $E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \; u(c_t)$

Investor beliefs represented by $\mathcal{P}$.

All bonds with $n > 1$ are resold one period after being purchased.
Budget constraint:

$$P_t c_t + \sum_{n=1}^{N} Q_t^n B_t^n = w_t P_t + \sum_{n=1}^{N} Q_t^{n-1} B_{t-1}^n$$

$Q_t^0 = 1$

A model of Bond Prices

Assume bond limits:

$$\overline{B} \geq B_t^n \geq \underline{B} \tag{2}$$

Inflation process:

$$\log P_t - \log P_{t-1} \equiv \pi_t \;\; = \;\; \mu(1-\rho) + \rho\pi_{t-1} + \eta_t^\pi$$

$$\eta_t^\pi \sim \mathcal{N}(0, \sigma^2_{\eta^\pi}), \text{ iid.}$$

The Model

Equilibrium conditions:

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2$$

The short rate Y_t^1 is exogenous, given by

$$Y_t^1 = -\log(\delta) + \pi_t^e + \xi_t$$

ξ_t a mean-zero exogenous process (policy shock).
So bond limit is binding for $n = 1$.

Roadmap

1. Yields Data, Some Puzzles
2. Literature
3. A Model
4. Internal Rationality
5. A Simple Model of Learning about the Slope

Investors' probability space under IR

$$\Omega \equiv \Omega_Q \times \Omega_{\pi,\xi}$$

where Ω_Q contains possible histories of $Q = [Q^n]_{n=1}^N$, and $\Omega_{\pi,\xi}$ contains possible histories of inflation π, ξ.

Investors have a belief system $\mathcal{P}$ with (consistent) probabilities on Ω.

RE is a special case when agents' beliefs about π, ξ are correct and they know the true pricing function determining Q.

Bayesian/RE a special case when agents' don't know everything about π, ξ but they know the true pricing function determining Q.

Model with learning about Q

However if agents don't know the pricing function prices are given by

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2 \tag{3}$$

Therefore we need to state investors' beliefs about joint distribution of Q, π.

Simple case

Assume for the rest of the talk

-ξ_t iid mean zero

-no risk aversion: $u(c) = c$

-Investors know the inflation process

- $\rho = 1$

RE prices

Under RE

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2$$

becomes

$$\begin{aligned} Q_t^{RE,n} &= \delta^n \; E_t \left(\frac{P_t}{P_{t+n}} \right) \\ Y_t^{RE,n} &= -log \delta + \pi_t \end{aligned}$$

Model with learning about Q

Agents' beliefs about Q's are given by a perceived yield curve (in blue we show investors beliefs equations)

$$Y_t^n = -log\delta + \pi_t^e + \mathbf{s}_t n + u_t^n \tag{4}$$

$\mathbf{s}$ a perceived, unobserved, "underlying slope".

In other words:

- Investors know the level of the yield curve and how inflation determines yields
- but (different from RE) agents believe there is a time-varying slope $\mathbf{s}_t$

A popular myth among academics:

"Investors with beliefs about bond prices Q are irrational."
The justification for this myth:
If investors are rational they can apply LIE:

$$Q_t^n = \delta^n \ E_t^{\mathcal{P}} \left(\frac{u'(c_{t+n})}{u'(c_t)} \frac{P_t}{P_{t+n}} \right)$$

And if they know inflation process then

$$Q_t^n = \delta^n \ E_t \left(\frac{u'(c_{t+n})}{u'(c_t)} \frac{P_t}{P_{t+n}} \right) = Q_t^{n,RE}$$

But:

Two reasons why LIE does not apply

1. LIE can not be applied with individual knowledge, it needs market knowledge.
   (Adam Marcet 2011)
2. Under IR $u'(c_{t+n})$ depends on future prices thus on price beliefs, so $E_t^{\mathcal{P}}$ can not become E_t
   (Adam, Marcet, Beutel, 2017)

Therefore investors with a consistent system of beliefs $\mathcal{P}$ do not see anything that contradicts their price beliefs.

Roadmap

A Simple Model

Recall investors' perceived yield curve

$$Y_t^n = -\log\delta + \mathbf{s}_t n + \pi_t^e + u_t^n$$

To complete investors' beliefs assume investors believe

$$\mathbf{s}_t = \mathbf{s}_{t-1} + w_t$$

Optimal filtering implies

$$E_t^{\mathcal{P}}(\mathbf{s}_t) \equiv \widehat{\mathbf{s}}_t = \widehat{\mathbf{s}}_{t-1} + g(s_{t-1} - \widehat{\mathbf{s}}_{t-1})$$

where s_t is the best estimate of period-t slope:

$$s_t = E_t^{\mathcal{P}}(\mathbf{s}_t \mid Y_t^N, ..., Y_t^1)$$

to be specified later. Then

$$E_t^{\mathcal{P}}(Y_{t+1}^n) = -\log\delta + \widehat{\mathbf{s}}_t n + \pi_t^e$$

Learning Equilibrium

Then, equilibrium under learning with a log-linear approximation, actual yield curve (equations that hold in equilibrium in brown)

$$Y_t^n = -\log \delta + \widehat{\mathbf{s}}_t \; n \left(\frac{n-1}{n} \right)^2 + \pi_t^e$$

Beliefs "near-rational"

Recall perceived yield curve:

$$Y_t^n = -\log \delta + \widehat{\mathbf{s}}_t n + \pi_t^e + \widehat{u}_t^n$$

Conceptually not a large deviation from RE:

- beliefs close to RE beliefs if $\sigma_w \approx 0$.
- just because agents believe there is a time-varying slope there is a time-varying slope (see below)
- model converges to RE (see below)
- perceived yield curve similar to actual yield curve, beliefs hard to reject

A simple estimate s_t (the perceived slope at t)

In principle investors should use all maturities Y_t^n to estimate $\mathbf{s}_t$.

A simple case:

assume $var^{\mathcal{P}}(u_t^m) << var^{\mathcal{P}}(u_t^n)$ for a reference bond of maturity m.

GLS means use only $n = m, 1$ to filter $\mathbf{s}$ with t-information:

$$E_t^{\mathcal{P}}(\mathbf{s}_t \mid Y_t^N, ..., Y_t^1) = s_t = \frac{Y_t^m - Y_t^1}{m - 1}$$

Learning Equilibrium with "simple" estimate s_t

Plugging in equilibrium yields

$$Y_t^n = -\log(\delta) + \widehat{\mathbf{s}}_t \; n \left(\frac{n-1}{n}\right)^2 + \pi_t^e$$

$$s_t = \frac{Y_t^m - Y_t^1}{m-1} = \widehat{\mathbf{s}}_t \; \frac{m-1}{m} - \frac{\xi_{t-1}}{m-1}$$

$$\widehat{\mathbf{s}}_t = \widehat{\mathbf{s}}_{t-1} + g(s_{t-1} - \widehat{\mathbf{s}}_{t-1})$$

So, under self-referential learning:

$$\uparrow s_t = \tfrac{Y_t^m - Y_t^1}{m-1} \implies \uparrow \widehat{\mathbf{s}}_{t+1} \implies \uparrow Y_{t+1}^n \implies \uparrow s_{t+1} = \tfrac{Y_{t+1}^m - Y_{t+1}^1}{m-1} \; \ldots$$

The T-map

When agents' perceived expectation (PLM) is

$$E_t^{\mathcal{P}} \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} = \widehat{\mathbf{s}}_t$$

then the true expectation (ALM) is

$$E_t \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} \equiv \frac{m-1}{m} \widehat{\mathbf{s}}_t \equiv T(\widehat{\mathbf{s}}_t)$$

Convergence to RE

We have $T' < 1$.
Standard theorems in the learning literature imply that model converges to RE:

$$Y_t \to Y_t^{RE}$$

if either

- OLS instead of constant gain ($g = 1/t$) or
- $g \to 0$

(Marcet and Sargent (1989), Williams (2017), Evans Honkapohja (2002))

Fluctuations around RE

Furthermore, for $g > 0$, plugging in equilibrium value of s_t

$$\widehat{\mathbf{s}}_t = \left(1 - \frac{g}{m}\right) \widehat{\mathbf{s}}_{t-1} - g \frac{\xi_{t-1}}{m-1}$$

So $\widehat{\mathbf{s}}_t$ very persistent and $E(\widehat{\mathbf{s}}_t) = 0$.

The learning model fluctuates around RE.

Collecting equilibrium relationships

$$\widehat{\mathbf{s}}_t = \left(1 - \frac{g}{m}\right)\widehat{\mathbf{s}}_{t-1} - g\frac{\xi_{t-1}}{m-1}$$

$$Y_t^n = -\log(\delta) + \widehat{\mathbf{s}}_t\,\frac{(n-1)^2}{n} + \pi_t^e$$

$$xr_{t+1}^n = \left[2n - 3 + (n-2)^2\frac{g}{m}\right]\widehat{\mathbf{s}}_t$$

$$+(n-1)\eta_{t+1}^{\pi} + \left[\frac{(n-2)^2 g}{m-1} - 1\right]\xi_t$$

$$E_t^{\mathcal{P}}(xr_{t+1}^n) = -\log\delta - \xi_t$$

With this we can easily compute all the variances and correlations.

Generic Properties of the Learning Model

In this simple case with unit root inflation we see analytically:

- $var(\widehat{\mathbf{s}}_t)$ contributes to increase $var(Y^n)$ and $var(xr^n)$. More so for large n.
- Consistent with RE regression:
  - $\widehat{b}_n^{xr} > 0$, increases with n
  - $\widehat{b}_n^{xr} > \widehat{b}_n^{\mathcal{E}}$
  - $\widehat{b}_n^{xr} - \widehat{b}_n^{\mathcal{E}}$, increases with n

$\widehat{b}^{xr}_n > \widehat{b}^{\mathcal{E}}_n$

Recall PLM and ALM:

$$Y^n_t = -\log\delta + \left(\frac{n-1}{n}\right)^2 \widehat{\mathbf{s}}_t \; n + \pi^e_t$$

$$Y^n_t = -\log\delta + \mathbf{s}_t n + \pi^e_t + u^n_t$$

So forecasts of future yields

$$E_t(Y^n_{t+1}) = -\log\delta + \left(\frac{n-1}{n}\right)^2 \left(1 - \frac{g}{m}\right) \widehat{\mathbf{s}}_t \; n + \pi^e_t$$

$$E^{\mathcal{P}}_t(Y^n_{t+1}) = -\log\delta + \widehat{\mathbf{s}}_t n + \pi^e_t + u^n_t$$

So agents *over*predict the influence of $\widehat{\mathbf{s}}_t$ on future yields.
Therefore investors *under*predict the influence of $\widehat{\mathbf{s}}_t$ on future *xr*
It can explain that in the data $\widehat{b}^{xr}_n > \widehat{b}^{\mathcal{E}}_n$.

Starting to fit a Calibrated model

- inflation is AR(1) with $\rho < 1$.
- use a yearly model
- Calibrate to sample values σ_{π^e}, ρ, σ_{η^π} ,σ_ξ for $\xi = Y^1 - \pi^e$.

Only one free parameter g.
Fiddled with it to select $g = 0.3$

Comparing with the data: Calibration

Table 7: Model fit

	$3y$			$10y$		
	data	*RE*	*Learn*	*Data*	*RE*	*Learn*
σ_{Y^n}	291	85	96	258	58	175
$\rho_{Y^n,-1y}$	0.91	0.94	0.94	0.93	0.94	0.97
σ_{xr^n}	254	56	185	871	201	642
$\widehat{b}_n^{xr}$	3.22	0	2.02	24.6	0	7.01
$\widehat{b}_n^{\mathcal{E}}$	1.02	0	1.93	2.5	0	4.00

To be done

- policy shock ξ serially correlated as in the data
- monthly model
- learning about the intercept in the yield curve
- risk aversion
- disagreement
- learning when investors observe factors

Conclusion

A very simple model suggests learning about the slope helps to explain the data.

It is consistent with RE regressions

Moves second moments in the right direction.

But room for improvement!