Peer Group Externalities and Learning Incentives:

A Theory of Nerd Behavior

by

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Abstract

This paper investigates peer group externalities in a model where individuals seek to learn their abilities to make career decisions. Such externalities are regarded as being of importance in understanding educational performance and there are many studies, beginning with the Coleman Report, which have found them to be empirically significant. The basic idea is that individuals respond positively to being grouped with more industrious individuals. In our model, individuals receive test scores which are imperfect indicators of their ability. Some sources of uncertainty, such as how difficult the test is, are common to the group. By having peers who put in effort, one can obtain a more precise estimate of this source of uncertainty. Effort by one individual may thus convey an externality for others who take the same test. Our model displays two key properties. First, utility is not concave in effort. This is quite standard in this type of model. Second, there will be multiple symmetric Nash equilibria in effort for some values of effort costs. One of these equilibria may have individuals putting in no effort at all. Thus identical populations could be found putting in quite different effort levels in equilibrium.

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I. Introduction:

This paper develops a simple model of peer group effects based on learning externalities. It is built as a partial explanation for the widespread finding that externalities are important in understanding educational performance. It may also have other applications in situations where there are interdependent learning decisions. We motivate the idea in the context of the many studies, beginning with the Coleman Report (see Coleman et al. (1966)), which have found peer group externalities to be empirically significant, the main idea being that individuals respond positively to being grouped with more industrious individuals.

Our paper shows that such externalities can be explained using a model where individuals are uncertain about their own abilities and face a "shock" which is common across peers and interacts in a particular way with an individual's ability. The model displays strategic complementarities in effort choice and hence the possibility of coordination failures resulting in low effort equilibria. We also discuss other models of such externalities and how different assumptions in our model affect the results. To anticipate, we conclude that the nature of peer group externalities in learning models is sensitive to the particular specification of the model. However, in comparison with other models, our approach does offer a parsimonious approach and, we believe, a reasonable model of the phenomenon. We also discuss some empirical implications which could set our model apart from others.

The structure of the paper is as follows. In the next section, we lay out the main ideas in our approach and relate it to the literature. Section III lays out the formal model and section IV develops the main results. In section V we discuss the specificity of the assumptions needed to derive our results and alternative approaches that we could have taken. Section VI discusses further issues that can be addressed using our model. We consider (i) the use of prizes to improve incentives in peer group models, (ii) the impact of matching individuals with different levels effort motivation and (iii) the issues which arise in choosing between school based and national testing of students. Concluding remarks are offered in section VII.

II. Overview of The Literature

Our model focuses on a world where education serves as a process by which individuals learn their abilities. It is a somewhat different, therefore, to that taken in the two paradigms of educational choice in the existing literature. Individuals do not wish to signal a known ability level in our model[1] nor do they acquire skills in the educational process[2]. The idea here is that individuals invest in education since they are intent on making the occupational choice which is most appropriate to their ability.

The education process is characterized as a "test" and the output is a grade used to make a career choice. Individuals' test scores depend upon effort, ability and two random variables. They are used to form an imprecise estimate of ability. Tests suffer from two sources of "error"; an idiosyncratic risk[3] and a common risk, reflecting the difficulty of the test. This common component, as we shall see below, is what makes peer groups important.

Most systems of grading used in practice, convey some information about an individual's performance relative to the group taking the test. Even the normal "grading curve" where individuals are graded on a scale of A through F is not absolute across schools – different schools requiring different performance standards to achieve the same grade. This makes it difficult for any individual to know their ability in the wider population using their grades at a particular institution.[4]

We construct a model where the task of inferring one's ability from a test score is easier if an individual has a peer group consisting of hard workers. The following explains the main. Suppose that I know that all my peers are likely to be of high ability and work very hard, and that when I also work, I receive a grade A. This provides me with a good reason for supposing that I am also of high ability. Hence, being mixed with good individuals who put in effort makes it more likely that I will learn more about my abilities by working hard myself. Thus my incentive to put in effort is affected by that of others. In a world where my peers are slacking, a grade A is worth much less and has lower informational content. Thus, if I expect lower effort from my peers, I shall also tend to put in less effort myself. There will be schools where everyone is optimally putting in high effort and others where everyone is putting in low effort. Furthermore, one might find that the same individual would find it worthwhile to put in effort with one set of peers but not with another.

Some of these ideas are not new. Similar observations lead Davis (1966) to draw an analogy between a college campus and a frog pond. As in a frog pond individuals will tend to look to their peers as immediate points of reference. He used data from a large sample of graduating college seniors and

> "offers the following interpretation of (his) data: (a) In making career decisions regarding the high–performance fields (which generally require graduate training), the students judgment of his own academic ability plays an important role. (b) In the absence of any objective evidence, students tend to evaluate their academic abilities by comparison with other students. (c) Most of the other students one knows are those one's own campus, and since GPA's are reasonably public information, they become the accepted yardstick. (d) Comparisons across campuses are relatively rare, and where they take place it is difficult to arrive at an unambiguous conclusion because institutional differences are not well publicized; even when these differences are known, there is no convenient scale comparable to GPA for drawing conclusions. (e) Since more conclusions are drawn on the basis of GPA standing on the local campus than by comparison with students on other campuses, GPA is a more important variable in influencing self–evaluations and consequently, career decisions." Davis (1967) page 25.

Davis describes the above as an account based on "relative deprivation" – the idea being that, at schools with high average GPA's, people tend to feel relatively less happy with their own GPA. The reason for this is that individuals are prone to evaluate their position relative to others.[5] Such ideas are occasionally taken seriously in economics, as in the work of Akerlof (1976, 1980, 1982)[6]. He, for example, discusses models of job effort in which individuals prefer to work hard only as long as others also do. In common with the model developed in this paper, models based on this assumption also exhibit multiple equilibria.[7]

For most arguments of this kind, the tendency of individuals to use reference groups is taken as a datum of social life. In contrast to the above models, our analysis does not presume that preferences of agents are interdependent. Instead, we derive a behavioral interdependence from a theory of testing when preference interdependence is absent. We do not believe, however, that the latter is uninteresting or unimportant. We simply wish to know how far one can go towards explaining peer group influences without invoking such assumptions. It should not be surprising that our model has many similar implications to some sociological theories of reference group behavior.[8] However, the conclusions that we draw relating peer group effects to the type of test used to evaluate individuals would be hard to justify sociologically and offers a potentially testable implication of our model against one based entirely on preference interdependence.[9]

III. The Model

We consider two person peer groups which can be thought of as schools[10]. Each individual in a school takes the same test, performance on which depends upon random influences as well as her effort and ability. Effort committed to preparing for the test is assumed to be known by both group members.[11] After the grades are revealed, individuals learn both scores. This information is used to form an estimate of ability which informs an individual's career choice.[12]

Individual i's test score depends upon her ability (θ_i), effort (ℓ_i) and two random variables – idiosyncratic risk (ϵ_i) and common risk (η). Effort is best thought of as being the amount of preparation that an individual engages in prior to taking the test. The variable η measures how hard the test is[13], while ϵ_i is luck (good or bad). Since individuals do not know their ability levels, θ_i is also random. We make

Assumption 1: The random variables $(\theta_1,\theta_2,\eta,\epsilon_1\epsilon_2)$ are independently and normally distributed, i.e., $\epsilon \sim N(0,1)$, $\eta \sim N(0,\psi^2)$ and $\theta_i \sim N(\bar{\theta}_i,\sigma^2)$.

While the mean ability level can vary between the two individuals, all other parameters of the distributions are common.

Our most important assumption governs the production technology through which the random variables are combined with effort to produce a test score x_i.

Assumption 2: The production technology is linear in effort and the common component interacts *multiplicatively* with effort, i.e.

$$x_i = (\theta_i + \eta)\ell_i + \epsilon_i. \tag{3.1}$$

The key feature of (3.1), to which we return in section V, is that the interaction between η and effort is multiplicative. Hence, if $\ell_i = 0$, then the test score just depends on luck.

The pair of test scores – $x \equiv (x_i,x_j)$ – is used to make career decisions. Our model of occupational choice is very simple. Individuals are employed by firms which provide the capital and within these firms individuals pursue particular occupations, of which there is a continuum indexed by γ.

Assumption 3: An individual of ability θ who enters occupation γ generates surplus equal to

$$u(\gamma,\theta) = \gamma - \frac{\alpha}{2}(\theta - \gamma)^2. \tag{3.2}$$

Occupations are ordered according to their productivity – a higher γ, other things being equal, being associated with greater surplus. However, individuals also wish to locate in occupations which are appropriate to their ability level. This is represented by the second term in (3.2). An individual of ability θ who takes a job which is different from his true ability generates less surplus. This works in both directions – high ability people will be bored in low γ jobs, while low ability people will be stretched by taking high γ jobs.

Firms claim part of the surplus in return for providing capital. However, the sharing rule between firms and workers is given, so that individuals will be allocated to careers to maximize the surplus they generate. Given an estimate of θ_i based on the linear regression of θ_i on x and ℓ ($\equiv (\ell_i,\ell_j)$) (see Proposition 1 below for details), the optimal occupational choice will be

$$\gamma_i^*(x,\ell) = \underset{\gamma}{\text{argmax}} \, (E_\theta^i\{u(\theta,\gamma) |\, x,\ell\}), \tag{3.3}$$

where $E_\theta^i\{\cdot\,|\,\cdot\}$ denotes taking the conditional expectation over the subscripted variable for individual i. Letting $\overline{\theta}_i(x,\ell) \equiv E_\theta^i\{\theta |\, x,\ell\}$, the first order condition can be arranged to yield

$$\gamma_i^* = \overline{\theta}_i(x,\ell) + 1/\alpha. \tag{3.4}$$

Thus an individual will choose a career which stretches her ability by a factor $1/\alpha$, reflecting the costs of being "misallocated".

Substituting (3.4) into (3.3) yields

$$V^i(x,\ell) \equiv E_\theta^i\{u(\theta,\gamma_i^*) |\, x,\ell\} = \overline{\theta}_i(x,\ell) + 1/2\alpha - \alpha \text{Var}\{\theta |\, \ell\}/3. \tag{3.5}$$

The last term denotes the conditional variance of θ which is independent of x and i under Assumptions 1 and 2 (see Proposition 1 below for details).

We consider a linear cost of effort and seek a Nash equilibrium in effort levels. We have not yet shown that such equilibria exist and even if one does, it may not be unique. Such concerns are important and we return to them below. To begin with, we focus on symmetric Nash equilibria where individuals share the same cost of effort. Since individuals choose effort before x is known, we take expectations of (3.5) over x to find the relevant "indirect" utility function. Define this as $W^i(\ell) \equiv E_x\{V^i(x,\ell) \mid \ell\}$. The effort choice will satisfy

$$\text{(3.6)} \qquad \overset{*}{\ell}_i(\ell_j;c) \in \underset{\ell_i}{\operatorname{argmax}} \{W^i(\ell) - c\ell_i\},$$

where c is the marginal cost of effort, satisfying $c \in [\underline{c},\infty)$, where $\underline{c} > 0$. Observing that $E_x\{\overline{\theta}_i(x,\ell) \mid \ell\} = \overline{\theta}_i$, the maximization problem becomes

$$\text{(3.6}'\text{)} \qquad \underset{\ell_i}{\operatorname{Max}} \{\overline{\theta}_i + \frac{1}{2\alpha}\left[1 - \alpha^2 \operatorname{Var}\{\theta \mid \ell\}\right] - c\ell_i\}.$$

Thus effort is chosen to minimize the variance of θ given the cost of effort; effort reduces the costly uncertainty of being mis–allocated in one's career choice.

A Nash equilibrium in effort levels is found in the usual way by examining the reaction functions derived from (3.6′). Provided that the function $W^i(\cdot,\ell_j)$ is quasi–concave in ℓ_i, then effort levels found by looking at the first order conditions associated with (3.6′) will constitute the solutions. However, our model is not necessarily well behaved, and we need to check whether it is better to put in any effort all. Thus, we compare the consumer's utility level at a strictly positive pair of Nash equilibrium effort levels – $(\overset{*}{\ell}_i,\overset{*}{\ell}_j)$ – with that achieved by putting in no effort at all. This depends upon the sign of

$$\text{(3.7)} \qquad \frac{\alpha}{2}\left[\operatorname{Var}\{\theta \mid (0,\overset{*}{\ell}_j(0;c))\} - \operatorname{Var}\{\theta \mid (\overset{*}{\ell}_i,\overset{*}{\ell}_j)\}\right] - c\overset{*}{\ell}_i(\overset{*}{\ell}_j;c) > 0,$$

where $\overset{*}{\ell}_j(0;c)$ denotes individual j's best response when i is putting in no effort.

This completes the basic model. The next section is devoted to exploring properties of the equilibrium and to expositing the peer group externalities which result.

IV. Properties of the Equilibria

Our first task is to use the linearity and normality assumptions to determine the statistical inference problem which an individual must solve to estimate her ability, i.e. forming an expectation of θ_i conditional on x and ℓ. Using standard results for a joint normal distribution, we obtain:

Proposition 1: The expectation of individual i's ability conditional on (x,ℓ) is normally distributed with mean and variance given by

$$\overline{\theta}_i(x,\ell) = \overline{\theta}_i + \beta\left[\left(\frac{x_i}{\ell_i} - \overline{\theta}_i\right) - \gamma\left(\frac{x_j}{\ell_j} - \overline{\theta}_j\right)\right] \text{ and } \mathrm{Var}(\theta|\,\ell\} = \frac{\kappa\sigma^2}{\kappa + \sigma^2},$$

where $\beta = \sigma^2/(\kappa+\sigma^2)$, $\gamma = \psi^2\ell_j^2/(1+(\psi^2+\sigma^2)\ell_j^2)$ and $\kappa(\ell_i,\ell_j) \equiv \dfrac{1+\psi^2(\ell_i^2+\ell_j^2)+\sigma^2(\ell_j^2(1+\psi^2\ell_i^2))}{\ell_i^2((\psi^2+\sigma^2)\ell_j^2+1)}$.

Proof: See Appendix. □

This conditional expectation may be explained as a two stage least squares estimate of θ_i. At the first stage, η is regressed on x_j/ℓ_j and its *predicted* value $(\hat{\eta})$ is used at the second stage regression of θ_i on x_i/ℓ_i and $\hat{\eta}$. The parameter γ is the regression coefficient obtained at the first stage, while β is that obtained at the second. The result is convenient for obtaining a precise expression for the variance of θ conditional on effort levels.

Next, we use Proposition 1 to investigate the Nash equilibrium in effort levels. To do so, we need some properties of the function $W^i(\ell)$ defined above. These are given in:

Proposition 2: The utility function $W^i(\ell)$ has the following properties

(a) it is non–decreasing in ℓ_i with $\partial W^i(0,\ell_j)/\partial \ell_i = 0$ for all ℓ_j.

(b) increasing in ℓ_j for all ℓ_i.

(c) there exists an $\hat{\ell}_i$ such that $W^i(\cdot,\ell_j)$ is convex in ℓ_i for all $\ell_i \leq \hat{\ell}_i$ and concave otherwise.

(d) $\partial^2 W^i(\ell)/\partial \ell_i \partial \ell_j \geq 0$, with strict inequality if $\ell_i > 0$.

Proof: See Appendix. □

This Proposition enables us to draw some simple diagrams to illustrate the results. Figure 1 illustrates an individual's utility and marginal utility functions together. An important feature of our model is that utility is not concave in effort. This, combined with the third result, shows that we have the elongated "S" shape for the utility function as illustrated in Figure 1.

The second and fourth results relate to the peer group effect in our model. Individual i is better off if individual j puts in effort. Property (d) of the Proposition says that the *marginal* value of effort is also higher to individual i if j puts in effort, so that effort levels are strategic complements.

Both of these effects depend on ψ^2 being positive. In fact, it is easy to check – see the proof of Proposition 2 in the Appendix for further details – that $\partial \kappa(\ell)/\partial \ell_j$ is proportional to ψ^2. Thus the peer group externality is stronger, the larger is ψ^2 and would disappear completely were the variance of the common component equal to zero. This makes sense, since it is the common shock to individuals' test scores which makes the other individual's effort valuable.

The presence of a peer group externality raises the possibility of there being multiple Nash equilibria in effort. There are two main possibilities. First, since reaction functions are upward sloping, there may be multiple interior symmetric equilibria, each involving a different level of effort. However, even if there is a unique interior equilibrium, the non–convexity that we demonstrated in Proposition 2, together with the peer group effect, gives rise to the possibility of a there simultaneously existing an equilibrium at a corner solution where effort is zero and an interior equilibrium with positive effort. It is this latter possibility to which we will direct most of our attention. This is formally stated in

Proposition 3: All equilibria are symmetric. Further, if $\underline{c}$ is low enough, then we can divide the values of c into three intervals:

(i) For $c \in [\underline{c}, c_A)$ the only equilibria have $\ell_i > 0$, $i = 1,2$.

(ii) For $c \in [c_A, c_B]$ there are at least two distinct equilibria one of which has $\ell_i = 0$, $i = 1,2$.

(iii) For $c \in (c_B, \infty)$ the only equilibria have $\ell_i = 0$, $i = 1,2$.

Proof: See Appendix. □

This Proposition is illustrated in Figure 2. Each curve represents the marginal utility of effort. The lower one is drawn assuming that individual j is putting in no effort while the higher one assumes a positive level of effort $- \ell_j^*$. Consider the lower of the two curves. We wish to compare choosing $\ell_i^*(0;c)$ and zero by looking at the individual's utility level net of effort cost. Whether it is worthwhile putting in effort depends upon the relative sizes of the areas A and B. We have drawn the case where A > B so that individual i prefers choosing effort equal to zero and since individuals are identical, this is a Nash equilibrium.

Next consider the higher of the two curves. If this is the marginal utility of effort schedule that individual i faces, then either zero or $\ell_i^*(\ell_j^*;c)$ is the effort level which is chosen. The individual will prefer the latter provided that $B + B' > A - A'$, which is true in the diagram as we have drawn it. This will be a Nash equilibrium if $\ell_j^* = \ell_j(\ell_i^*;c))$, i.e. both individuals are choosing effort levels according to their best response functions. Thus there is s second equilibrium where, anticipating that individual j will put in ℓ_j^* (> 0) units of effort, individual i will optimally put in ℓ_i^* (> 0).

The Proposition asserts that there is always some range of costs for which this argument can be made. This too is evident from the diagram. Consider two curves, one with zero effort by individual j and the other where individual j is putting in some effort, then we can always draw a family of lines as we have in Figure 2, such that if zero effort is anticipated, then it is self fulfilling while if some effort is expected from individual j, then some effort is made by individual i.

The possibility of multiple effort equilibria is driven by the peer group externality. Hence, apparently homogeneous populations could be found to be putting in different effort levels in equilibrium and both members of the group would be better off if some coordination between individual effort levels were possible. Thus we may have a kind of coordination failure familiar from models with strategic complementarities.

There are conditions under which the equilibrium is unique. This will be the case if the cost of effort lies outside the interval $[c_A, c_B]$. For costs below c_A, the zero effort equilibrium disappears and we have only positive effort equilibria. There may, however, be many such equilibria – our assumptions do not guarantee uniqueness. Hence, there may still be some scope for measures which focus the group toward the high effort outcome. If the cost of effort exceeds c_B, then a zero effort equilibrium is all that survives. These results can be interpreted by thinking of c as a measure of an individual's motivation to work – perhaps reflecting his family background. If the peer group consists only of highly motivated individuals, then the only equilibrium has both individuals putting in effort in equilibrium, while for a poorly motivated group, there is a unique equilibrium where individuals put in no effort at all. Matters would be made more interesting, of course, were we to consider mixing together individuals who have different propensities to work and this issue is pursued in section VI.

V. Discussion

It is clear that our results depend in an important way on the specification of the technology for generation of test scores. If, for example, the specification in (3.1) were replaced with the slightly altered

$$(3.1') \qquad x_i = (\theta_i+\eta+\epsilon_i)\ell_i,$$

then there would be no peer group effect. The effect would also be eliminated by replacing (3.1) with

$$(3.1'') \qquad x_i = \theta_i\ell_i + \eta + \epsilon_i.$$

Now it might be argued that either of the above specifications is as plausible as (3.1) as a model of test scores. There is indeed no really good reason why the size of the idiosyncratic effect should not be affected by the effort choice (as in (3.1′)) or that the common shock should not have a component which is independent of effort (as in (3.1′′)). However, we could replace (3.1) by the more general form of technology

$$(3.1''') \qquad x_i = (\theta_i + \eta_1 + \epsilon_{1i})\ell_i + \eta_2 + \epsilon_{2i},$$

where θ_i, η_1, η_2, ϵ_{1i} and ϵ_{2i} are all independently distributed. This encompasses all of the alternatives above. It can also be checked that results of the kind that we demonstrated earlier can be obtained using (3.1′′′). The key requirement is that there be some idiosyncratic shock which is not affected by the level of effort.

A second objection to our specification is the assumption of normality, the problem being that over some range of the realization of this the return to effort will be negative – the latter being implausible. This is a valid objection, although as long as the random shocks are sufficiently positive and their variances sufficiently small, the probability of a negative realization for θ_i and η will be quite small. As a consequence our results will not be too unreasonable as an approximation of the case where we explicitly truncate the distributions of θ and η at zero.

A further possible criticism of our line of attack on the problem is that it is too complicated for the problem at hand. An alternative model which can generate positive externalities between effort choices is one in which individuals in school know their abilities, but face an employer who wishes to give a job to the more able candidate. A natural rule for the employer to adopt is them to give the job to that individual who obtains the highest test score, thus creating a kind of tournament[14]. In this framework, one agent working harder will tend to make it necessary for others to do so as well in order to have a reasonable chance of winning the job. This suggests the possibility of there being positive externalities in this case too.

Note however, that the presence of positive externalities does not by itself guarantee the existence of multiple equilibria. For simple specifications of the cost of effort functions and distributions of the test scores, this model typically either yields a unique equilibrium where both people work "too hard" or else an asymmetric equilibrium where one person works hard and the other gives up (this is more likely to happen of there are small *ex ante* asymmetries between the agents as in models of races). Thus, while it is possible to obtain multiple equilibria in such tournament models, it requires somewhat less natural specifications of the distribution of grades than we have taken, or else some kind of non–convexity (i.e., a fixed cost) in the effort function.

We therefore conclude that this alternative approach is not necessarily simpler than our specification. It is also not necessarily more realistic – it depends upon the assumption that good jobs are scarce relative to a particular school. Even at the very best schools, this seems tenuous and is certainly unreasonable for most of the occupations which people enter. In addition, this picture requires a model in which individuals earn rent in the higher paying jobs. Our model works with a more standard view in which individuals earn something closer to their marginal products.

An interesting way of extending the model would be to introduce some elements of asymmetric information about the ability of the individual who is making effort and his potential employee. This would introduce some elements of signaling into the model. This would certainly open up set a rich possibilities not available here, but we prefer to defer any interesting extensions in this direction for future work.

Thus to conclude, it should be clear that the present model is not the unique framework capable of generating the kind of externality that we are interested in here and, moreover, there are particular assumptions which are necessary. On the other hand, we hope to have convinced the reader that the approach taken here deserves serious consideration and that alternatives are not as straightforward as might be thought at first blush.

VI. Further Issues

This section discusses three further issues which our model can analyze. The first concerns the design of an optimal "competitive" school structure in the presence of peer group externalities; we suggest that reward structures can be thought of as "pricing" the peer group externality. The second concerns the choice of the class mix from among students with different levels of motivation; we consider the impact of pairing two individuals with different values of c. Finally, we investigate whether tests should be designed at the local (i.e. school) or national level.

VI.1 Prizes

The model suggests that use of rewards related to school performance *per se* might play an important role in the presence of peer group externalities by enhancing the incentive to work hard. Thus, even though the learning process is not intrinsically competitive, it may be worthwhile having a competitive school system. In effect, this "prices" the peer group externality. It is related to results in the multi–agent incentive models which show that it is optimal to offer an incentive scheme based on the whole vector of output levels, as opposed to just basing rewards on individual output.[15] Most crudely this would be a rank order reward scheme where the individual with the highest test score is given a prize.

More generally, any reward scheme related to (x_i,x_j) can potentially eliminate the equilibrium with zero effort by offering individuals the appropriate extra incentive to put in effort. Thus even though the structure of rewards in the career choice are not competitive, the presence of an externality may motivate introducing some competition into the learning process. Thus, we can get to a more cooperative outcome by introducing competition.

VI.2 Mixing

Our analysis above concentrated on homogeneous peer groups. Here, we allow for pairings of individuals with different values of c, which allows us to analyze non–homogeneous groups. First, consider an individual whose c (call it c_H) is high enough so that the unique symmetric equilibrium when she is mixed with another identical individual has effort equal to zero. Thus in terms of Proposition 3, $c_H > c_B$. Suppose that we mix this individual with another whose c (call it c_L) is such that, were he mixed with another identical individual, would yield an "effort making" equilibrium (i.e., $c_L < c_A$). Depending upon the values of c_H and c_L, there are three possible outcomes:

Case 1: If c_H and c_L are both high enough, then there will be a unique Nash equilibrium where neither puts in any effort. The result of mixing is thus Pareto inferior to having these individuals in separate schools.

Case 2: If c_H is high but c_L is reasonably low, then the high cost individual continues to put in zero effort while the low cost individual does put in effort[16]. Hence, despite the low cost individual providing a positive externality for the high cost individual, it is not large enough to induce the high cost individual to invest some effort. This case is also Pareto inferior to putting individuals in homogeneous groups since the high cost individual is no better off and the low cost individual is worse off than she would be with an identical peer.
Case 3: If c_H and c_L are low enough, then the both individuals may invest in equilibrium. The high cost individual is better off than he would be in a homogeneous group and is investing effort. The low cost individual is still putting in effort but at a lower level than in a homogeneous group. The latter is therefore worse off. Hence, this case leads to a redistribution from the low cost to the high cost individual. Even though we do not have a Pareto improvement, there may one available if we could arrange a transfer from the high cost to the low cost individual.

This confirms the view that mixing highly and poorly motivated individuals in a learning environment may sometimes pay, essentially when the lowest c is not too low. There are some individuals who will put in effort at school provided that they are paired with other individuals who are more motivated than themselves. If paired with individuals with similar levels of motivation, this will not be true. This analysis also gives qualified support to the view that educational mixing of some sort can be beneficial. The main qualification arises from considering our cases 1 and 2 above. Here mixing resulted in a Pareto inferior outcome. A highly motivated individual was dragged down by being matched with a poorly motivated one without being induced herself to start trying.

Our results on matching individuals with different motivation levels are summarized in Figure 3[17]. The 45° line represents the case of identical individuals with which we began. Case 1 that we described occurs when c_i and c_j are in the middle right hand region of the figure in the area where $\ell_i^* = \ell_j^* = 0$. Cases 2 and 3 refer to the bottom right hand region of the diagram where individual i continues to put in effort while individual j may not depending how high her cost of effort is. More generally, the diagram divides the space (c_i,c_j) into regions where different equilibria are possible.

The model with different cost individuals might also also be used to underpin the kind of sorting model questions investigated in Benabou (1991). He he supposes that the price of housing adjusts to sort individuals of different types into communities. Individuals will be willing to pay to sort in our model if

$$(6.1) \qquad U(c_H,c_H) - U(c_H,c_L) \geq U(c_L,c_L) - U(c_L,c_H),$$

where $U(c_i,c_j)$ is the utility of a type i when grouped with a type j. It is straightforward to check that this holds if $\partial^2 U/\partial c_i \partial c_j \geq 0$. It is difficult to establish analytically exactly when this condition holds in our model, but it certainly can and our approach may therefore constitute a model to underpin Benabou (1991)'s more reduced form approach.

VI.3 National versus School Specific Testing

Finally, we consider the appropriate level at which to gather test information, i.e. whether school or national tests provide a better system for assessing student potential. Both types of test are common in practice. Hence, in the United States, there are also national tests such as the SAT or GRE which are used extensively in school admissions in addition to GPA's. The motivation for having such a test fits somewhat closely with the model that we are using here – "that college bound students can use their SAT scores to help select a college that matches their ability" (Crouse and Trusheim (1988) page 72). Another professed advantage of national testing relates is that "the introduction of tests resulted in a substantial increase in opportunities for educational advancement of low income students by providing a credible demonstration that many such students from schools without reputations for educational excellence could succeed in the more demanding programs of the most selective institutions." (Educational Testing Service

(1980) page 10) and that "an external examining system allows colleges to compensate more fully for differences in high school grading systems." (Crouse and Trusheim (1988) page 71). The idea is that individual performance is unhinged from peer group performance in a national test. An individual who does well in a poor school, i.e. one where she has a poor peer group, stands little chance of making a good selection decision. Neither she, nor anyone else trying to infer her ability from her test score, would have a very precise estimate of her ability using only a school based score. By contrast, the SAT score is much more reliable since it is based on a large sample of individuals and it does not matter so much whether or not an individual's immediate peer group is putting in effort when one is preparing for such a test.

This, at least, is the theory and can be captured in our model by noting that the conditional variance of η will be lower with a more national based test. To see this, imagine increasing the size of an individual's peer group while maintaining the same structure of shocks across individuals that we outlined above. As the number of individuals grows then, since ϵ_i is uncorrelated across individuals and the strong law of large numbers holds, an individual will obtain a more and more precise estimate of η for any observation of the vector (x,ℓ). In the limit, uncertainty about η will disappear altogether.[18] This justifies modeling national tests as those for which ψ^2 is small. This seems to capture some of the arguments made above. In terms of the impact on effort. It is straightforward to check that

$$\text{(6.2)} \qquad \frac{\partial^2 W(\ell)}{\partial \ell_i\, \partial \psi^2} = \frac{\sigma^4}{(\kappa+\sigma^2)} \cdot \frac{2(1+(\sigma^2 \ell_j^2)^2)}{\ell_i^3(1+(\sigma^2+\psi^2)\ell_j^2)^2} > 0,$$

which tells us that using a test which is less noisy raises the marginal gain from putting in effort. Thus the curve in the bottom part of Figure 1 will shift upwards, increasing the level of effort put in at any positive effort equilibrium and the chance that such an equilibrium (as opposed to the zero effort equilibrium) will be preferable.

The use of SAT scores is, however, controversial (see, for example, Crouse and Trusheim (1988)). The arguments are often subtle and complex and go beyond the issues raised here. One of the objections to the use of SAT scores rather than more school specific examinations may however be made formally precise in our model. This argument concerns the fact that SAT scores may be rather sensitive to curriculum and instructional variations among high schools (Crouse and Trusheim (1988) page 157). Thus it is doubted whether individuals from different high schools really are put on a more equal footing by the SAT. There are two ways of thinking about this. First, if the curriculum variation is systematic at the high school level, then the argument questions the very idea that the conditional variance of η actually falls if individuals look to the SAT for making decisions, since there will still be good school and bad school effects.

A second way of interpreting this idea in our model, is by arguing that a reduction in the variance of η can only be "bought" by increasing the variance of ϵ.[19] It is straightforward to check that increasing the variance of ϵ lowers expected utility. However, its effect on $\partial W/\partial \ell_1$ is ambiguous. An increase in the pure noise element of the test has two opposing effects; it makes any signal that an individual obtains from the test noisier and thereby increases her incentive to apply effort to reduce the noise. It also reduces the value of effort put in by her peers who, for any given effort level, provide less information about η. The net effect on effort is therefore ambiguous.

Our model, therefore, gives a precise but qualified case in favor of adopting national tests. Tests which costlessly reduce the variance of common shocks reduce the peer group effect, but may be very difficult to find in practice. Considering these issues does give an implication of the model, which sets it apart from one based on preference interdependence, since it predicts that peer group effects should be less important after the introduction of national testing. This suggests a potentially testable difference between our model of peer group externalities and one based on preference interdependence.

VII. Concluding Remarks

This paper has investigated peer group externalities in a simple model where education is viewed as a process by which individuals learn their abilities. The importance of the peer group arises because individuals face a common uncertainty about the validity of their test scores as an indicator of their abilities which affects the productivity of effort. If an individual's peers put in effort, then she obtains a more precise estimate of her abilities for every unit of her effort.

We set up a simple model with linear costs of effort, normally distributed random variables and a linear production function. There are two main interesting features of this set–up. First, utility is not concave in effort. Second, for some range of costs there will be multiple equilibria. The latter possibility is driven entirely by the peer group externalities. This suggests the possibility of a pessimistic equilibrium "trap" in which individuals would be better off investing effort to learn their ability, but do not do so because they do not expect their peers to do the same. Under certain conditions, judicious mixing of different individuals can remedy this problem, although inappropriate mixing is Pareto inferior.

The paper's model is unashamedly simple and, as we have been at pains to point out, there are some special assumptions which play a key role in generating a positive peer group effect. There are however more general structures which preserve the essence of our model. The ideas in this paper may also be relevant for other situations where individuals learn in an interdependent environment.

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Appendix

Proof of Proposition 1: We are interested in the joint distribution of θ_i, x_i/ℓ_i and x_j/ℓ_j. Since all three of these variables are normally distributed, their joint distribution will also be normal. It is easy to check that the variance–covariance matrix of these three variables is

$$\begin{bmatrix} \sigma^2 & \sigma^2 & 0 \\ \sigma^2 & \nu_i & \psi^2 \\ 0 & \psi^2 & \nu_j \end{bmatrix} \quad \text{where } \nu_k \equiv \sigma^2 + \psi^2 + 1/\ell_k^2,\ k \in \{i,j\}.$$

Define $\Sigma \equiv \begin{bmatrix} \nu_i & \psi^2 \\ \psi^2 & \nu_j \end{bmatrix}$, then a standard result for normal distributions (see, for example, Theil (1971, p. 189)) says that

$$E\{\theta_i \mid x/\ell\} = \bar{\theta}_i + [\sigma^2,0]\Sigma^{-1}\begin{bmatrix} (x_i/\ell_i - \theta_i) \\ (x_j/\ell_j - \theta_j) \end{bmatrix}$$

and

$$\mathrm{Var}\{\theta_i \mid x/\ell\} = \sigma^2 - [\sigma^2,0]\Sigma^{-1}\begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}.$$

Substituting and rearranging now yields the result. □

Proof of Proposition 2: All of these results are proved using simple differentiation after noting the result in Proposition 1. Thus

$$W^i(\ell) = \bar{\theta}_i + \frac{1}{2\alpha} - \alpha\frac{\kappa\sigma^2}{(\kappa+\sigma^2)}.$$

First note that

(A.1) $$\partial\{\kappa\sigma^2/(\kappa+\sigma^2)\}/\partial\kappa = \sigma^4/(\kappa+\sigma^2)^2 > 0$$

and

(A.2) $$\partial^2\{\kappa\sigma^2/(\kappa+\sigma^2)\}/\partial\kappa = -\,\sigma^4/(\kappa+\sigma^2)^3 < 0.$$

To prove part (a) it now suffices to note that $\partial\kappa/\partial\ell_i = -2/\ell_i^3$. Part (b) follows from the fact that $\partial\kappa/\partial\ell_j = -\,\psi^4 2\ell_j/(1+(\psi^2+\sigma^2)\ell_j^2)^2 < 0$. Part (c) requires us to use the fact, obtained by differentiating the expression for $\kappa(\ell_i,\ell_j)$ given in Proposition 1 to yield

(A.3) $$\frac{\partial^2 W(\ell)}{\partial(\ell_i)^2} = \frac{2\sigma^4}{(\kappa+\sigma^2)}\cdot\frac{1}{\ell_i^4}\left[\frac{4}{(\kappa+\sigma^2)\ell_i^2} - 3\right].$$

Note that, since $\lim_{\ell_i\to 0}(\kappa+\sigma^2)\ell_i^2 \to 1$, the utility function is convex for small ℓ_i and concave thereafter. Part (d) follows from (A.2) after using $\partial^2\kappa/\partial\ell_i\partial\ell_j = 0$ and $\partial\kappa/\partial\ell_j < 0$. □

Proof of Proposition 3: (Sketch) Our strategy for proving the result is to characterize the best reply correspondence $\ell_i(\ell_j;c) \equiv \underset{\ell_i}{\operatorname{argmax}}\{W(\ell_i,\ell_j) - c\ell_i\}$. Since the maximum value of $W(\cdot)$ is σ^2, ℓ_i and ℓ_j must both be less than $\sigma^2/\underline{c}$. The best response correspondence therefore maps $[0,\sigma^2/\underline{c}]$ into $[0,\sigma^2/\underline{c}]$. We can therefore restrict attention to values of ℓ_j is this interval. With this domain, $W(\ell_i,\ell_j) - c\ell_i$ is a compact valued function and we know also that it is continuous in ℓ_i, ℓ_j and c. By Berge's Theorem, the best correspondence is therefore upper hemi–continuous in ℓ_j and c and *a fortiori* continuous when it is single valued. Referring to Figure 1 in the text, it is clear that the best response for any given value of ℓ_j, is either 0 or at ℓ_i^*. Which of these points will be chosen, can be discerned by comparing the areas under the curves as we do in Figure 2. At a strictly positive $\ell_i^*(\ell_j;c)$, if it exists, it is clear that $\partial W(\ell)/\partial\ell_i = c$ and $\partial^2 W(\ell)/\partial(\ell_i)^2 < 0$. Note also that if ℓ_j goes up, then (by Proposition 2(d)) $\partial W(\ell)/\partial\ell_i$ moves up, so that ℓ_i is increasing in ℓ_j. What is more, if $\ell_i(\bar{\ell}_j;c) = \ell_i^*(\bar{\ell}_j;c)$ for some $\bar{\ell}_j$ and c, then $\ell_i(\ell_j;c) = \ell_i^*(\ell_j;c)$ for all $\ell_j \geq \bar{\ell}_j$. So the best response correspondence is *always non–decreasing.* This immediately implies that all equilibria must be symmetric (since the game is symmetric).

Now consider how $\ell_i(0;c)$ varies with c. If $\underline{c}$ is low enough, then for c close enough $\underline{c}$, it must be the case that $\ell_i(0;c) = \ell_i^*(0;c)$. In fact, if $\underline{c}$ is low enough, there is clearly a whole interval $[\underline{c},c_A)$, say such that $\ell_i(0;c) = \ell_i^*(0;c)$ for c in this interval. In this case, the best reply correspondence has the shape given by the continuous line in Figure A1. Note that it has to be continuous and there will be at least one equilibrium at which $\ell_i > 0$, but no equilibrium at $\ell_i = 0$. The ℓ^* in the above formulations, will be that at which $\ell_i(0;c) = \{0, \ell_i^*(0;c)\}$, i.e. it will be the value at which individual i is indifferent. In this case, there are clearly two equilibria, one of which has $\ell_i = 0$ and the other with $\ell_i > 0$.

As we raise c above c_A, the best response correspondence will have the shape given by the dotted line in Figure A1. Hence, $\ell_i^*(\ell_j;c)$ will be at zero for low values of ℓ_j but will be at $\ell_i^*(\ell_j;c)$ for higher values of ℓ_j. Clearly, by the continuity of $\underset{\ell_i}{\mathrm{Max}} \{W(\ell_i,\ell_j) - c\ell_i \mid \ell_i \geq 0\}$ with respect to c, the point at which the best response correspondence jumps will move continuously with c. Since for $c = c_A$, the jump is at $\ell_j = 0$, for c close to c^*, the jump will be at ℓ_j close to zero. Since the jump is always a discrete jump, ($\ell_i^*(\ell_j;c)$ is bounded away from 0), for values of c just above c_A, we must have $\ell_i^*(\ell_j;c)$ greater than ℓ_j at the jump point. Since $\ell_i^*(\ell_j;c)$ is continuous, there will be an equilibrium with $\ell_i > 0$. So for values of c not too far above c_A, there will be a least two equilibria.

In fact as c increases, the best reply correspondence moves down and the jump point moves to the right. The number of equilibria with $\ell_i > 0$, therefore cannot increase. In fact, by continuity, there will be a value of c, c_B (say) such that for $c > c_B$, the best response map will be as in the dashed line in Figure A1 and there will be only one equilibrium (0,0). The cost level c_B, will be that at which the best response correspondence is below the 45° line everywhere except at one point. This completes the argument. □

End Notes

[1]See Arrow (1973) or Spence (1973) for the signaling view of education.

[2]This is the focus of the human capital approach to educational decisions pioneered by Becker (1975). Our model may be regarded as a model of human capital acquisition if the latter is broadly defined. An individual's knowledege of her own abilities might be thought of as being part of her human capital.

[3]This can be thought of as representing how an individual feels on a particular day or, perhaps, how the grader feels.

[4]This is not to suggest that there are good reasons why local tests, i.e. those which test individuals relative to a narrow group, are used in practice. First, national testing tends to reduce teacher autonomy. Second, individuals often demand special variations in the teaching environment, as in the case of religious schools, which make it difficult to operate on a universal curriculum.

[5]The earliest studies of this phenonmenon were based on the attitudes of American soldiers in World War II. One major finding was that services with the highest promotion rates also experienced the greatest level of discontent – the explanation being that individuals compared their lot with other personnel who had been promoted.

[6]Akerlof suggests in all of these papers that his central departure from the Arrow–Debreu paradigm is supposing that individuals care about what others think. Thus individuals "have positive utility from obeying social customs" Akerlof (1976) page 617.

[7]This is true, for example, in the model of Akerlof (1980).

[8]One should note, in addition, that there are many different views about the origin of peer influences within sociology. A useful review of a some of them can be found in Hirschi (1969).

[9]There are a number of empirical papers which have analyzed the peer group effect in education. Hanushek (1986)'s survey of the economics of schooling discusses the empirical determination of such effects in some detail. There are also some theoretical treatments of the implications of peer group externalities (for example, by Arnott and Rowse (1987) and De Bartolome (1990)) but these examine only the consequences of peer group externalities, taking their existence for granted. Our paper will examine how these effects can be given a theoretical foundation.

[10]Restriction to two person groups is for analytical convenience and in no way essential to the story which follows.

[11]Thus whether someone is paying attention in class, or completing their homework is assumed to be observable.

[12]Our model bears some formal resemblance to those developed to analyze multi–agent incentive problems (see, for example, Lazear and Rosen (1981), Holmstrom (1982), Green and Stokey (1983) and Nalebuff and Stiglitz (1983)). Just as in these models, and the related work on herd behavior in investment by Scharfstein and Stein (1990), it is the common component of the uncertainty which generates interesting differences from single agent settings. Unlike much of the multi–agent incentive literature, however, we do not have individuals competing for rewards. Although, as we discuss in section IV, some system of prizes may make sense as a means of pricing the peer group externality.

[13]A higher η means that the test is easier.

[14]This is clearly the optimal decision rule for an employer of two agents who appear *ex ante* identical. If they are not identical, the optimal decision may involve some bias (see Meyer (1991) for further discussion). It is unclear whether such a framework would make it easier or more difficult to obtain multiple equilibria in comparison with that adopted here.

[15]See, for example, Holmstrom (1982) and Green and Stokey (1983).

[16]Note that for some values of the c's, both this equilibrium and the zero effort equilibrium may exist.

[17]We have drawn the diagram so that, within the range if c's illustrated, all types of equilibria exist. This may not however be the case were the c's constrained to lie in some finite interval.

[18]This result is essentially the same as Theorem 9 in Holmstrom (1982) and can be proved along similar lines.

[19]This can be put in the language of the educational testing literature (see, for example, Brown (1983)). This literature would define the reliability of a test (in terms of our model) as $\ell_1^2\sigma^2/(\ell_1^2(\sigma^2+\psi^2)+\pi^2)$, where π^2 is the variance of ϵ. This is the ratio of the variance of test scores if the test measured without error to that obtained when the test measures with error. Thus even if ψ^2 increases, the test reliability may not increase if π^2 also rises.