Financial Stabilisation Policies in a Credit Crunch: Zombie Firms and the Effective Lower Bound*

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Abstract

Does looser financial regulation during recessions stifle the forces of creative destruction or does it help to avoid job losses and a weak aggregate demand? This paper analyses the effects of a limit on firms' borrowing restricting debt to a fraction of their profits. Constrained firms can invest less, and demand for investment is lower than available savings, so a reduction in the interest rate helps reestablish an equilibrium, by inducing unconstrained firms with lower productivity to start production. The constrained equilibrium features too many low-productivity firms: *zombies*. They generate a negative spillover on the borrowing capacity of more productive firms, as they contribute to reducing the value of profits for all firms, by inflating labour costs. When the interest rate is at the effective lower bound, the opportunity cost of operation is artificially high, so the economy features less investment. As fewer low-productivity firms invest, future aggregate productivity is improved, however aggregate demand is low in the present, and output is demand-determined. While liquidating zombie firms away from the lower bound can improve the efficiency of the allocation, it can be counterproductive at the lower bound, as these firms are not zombies but make use of idle resources, boosting output and welfare.

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1 Introduction

A decade of low interest rates and a pandemic-induced recession raise a classic question on the extent to which policymakers should intervene to support or liquidate inefficient firms. On the one hand, the Schumpeterian viewpoint underscores the cleansing effect of recessions: bailing out businesses may create zombie firms and generate other supply-side inefficiencies. On the other hand, Keynesians argue that intervening to stimulate the economy in a crisis is especially beneficial, considering that aggregate demand may be insufficient. This paper presents a model of the tension between demand management concerns and efficiency of supply by offering a theory of zombie firms that accounts for the effective lower bound (ELB), and analyses its efficiency properties.

Zombie firms were one of the main concerns during the Japanese lost decade, while recent evidence suggests their role in driving down aggregate productivity in European countries around the time of the Great Recession.[1] Low interest rates are often indicated as one of the main causes of the survival of weak firms.[2] However, how is the incentive of these zombie firms to operate influenced by the presence of an effective lower bound preventing further reductions in the interest rate? From a policy perspective, does the ELB affect the optimal financial stabilisation policies that should be pursued in relation to zombie firms?

This is important because the monetary authority in charge of setting interest rates does not always work in close collaboration with the financial authority setting financial regulation, bailout and resolution policies.[3] As a result, the objectives established for each institution do not necessarily account for the potential spillovers or complementarities with the other authority's targets. Is there a risk that each authority, by aiming to achieve their individual goals, may interfere with the job of the other institution, or are they in fact facilitating each other's success?

This paper addresses these questions within the context of a simple theoretical framework featuring a restriction in credit conditions. It considers the potential for financial stabilisation policy interventions, such as liquidation or bailout policies, both when the interest rate is free to adjust in response to the shock and when it hits the lower bound. In the model, a credit crunch restricts the borrowing capacity of all firms to a fraction of their earnings.[4] As a result,

[1] Definitions of *zombie firms* vary across studies, but tend to refer to low profitability firms kept afloat due to unusually favourable borrowing conditions. For evidence, see Caballero, Hoshi, and Kashyap (2008) in Japan in the 1990s and e.g. Adalet McGowan, Andrews, and Millot (2018) in OECD countries in the 2000s.

[2] See e.g. Banerjee and Hofmann (2018).

[3] As examples, in the UK these two authorities are different branches of the same institution, the Bank of England, while in the Euro Area, the European Central Bank (ECB) is a formally distinct body from the European System of Financial Supervision. The ECB however provides input to the European Systemic Risk Board, as set out in EU regulations.

[4] The credit constraint considered is an earning-based constraint, as in Drechsel (2018).

constrained firms have to lower their investment relative to the first best. In equilibrium, the real interest rate falls so as to offset the direct impact of the shock, by inducing less productive firms, which would otherwise be inactive, to enter the market. Aggregate investment is preserved, but this comes at the expense of lower productivity, due to capital being operated by less efficient firms. As the efficiency of production falls, real wages are lower. This in turn raises firms' future profits and loosens their borrowing limit, further offsetting the initial shock.

This outcome is not constrained efficient.[5] A policymaker that takes the financial friction as given, but internalises the effect of individual choices on prices, can intervene to improve on the allocation by liquidating some low-productivity firms. The allocation can be improved through policy interventions because of the interaction of two inefficiencies. First, changes in interest rates and wages have a heterogeneous impact on different agents. Importantly, they do not net out in the aggregate, because constrained entrepreneurs have a relatively higher marginal valuation of wealth than other agents. Second, there is a pecuniary externality associated with the borrowing limit when prices affect firms' borrowing capacity. In the model, investment by low-productivity entrepreneurs creates a negative spillover on more efficient producers. At the margin, one extra unit of investment from a low-productivity firm increases the aggregate stock of investment, pushing up labour costs, which lowers the value of productive firms' profits and so reduces their borrowing ability. The financially-constrained allocation features too many inefficient firms in operation: *zombies*. Bailout policies may be counterproductive when the interest rate is free to adjust to shocks.

When the economy is at the effective lower bound, these conclusions are reversed. Now, because the interest rate is too high relative to what it would be without the bound, firms invest less in physical capital. As a result, demand is lower than the production potential, and the pre-installed capital stock is not fully utilised. While productivity is high, since low-productivity firms are not operating as much capital, aggregate investment is inefficiently low, and output and welfare are low both in the present and in the near future: in the present due to the weak demand; in the future because of the low capital stock. Low-productivity firms do not internalise how by demanding more investment in the present, they can increase not just future output, but also current output and consumption as the increased demand leads to a larger utilisation of resources in the economy. Provided that these aggregate demand externalities are sufficiently strong to offset the negative spillovers generated by low-productivity firms, then the previous result is turned on its head: it is now in the policymaker's interest to induce more low-productivity firms to operate, so that aggregate investment can increase. They are no longer zombies, but rather effective users of idle resources that boost output and welfare.

[5] A formal definition of constrained (sub)optimality can be found in Geanakoplos and Polemarchakis (1986).

Next, I consider policies that aim to prevent or moderate the crisis before it hits the economy. I show that restricting the amount of debt that firms can access before the credit crunch can help the economy during the crisis:[6] if productive firms need to repay a smaller stock of initial debt, they can invest more once the shock hits, which alleviates the impact of the financial friction during the crisis. At the same time, however, these policies can also contribute to pushing the economy towards the ELB. The reasoning is as follows: in the aggregate, a smaller amount of bonds issued allows indebted but productive firms to invest more, while it induces low-productivity entrepreneurs, who are also savers, to invest less. This induces a more efficient allocation of resources in the future, which boosts the future cost of labour and contributes to reducing the return to investment. As the interest rate offered on financial markets corresponds to this physical return to investment, this induces a lower equilibrium real interest rate. Policies that are meant to moderate the effects of pecuniary externalities are not necessarily also helpful in preventing aggregate demand externalities: financial stabilisation policies may affect the availability of monetary tools when the shock hits.

The paper further considers some extensions, which emphasise the importance of specific assumptions. First, I introduce another, less capital-intensive sector. Fighting zombie firms in the capital-intensive sector comes at a cost of further exacerbating resource misallocation across sectors:[7] the redistribution of resources away from the constrained sector during a credit crunch is intensified when fewer low-productivity firms are allowed to operate. The capital-intensive sector shrinks even more than in the absence of interventions to liquidate zombies. Second, I consider the possibility of another, unconstrained type of agents in the economy. More specifically, in the main model workers are assumed to be hand-to-mouth, so completely unable to smooth out their resources overtime. However, when they are free to make optimal intertemporal consumption choices, the result on optimal liquidation of zombie firms in a credit crunch is overturned. Low-productivity firms' investment contributes to increasing future wages, and the higher cost of labour in the future allows for a redistribution of resources towards entrepreneurs with a higher valuation of wealth in the present. More generally, the presence of any other type of unconstrained agents in the economy is likely to affect the optimal policy for zombie firms.

Finally, the paper shows that the specific type of financial frictions constraining investment plays an important role in shaping the second-best distribution of active firms. If firms are constrained, not by future profits, but rather by the value of a collateral asset, then increased

[6] This type of interventions refers to, for example, cyclical leverage ratios or capital requirements imposed on financial intermediaries, which get passed on to borrowers through reduced loans issuance.

[7] In defining misallocation across sectors, a first best allocation is used as reference on the optimal relative sector sizes.

demand for investment by any type of firm will boost the price of collateral and so increase the borrowing ability of constrained firms. In all these cases, the beneficial effects of Schumpeterian liquidationist policies are reduced, and the policymaker should strive to promote more investment, rather than chasing zombies.

The rest of the paper is organised as follows: after relating this work to the relevant literature, section 2 presents the main framework used for the analysis. In section 3 a credit crunch is introduced, restricting firms' borrowing possibilities. After analysing the efficiency properties of the allocation, section 4 analyses a situation of liquidity trap, exploring what makes it likely to happen and what are its effects. It then considers the potential for policy interventions. Section 5 considers the possibility for interventions before a crisis can hit the economy; section 6 considers some extensions of the main model. Section 7 concludes.

1.1 Related Literature

This paper combines two main strands of the literature: one on misallocation induced by financial frictions, and another on the liquidity trap and demand shortages. The first underlines the supply-side cost of low-productivity firms operating in the economy; the second focuses on the beneficial effects of demand stimuli in a situation of low demand.

Among the papers that formalise how financial frictions can have an impact on the aggregate productivity of an economy by distorting the allocation of productive factors, this work is most closely related to Kiyotaki (1998), Aoki, Benigno, and Kiyotaki (2010) and Reis (2013). The setting in Kiyotaki (1998) is augmented with the introduction of workers supplying labour within the period. This is important as it introduces a price in firms' earnings, which depends on firms' choices and influences how much they can borrow. As a result of this modification, the constrained economy is constrained inefficient.[8] The two-sector framework of Aoki et al. (2010) and Reis (2013) is considered as an extension of the main model, in order to analyse the effects of misallocation not only within but also across sectors. The consequences of fluctuations in the price of a good on the sector's borrowing capacity have been extensively studied from the point of view of open economy models,[9] but they generate interesting effects in a closed economy too. Gopinath, Kalemli-Özcan, Karabarbounis, and Villegas-Sanchez (2017) use an analogous setting to analyse the effects of lower interest rates due to the euro convergence process on the distribution of productivity in southern European countries. The present work also proposes an efficiency analysis, stressing when there is scope for policy interventions and

[8] On the contrary, the baseline model in Kiyotaki (1998) is constrained efficient because the only price, the interest rate, does not depend on firms' choice variables.

[9] See e.g. Benigno, Chen, Otrok, Rebucci, and Young (2016) or Bianchi (2011).

when the misallocation is in fact constrained efficient.

Concerning the demand rationing, positive papers on the liquidity trap, such as Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2017), highlight the importance of debt accumulation in amplifying recessions induced by financial constraints. Farhi and Werning (2016) and Korinek and Simsek (2016) explore the normative implications of aggregate consumption demand externalities in the presence of constrained monetary policy. This work adds an analysis of intertemporal choices related to capital investment to their insights. This has two implications: first, borrowing arises endogenously through the financing of capital; second, a different source of aggregate demand externalities is explored in connection to capital investment. Differently from consumption externalities, what starts out as a demand deficiency can turn into a supply-side problem in the following period in presence of investment externalities. Among the normative papers, Rognlie, Shleifer, and Simsek (2018) propose a model to study the investment overhang of the great recession, and analyse the effects of the liquidity trap on misallocation and unbalanced recovery across sectors. This paper underlines the potential for misallocation not only across sectors but also within a sector.

This paper is also related to the literature on pecuniary externalities and financial stabilization policies, as in Jeanne and Korinek (2020); Bianchi and Mendoza (2018); Dávila and Korinek (2018); Benigno et al. (2016); Bianchi (2011); Lorenzoni (2008), as it also features pecuniary externalities connected to a borrowing constraint and unequal marginal rates of substitution (MRSs). This is combined with a demand externality and it can generate a policy tradeoff.

Drechsel (2018) and Lian and Ma (2020) document how widespread earning-based borrowing constraints are. However, the normative implications of this type of constraint have been studied very little, compared to the more popular collateral-type borrowing constraints. This work compares the policy implications of a cashflow constraint to a collateral constraint and shows that optimal interventions depend crucially on the type of constraint considered.

In the setting proposed in this work, *zombie* firms are defined as low productivity firms that generate a negative spillover on productive firms. The empirical literature has provided mixed evidence of spillovers from zombie to non-zombie firms. Using firm-level data in Japan up to the early 2000s, Caballero et al. (2008) find that investment and employment growth for healthy firms falls as the percentage of zombies in their industry rises. More recently, Acharya, Eisert, Eufinger, and Hirsch (2019) documented similar effects in Europe. Both investment and employment growth of healthy firms are found to be significantly lower compared to non-zombie firms active in industries with less zombies. Schivardi, Sette, and Tabellini (2017) use a different identification strategy to show that zombie firms as induced by low bank capitalisation have a negligible effects on the growth rate of healthy firms. They point at general equilibrium effects such as aggregate demand externalities to explain this difference with the rest of the relevant literature. This paper encompasses a potential explanation both for the presence and absence of negative spillovers from zombie to healthy firms, depending on the extent of aggregate demand externalities.

2 Framework

The economy features two dates: $t = 1, 2$; two types of agents populate the economy: workers and entrepreneurs. Entrepreneurs have access to a Cobb-Douglas production technology that employs capital and labour, $y_i = a_i(k_i)^\alpha(n_i)^{1-\alpha}$, where $i \in \{h, \ell\}$. The productive sector is therefore populated by different types of entrepreneurs, some with a high and some with a low fixed productivity component. They can consume, invest in productive capital and save or borrow on the financial market. Workers, on the other hand, do not have access to a production technology, but can supply labour with a certain disutility and enjoy consumption. In the following subsections, the problem of each agent operating in the economy is described in more details.

2.1 Entrepreneurs

High and low productivity entrepreneurs in the productive sector represent a share π_h and π_ℓ of the population respectively.[10] To ease the exposition, their problem is split into intra-temporal and inter-temporal decisions.

2.1.1 Static Choices

When starting the period with a positive amount of capital $k_i > 0$, firms solve a static problem of choosing the optimal level of employment n_i:[11]

$$\begin{aligned} & d_i = \max_{n_i} \; y_i - \omega n_i \\ \text{s.t.} \quad & y_i = a_i(k_i)^\alpha(n_i)^{1-\alpha} \end{aligned} \tag{2.1}$$

[10] In general, there can be switching across the two types of productivity, according to a transition matrix P, such that the share of both types of entrepreneurs in the population remains constant. However, the results presented here assume that entrepreneurs maintain their type throughout their lifetime.

[11] Throughout the paper, capital letters are used to indicate aggregate variables, while a prime superscript is used to indicate future variables.

where ω represents the wage. In choosing the optimal level of employment, the production technology represents the limit to the profit maximization problem. The firms' optimal choice of labour to employ is up to the point where the marginal product of labour equals the wage rate:

$$(1-\alpha)\frac{y_i}{n_i} = \omega$$

2.1.2 Dynamic Choices

Entrepreneurs choose the level of consumption $\hat{c}_i$, debt (if positive) or savings (if negative) in financial instruments b_i' and investment to start or continue running a firm in the following period k_i'. They solve the following problem:

$$V_{it}(z_i; S) = \max_{\hat{c}_i, k_i', b_i'} \log \hat{c}_i + \beta V_{it+1}(z_i'; S') \tag{2.2}$$

$$\text{subject to} \qquad \hat{c}_i + k_i' - \frac{b_i'}{R} = z_i, \qquad k_i' \geq 0$$

$$z_i' = a_i (k_i')^{\alpha} (n_i')^{1-\alpha} - \omega' n_i' - b_i'$$

$$b_i' \leq \frac{\theta}{1-\theta} z_i' \tag{2.3}$$

with $V_{i3}(\cdot) = 0$ in the last time period. $S = \{K_h, K_\ell, B, \theta\}$ is a vector of aggregate state variables, R is the gross real interest rate, while z_i is the entrepreneurs' net worth,[12] which is taken as given after having chosen the level of employment according to (2.1). The limit on debt in (2.3) requires that borrowing be at most a fraction $\frac{\theta}{1-\theta}$ of entrepreneurs' future net worth. This constraint can also be rewritten as depending on firms' output after labour costs, as in e.g. Drechsel (2018): $b_i' \leq \theta d_i'$.[13,14]

2.2 Workers

Workers supply labour to the economy. They do not have access to borrowing or lending, so that $B^w = 0 \ \forall t$, and are therefore hand-to-mouth consumers. Every period, they solve the

[12] After production, capital is assumed to fully depreciate.

[13] To see this, notice that $z_i' = d_i' - b_i'$. Plugging this expression in the initial borrowing constraint and rearranging shows the equivalence. See also the discussion in Drechsel (2018) on the irrelevance of stock vs. flow distinction for borrowing.

[14] Firms obtain credit based on expected earnings as opposed to the entire continuation value of the firm. One can think of this type of constraint as arising from the fact that it is not possible to continue to operate the production technology if the entrepreneur withdraws from the firm. Then, the only thing that can be recouped is the production net of labour costs at the time of repayment.

following problem:

$$W_t(S) = \max_{\hat{C},L} \log\left(\hat{C} - \frac{L^{1+\psi}}{1+\psi}\right) + \beta W_{t+1}(S') \tag{2.4}$$
$$\text{subject to} \qquad \hat{C} = \omega_t L_t$$

with $W_3 = 0$ in the last time period. The workers' preferences are as in Greenwood, Hercowitz, and Huffman (1988) (GHH) and they imply that labour supply features no wealth effect: $L^\psi = \omega$.

2.3 Equilibrium

It is now possible to define an equilibrium within this framework.

Definition 2.1. *An equilibrium is a path of allocations* $\left\{\hat{c}_{it}, \hat{C}_t, n_{it}, k_{it+1}, b_{it+1}, L_t\right\}$ *and prices and profits* $\{\omega_t, R_t, d_{it}\}$ *for all time periods* $t = 1, 2$ *and* $i = h, \ell$, *with* $\{b_{i1}, k_{i1}\}_{i=h,\ell}$ *given, such that entrepreneurs in the productive sector solve problems* (2.1) *and* (2.2), *workers solve problems* (2.4) *and markets clear.*

2.4 Three Allocations

For the analysis that follows, it is useful to describe three different allocations: a first best allocation with no frictions, a laissez-faire allocation with a financial constraint, and a second best allocation with financial frictions.

2.4.1 First best allocation

The allocation is first best when no financial friction affects the economy: entrepreneurs are able to borrow as much as they wish, and workers can access financial markets. In this case, the low productivity entrepreneurs prefer to become financiers, extending loans to the high productivity entrepreneurs to run their firms. The high productivity technology provides a return proportional to a_h, and productive firms have to offer this return on any loans they take out, in order for the financial market to be in equilibrium.[15] Therefore, by extending loans to the high productivity entrepreneurs, the financiers have access to a higher return than they

[15] A higher interest rate would lead to zero loan demand, as it would generate losses on each unit borrowed. A lower interest rate causes an infinite demand for borrowing as firms could make positive profits that way.

could achieve by operating their own technology:

$$R = \mathrm{MPK}'(h) \equiv \frac{\alpha a_h}{\left(\hat{K}'_h\right)^{1-\alpha}} \tag{2.5}$$

where $\hat{K}'_h \equiv \frac{K'_h}{N'_h}$ is the individual capital-labour ratio. The constant return to scale technology implies that only high-productivity entrepreneurs run firms, while zero low-productivity firms produce. Additionally, entrepreneurs as well as workers are able to transfer resources intertemporally at the same rate. This implies that their marginal rates of substitution (MRSs) are equated:

$$\beta\frac{\hat{c}_h}{\hat{c}'_h} \equiv \mathrm{MRS}'(h) = \mathrm{MRS}'(\ell) = \mathrm{MRS}'(w) = R^{-1}$$

Employment demand is efficient, and the equilibrium wage rate ensures that labour demand and labour supply are equal. Production productivity is affected by how capital is distributed among entrepreneurs. Because only high-productivity firms operate, the TFP in this economy is high at a_h.

Figure 1 provides a graphical representation of the market for financial instruments and capital investment, both in the first best and in the laissez-faire constrained allocation.[16] The left side panel plots the equilibrium in the market for bonds, taking the quantity of capital investment as given. Using the first order conditions for bonds of the two types of entrepreneurs and aggregating over type, it is possible to obtain expressions for the demand and supply of bonds as functions of the interest rate.[17]

$$(\mathrm{MRS}'_h)^{-1} = R = (\mathrm{MRS}'_\ell)^{-1}$$

$$(\mathrm{MRS}'_h)^{-1} = \frac{\hat{C}'_h}{\beta\hat{C}_h} = \frac{D'_h - B'}{\beta\hat{C}_h} = R, \qquad (\mathrm{MRS}'_\ell)^{-1} = \frac{\hat{C}'_\ell}{\beta\hat{C}_\ell} = \frac{D'_\ell + B'}{\beta\hat{C}_h} = R$$

At point A, the inverse of the marginal rates of substitution of active entrepreneurs and financiers meet, showing the equilibrium when the economy is unconstrained. The right-hand panel plots the marginal products of capital for the two types of entrepreneurs. For given level of capital investment, the marginal product of capital for high-productivity entrepreneur is always above that of low-productivity entrepreneurs. Given the interest rate that clears the bond market, the efficient level of capital-labour ratio for the h entrepreneur is at point A, while low-productivity entrepreneurs do not invest, given that their marginal productivity is lower.

[16] For simplicity, the case where workers optimally do not demand nor offer any bonds is considered here.
[17] Debt and savings are relabeled as follows: $B = \pi_h b_h,\ B_\ell = \pi_\ell b_\ell = -B$.

Figure 2 represents the labour market. As it is standard, labour demand is decreasing in the wage rate, but the position of the curve depends on TFP. In an unconstrained equilibrium, the economy is at point A. On the right side panel, any level of future wage rate corresponds to a current rate of interest. As future labour costs increase, the return to production falls. This relationship, too, is effected by productivity.

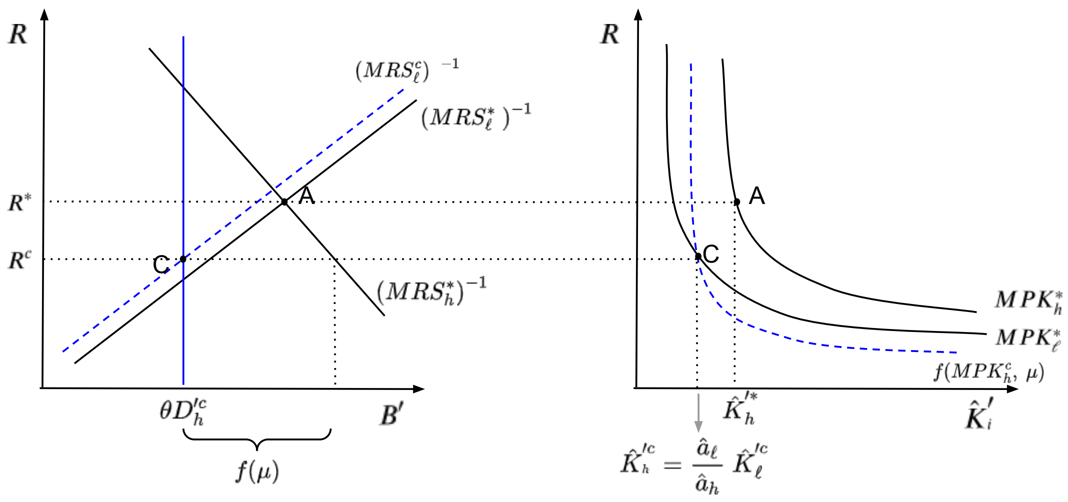

Figure 1: Capital and debt choices in a first best (A) and constrained (C) allocation.

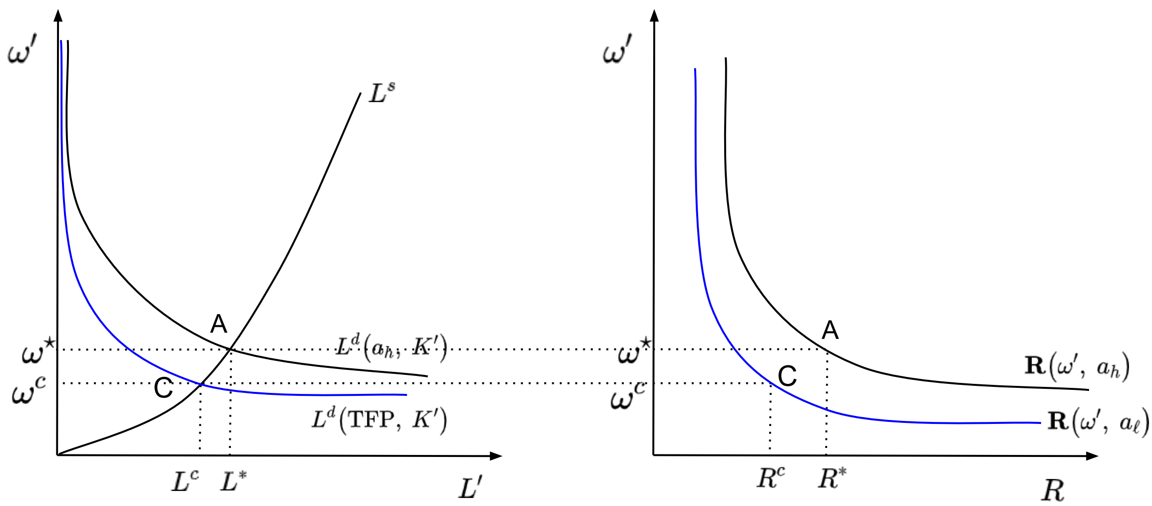

Figure 2: Labour market clearing and relationship between prices in a first best and constrained equilibrium (blue line).

2.4.2 Financially constrained, laissez-faire allocation

Consider a constrained allocation where the debt limit (2.3) is binding and workers do not have access to means of borrowing or saving. Due to the financial constraint, firms are not free to borrow as much as they wish from financiers. Likewise, workers would also like to borrow but cannot.[18] The marginal rate at which resources can be transferred intertemporally is then no longer the same for all agents in the economy:

$$\begin{aligned} R^{-1} = \mathrm{MRS}'(\ell) =& \mathrm{MRS}'(h) + \mu_h \hat{c}_h > \mathrm{MRS}'(h) \\ =& \mathrm{MRS}'(w) + \mu_w \hat{C} > \mathrm{MRS}'(w) \end{aligned}$$

with μ_i the Lagrangian multipliers associated with the borrowing constraints: $B^{w\prime} = 0$ and (2.3). Due to the debt limit restricting the resources available to the high-productivity entrepreneurs, these agents have a relatively lower marginal rate of substitution than low-productivity firms: their marginal valuation of wealth is high at time 1, and low at time 2.

The interest rate consistent with market clearing is lower than in a perfect allocation, so as to ensure that all of the goods that could potentially be produced can be consumed and invested. In particular, if the interest rate is sufficiently low, financiers start investing in productive capital. So long as the interest rate on loans is exactly equal to the return of the ℓ technology, these entrepreneurs are in fact indifferent between investing in productive capital or in financial markets. For appropriate initial conditions, they will do both in equilibrium.

$$R = \mathrm{MPK}'(\ell) \equiv \frac{\alpha a_\ell}{(\hat{K}'_\ell)^{1-\alpha}}$$

This implies that the amount of low-productivity firms that invest and produce is no longer zero. A binding borrowing limit generates a redistribution of capital within the productive sector from high to low productivity firms. As a result, TFP is lower than in a first best allocation. The lower efficiency in aggregate production in turn results in a lower equilibrium wage level.

The left panel of figure 1 shows that a borrowing limit constrains the maximum amount of bonds that can be demanded for any level of the interest rate to the blue vertical line. Correspondingly, financiers partly invest in their production technology, reducing the supply of bonds for any level of the interest rate. The constrained laissez-faire equilibrium is at point C, where the equilibrium interest rate is now lower. The borrowing constraint also implies that the MPK of the high-productivity entrepreneur no longer corresponds to the interest rate. Productive

[18] I consider the case where workers would like to borrow to consume more in the present as the baseline. This can arise for example from pre-existing debt they have to repay. Section 6.2 considers the opposite case.

firms perceive an additional benefit of investment: besides the additional resources available in the future period, they have also access to more resources today thanks to a more relaxed borrowing limit. This implies that the relationship between the gross interest rate and the MPK is now a shifted down version of the original line, accounting for the Lagrangian multiplier on the borrowing constraint.[19] The constrained equilibrium is at point C, in correspondence to the new level of interest rate, where the h firms only invest up to $\hat{K}_h^{\prime c}$, while low productivity firms invest an amount that is proportional to this capital-labour ratio, but weighted for the distance in productivities of the two types of firm.

The lower TFP generated by the limit on debt affects the labour market too, as can be seen in figure 2. For any level of wage, the lower efficiency in production induces a lower level of labour demand. The constrained laissez-faire equilibrium is where the blue line meets the labour supply, where both the equilibrium wage and the level of employment is lower.

2.4.3 Financially constrained, second best allocation

Consider a planning problem where a social planner can choose the allocation for one period, subject to the same financial constraints as the decentralised economy, and lets the competitive equilibrium be realised thereafter.

Definition 2.2. *A constrained efficient or second best allocation is the solution to the following problem:*

$$V_t^P(K_h, K_\ell; \theta) = \max_{\tilde{C}, \hat{c}_i, K_i', B_i'} \Bigg\{ \sum_{i \in h,l} \chi_i \pi_i \left[\log \hat{c}_i + \beta V_{it+1}(z_i'; K_h', K_\ell', B') \right] + + \log \tilde{C} + \beta W_{t+1}(K_h', K_\ell', B') \Bigg\} \tag{2.6}$$

$$\textit{subject to} \qquad \tilde{C} + \sum_{i \in h,l} \pi_i \left(\hat{c}_i + K_i' \right) = Y - (1+\psi)^{-1} L^{1+\psi};$$

$$B' \leq \theta \left(Y_h' - \omega' N_h' \right)$$

with $\tilde{C} \equiv \hat{C} - (1+\psi)^{-1} L^{1+\psi}$*,* $Y = \sum_i a_i k_i^\alpha n_i^{1-\alpha}$*, and where the planner internalises how current choices affect prices.*

Differently from private individuals, the social planner takes into account how prices are formed. The principle of optimality applies here, so that any inefficiency internalised by the

[19] In particular, the blue dashed line plots the constrained choice of capital for high-productivity entrepreneurs: $R = f\left(\mathrm{MPK}_h^c, \mu\right) \equiv \left[\frac{1}{\mathrm{MPK}_h^c} + (1-\theta)\mu c_h \right]^{-1}$.

planner but not by private individuals is connected to prices either entering the borrowing limit, or the budget constraint of agents who don't all share the same marginal valuation of wealth. For example, there are pecuniary externalities connected to changes in wages hitting workers and producers in opposite ways. If there is a difference in how these agents value wealth at the margin, then this creates the opportunity for a Pareto improvement. These forces can act to either reinforce the effect of pecuniary externalities arising from the borrowing limit, or they can go in the opposite direction.

The following two assumptions describe the limits and possibilities of interventions that are implicit in the social planner's problem: 1) the social planner does not have sufficient instruments to completely undo the financial frictions; 2) the social planner can utilise enough instruments to perfectly implement the second best allocation.

The first point implies that the borrowing limit has to be satisfied in both the second best and the laissez-faire economy. For example, it may not be possible for policy makers to completely undo the moral hazard and limited commitment problems that generate the credit crunch. Because the only advantage of the social planner compared to single individual is to internalise how prices depend on choice variables, if prices are not a function of choice variables in equilibrium then the laissez-faire allocation is second best.[20]

The second point clarifies that, while a first order concern in practice, problems of imperfect implementation are abstracted from here. Rather than focusing on how to best use one particular policy instrument, this work looks at what are all the possible margins of interventions in the laissez-faire economy. In practice, the social planner might not have sufficient instruments or information to achieve the second best allocation. As an example, interventions after the crisis may consist of resolution policies, but there may not be a way to directly subsidise a firm's investment. Nevertheless, given the multitude of instruments at policymakers' disposal, and the continuous introduction of new tools, it is useful to single out all the margins where intervention could be beneficial.

The definition of these three allocations will be useful in the analysis that follows, where a shock moves the economy from a fully unconstrained to a financially constrained setting. I will then compare the laissez-faire allocation to a second best allocation.

3 A Financial Crisis

Consider the effect of a credit crunch restricting firms' access to credit. In particular, assuming a perfect allocation of resources at date 1, consider the effect of a low θ, which precludes efficient

[20] See appendix B.3 for an example where the financially constrained laissez-faire allocation is second best.

firms from borrowing as much as needed for date 2. First, I show that a financial crisis can only occur if $\theta < 1$, that is, firms cannot use the entire value of their profits to obtain credit. If profits can fully be recovered by potential lenders, then there is no financial friction and the economy is first best.

Lemma 1. *If $\theta = 1$ the allocation is first best.*

Proof. See appendix B.1. □

3.1 Analysis of the Financial Crisis

Date 1: ***During the crisis.***
Let the allocation of capital at the beginning of period 1 be perfect: all of the capital is owned by the more productive entrepreneurs, and they have an aggregate amount of debt B_h to repay. In turn, low productivity firms start the period receiving the returns from the loans they extended to productive firms.[21] At date 1, however, the productive sector is subject to a credit crunch. The restriction in borrowing means productive firms are no longer on their Euler Equation. The supply of savings exceeds the demand. A lower interest rate induces a lower demand for financial savings and an increased demand for investment from unconstrained agents. While the lower interest rate has the potential to induce a lower aggregate productivity, and impair the *quality* of investment, it also ensures that the *quantity* of capital invested in the aggregate is kept closer to the efficient level, by inducing the low productivity entrepreneurs to invest in setting up firms for production in the following period.
Date 2: ***After the crisis.***
After date 2, the world ends. Therefore, there can be no demand for debt, and no entrepreneur would want to take on any new investment: agents simply make their static consumption and labour choices. The financial friction of date 1 however implies that TFP at date 2 is lower.

Proposition 1. *A financial crisis at date 1 induces no change in aggregate output within the period, but lower aggregate productivity and production at date 2.*

Proof. See appendix B.2 □

At time 1, labour demand is chosen to solve problem (2.1), and for given level of pre-installed capital, it is unchanged compared to the frictionless case. Additionally, the GHH preferences imply that the supply of labour at time 1 is also unchanged. Hence, compared to the first

[21] One can imagine that the economy at date 0 was at an unconstrained, first best steady state. In section 5 this previous period is explicitly modelled.

best, output remains the same at time 1. The lower interest rate is what allows the low productivity entrepreneurs to pick up the slack, by absorbing the extra resources produced that can no longer be demanded for investment by the high-productivity entrepreneurs. As a result however, realised TFP following the credit crunch is reduced, as high productivity entrepreneurs are no-longer the only active firms, and part of the investment is carried out by less efficient firms. The aggregate productivity-weighted investment in the economy is therefore lower at time 2, which reduces the equilibrium real wage and the aggregate level of employment.

3.2 Interventions during the Crisis

This section compares a second best allocation as defined in subsection 2.2 to the laissez-faire constrained allocation. A binding borrowing constraint combined with a lower equilibrium interest rate induces the low-productivity entrepreneurs to enter the market and start production, which poses the question of whether the resulting distribution of active firms' productivity is constrained efficient, or whether a regulator might want to intervene to alter it. The following definition is useful for addressing circumstances in which the planner optimally chooses to reduce the number of low-efficiency firms in operation.

Definition 3.1. *Zombie firms are low-productivity firms that produce in the constrained laissez-faire economy, but remain inactive in a second best allocation.*

In a first best allocation the number of low-productivity firms that are active is zero. In this sense, all low-productivity firms investing in the financially constrained laissez-faire economy could be considered *zombie* firms, if a first best allocation is chosen as reference. However, in presence of a borrowing limit, the first best allocation can no longer be achieved, and therefore, it is useful to refer to a constrained efficient allocation in defining zombie firms.

Proposition 2. *The allocation in a financial crisis is not second best. Compared to a constrained efficient allocation the laissez-faire economy features overinvestment and zombie firms.*

Proof. See appendix B.4. □

To give an insight into this proposition, I will first show that the choices of the constrained workers and entrepreneurs is second best. I then show that the choice of investment for the low productivity entrepreneurs is not second best, and in particular, that in a second best, these firms optimally invest less.

The choice of the planner for debt at date 1 corresponds to the laissez-faire allocation, as it is always optimal for high productivity firms to borrow as much as possible.[22] On the other hand, there is no way for the planner to undo the financial friction that prevents workers from accessing financial markets. As a result:

$$\tilde{\mu} = \Delta\text{MRS}'(\ell, h) = \Delta\text{MRS}'(\ell, w), \tag{3.1}$$

where $\Delta\text{MRS}_{0,1}(h, \ell) \equiv \text{MRS}_{0,1}(h) - \text{MRS}_{0,1}(\ell)$ is the distance in marginal rates of substitution of entrepreneurs with high and low productivity and with $\tilde{\mu}$ the Lagriangian multiplier that the social planner attaches to the borrowing limit. Likewise, the choice of investment of the high-productivity firms is also constrained efficient: it is always optimal to let efficient firms invest as much as possible. However, the social planner's choice of capital for the low-productivity entrepreneurs is pinned down by the following optimality condition:

$$1 - \text{MRS}'(\ell)\text{MPK}'(\ell) = -\left[\tilde{\mu}\theta N_h' - \underbrace{\Delta\text{MRS}'(\ell, h)}_{>0} N_h' + \underbrace{\Delta\text{MRS}'(\ell, w)}_{\geq 0} L'\right] \frac{\partial \omega'}{\partial K_\ell'} \tag{3.2}$$

where

$$\frac{\partial \omega'}{\partial K_\ell'} = \frac{\alpha\psi}{\alpha + \psi} \frac{\omega' \hat{a}_\ell}{\sum_i \hat{a}_i K_i'} > 0.$$

The left-hand-side of this expression coincides with the decentralised optimal choice. The choice of the planner differs from private individuals as the right-hand-side is in general not zero. In particular, one extra unit of investment by a low-productivity firm generates up to three different spillovers that individuals do not take into account, all connected to an increase in wages. A higher level of aggregate investment in the sector in fact contributes to raising labour demand for every wage level, resulting in a higher equilibrium wage.

Firstly, there is a pecuniary externality connected to the cashflow constraint: a higher wage increases costs of production and contributes to reducing the value of firms' profits, therefore restricting the borrowing ability of high-productivity firms. Second, a change in wage affects the budget constraint of productive entrepreneurs, who have a lower marginal rate of substitution than low-productivity firms. The increased cost of production reduces the resources available to the more productive firms at time 2, when their valuation of resources is lower. This is beneficial in the aggregate, as it reduces the distance in MRSs. The two aforementioned effects go in opposite directions, but because $\theta < 1$, the latter one always dominates in the aggregate.

[22] By equating the two first order conditions of productive and unproductive entrepreneurs, the same optimality condition follows in the laissez-faire economy.

Third, a higher wage increases resources available to workers at time 2, when they value resources less because they are constrained today, so this is not welfare improving.

By combining expression (3.2) with (3.1) one can derive the sign of the aggregate effect of an increase in wages:

$$1 = \text{MRS}'(\ell)\text{MPK}'(\ell) - \tilde{\mu}\left[\underbrace{L' - (1-\theta)N_h'}_{>0}\right]\frac{\partial \omega'}{\partial K_\ell'}$$

In a second best allocation, the social planner optimally chooses a lower amount of investment of low productivity firms. The net negative effects of a higher wage in the form of reduced borrowing capacity for productive firms and more resources allocated to workers in a period when they value resources less are sufficient to always dominate the benefit of reduced resources to productive entrepreneurs in a period when they value resources less.

As a result, in a financial crisis, zombie firms arise. This force will now be assessed against the need to increase demand during a deleveraging phase that is not accompanied by a strong enough reduction in the real interest rate.

4 A Financial Crisis at the Effective Lower Bound

Consider now a situation where the real interest rate is bounded from below:

$$R \geq \rho \tag{4.1}$$

This constraint is exogenous and taken as given here, but it is in general consistent with a lower bound on the nominal interest rate, combined with nominal rigidities.[23] Without loss of generality, in what follows, the lower bound is normalised to 1.[24]

In normal times, the interest rate can adjust to ensure that the aggregate quantity of savings equals the total amount of investment:

$$Y - C = K'$$

Outside the lower bound, aggregate demand is sufficient to induce full capital utilisation, so labour demand is at the efficient level.[25] If however the interest rate that would be needed to

[23] Sticky inflation expectations can be motivated in various ways: a constant inflation target for monetary policy, the New-Keynesian framework, etc.

[24] This would correspond to a situation where prices are fully rigid.

[25] That is, such that $\omega = \text{MPN}$, with MPN the marginal product of labour.

clear the market is below the lower bound, then the real rate is constrained, aggregate demand for investment is too low, while demand for savings is too high. In this case, the pre-installed stock of capital cannot be fully utilised, so production is below the efficient level:[26]

$$\begin{aligned} d_i &= \max_{n_i} \bar{y}_i - \omega n_i \\ \text{s.t.} \qquad \bar{y}_i &= \frac{1}{\pi_i}\left(C + K' - \sum_j y_j\right) \leq a_i (k_i)^\alpha (n_i)^{1-\alpha} \end{aligned} \tag{4.2}$$

Firms are capable of producing more, given their technology and previous capital investment. However, because the real interest rate is relatively too high to clear the market, aggregate demand is insufficient and cannot absorb the entire amount of potential output. In this case, production is demand-determined, demand for labour is below the efficient level and the marginal product of labour is larger than the wage rate.

4.1 The Effects of a Liquidity Trap

An interest rate that is stuck at the lower bound, and is therefore inefficiently high, induces entrepreneurs to invest less than would be optimal. The level of investment of low-productivity firms outside the lower bound would be such that the optimal level of aggregate investment is maintained, in spite of the cashflow constraint limiting how much productive firms can invest. When the interest rate is at the lower bound, low productivity firms invest less:

$$\rho \equiv 1 = \frac{\alpha a_\ell}{(\hat{K}'_\ell)^{1-\alpha}} = \left[\frac{\hat{\alpha} a_\ell}{(\omega')^{1-\alpha}}\right]^{1/\alpha} \tag{4.3}$$

This expressions shows that low productivity firms investment will be chosen so as to ensure that the return offered on the financial market exactly corresponds to the return on their investment technology. But as the return offered on bond is too high at the ELB, they choose to invest less, maintaining a higher marginal product of capital. The choice of employment after the crisis is optimal, which implies that the capital labour ratio $\hat{K}'_\ell$ depend on future wages.[27]

[26] Equal rationing across all firms producing is assumed.

[27] No intertemporal decision has to be made in the last period, so the capital available at date 2 is used to full capacity in order to maximise consumption.

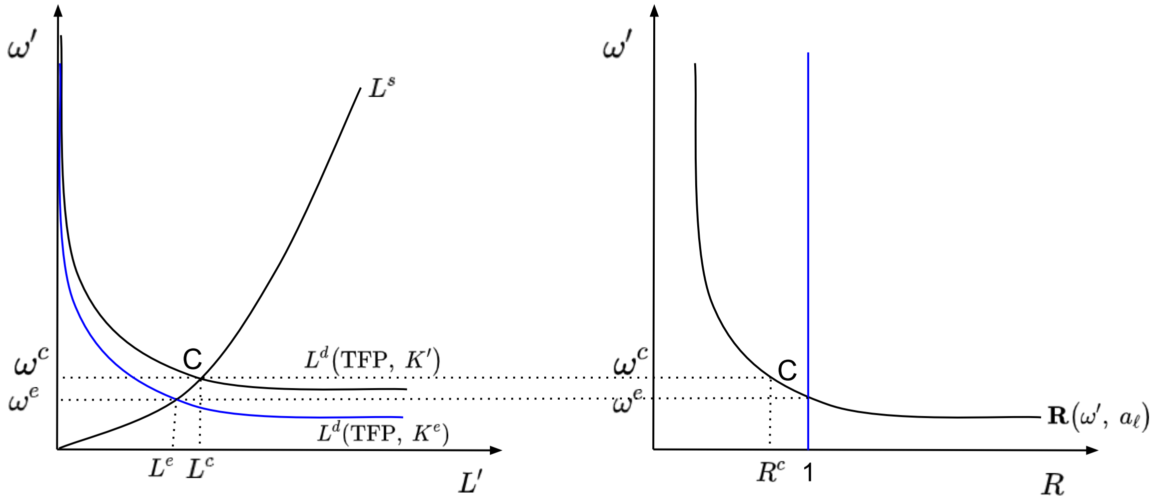

Figure 3: Labour market clearing and relationship between prices outside and at the ELB (blue line).

Figure 3 shows the labour market equilibrium and the relationship between future wages and current real rate at the lower bound. The effective lower bound implies that the interest rate cannot fall below a threshold of 1. The corresponding wage is lower than in a financial crisis where the lower bound is not binding. This is generated by a lower level of aggregate investment pushing the labour demand curve down. Lower future wages increase future earnings and for productive firms, this corresponds to a less binding borrowing constraint.

An interest rate above the optimal market clearing level has implications not just for aggregate investment but for consumption too. Unconstrained agents are on their Euler equation, but both the interest rate that is too large, as well as the fact that future consumption is lower due to the reduced current investment, induce a lower level of consumption demand of low-productivity entrepreneurs and financiers in the present.

As aggregate demand for both investment and consumption is low, the pre-installed level of capital can no longer be fully utilised: the full-capacity level of production cannot be absorbed. As a result, a lower level of labour demand than full employment arises. The wage rate falls in the present period to ensure that the labour market is in equilibrium. This induces workers to also demand less consumption. As less output is produced at time 1, high productivity firms have access to less resources for investment. The weak demand at time 1 turns into a supply-side problem in the following period: production is low at date 2, even though the economy escapes the lower bound, due to the low level of capital that was invested at date 1.

Proposition 3. *At the effective lower bound, a financial crisis generates a recession featuring* *lower employment and output at date 1, as production is demand-determined. Date 2 features a supply-driven recession, but fewer low-productivity firms operate.*

Proof. See appendix C.1 □

The proposition clarifies that while the low level of investment restricts production at time 1, it also induces fewer low-productivity entrepreneurs to invest in their production technology. While the average *quality* of capital investment can be higher, the *quantity* is inefficiently low, restricting production possibilities. In this sense, date 2 can see a higher level of productivity, yet it features a supply driven recession.

4.2 Interventions in a Financial Crisis at the Effective Lower Bound

When a financial a crisis brings the economy to the effective lower bound, productive firms continue to be constrained in their choice of debt and investment. The less efficient entrepreneurs, on the other hand, invest less due to the high opportunity cost of operation, and consume less in the present given their lower future available resources. The social planner problem is altered by the presence of an effective lower bound to account for both 1) investment demand limiting consumption demand of the low-productivity entrepreneurs (*Euler equation*), as well as 2) aggregate demand restricting current output. The solution of this revised planner's problem leads to the following proposition.

Proposition 4. *During a financial crisis where the economy is at the lower bound, the allocation at time 1 is not constrained efficient. Provided that the demand externalities are sufficiently strong, the laissez-faire economy features low aggregate output at time 2 and under-investment, as more low-productivity entrepreneurs should engage in production.*

Proof. See appendix C.2 □

During a liquidity trap, the interest rate is at one, which makes the opportunity cost of production so high that fewer low-productivity entrepreneurs run firms, inducing them to keep their savings on the financial market instead. While this contributes to a higher aggregate TFP, it also constrains demand, limiting the effective level of output below potential. This represents the demand-side concern that cannot be addressed via a lower interest rate in presence of a liquidity trap. From this point of view, a regulator would find it beneficial to encourage some investment from low-efficiency entrepreneurs, so as to moderate the extent of the rationing. On the other hand, the negative spillover that less efficient firms would normally create, both on the borrowing capacity of productive entrepreneurs as well as on the distribution of resources to agents with higher marginal valuation, plays a smaller role when these firms invest less. So the regulator has to trade off these two aspects: by inducing more low productivity entrepreneurs to start operation, the rationing of the demand is relaxed, but these negative externalities become more important.[28]

From the point of view of the planner, the optimal choice of investment for low-productivity firms at the lower bound is:

$$1 = \text{MRS}'(\ell)\text{MPK}'(\ell) - \tilde{\mu}\left[L' - (1-\theta)N_h'\right]\frac{\partial \omega'}{\partial K_\ell'} + \frac{\varepsilon}{\beta} + \gamma\zeta\frac{\partial \omega'}{\partial K_\ell'} \tag{4.4}$$

where ε and γ are the Lagrangian multiplier associated to the Euler equation of financiers and the demand constraint, while ζ is a function of parameters defined in the appendix. Additional investment of low productivity firms has two effects which private individuals do not take into account: first, it allows more consumption for the low-productivity entrepreneurs, by relaxing their Euler equation, which induces a higher demand. Second, it increases resources currently available by reducing the effect of the rationing, hence allowing productive firms to also invest and consume more. Provided that these effects are sufficiently strong, less productive firms should invest more.

5 Interventions Before the Crisis

In the analysis so far, the economy starts the period by facing a credit crunch. But much of the literature on financial regulation and financial stability tends to focus on potential policy interventions before a crisis can take place. In this section, I analyse such ex-ante interventions, considering a setting where a financial crisis occurs with probability one, and where everyone anticipates it will happen.[29] I show that while from a prudential perspective restricting debt can make the crisis less likely, a lower stock of debt has the potential to push the interest rate at the lower bound. If this happens, a lower stock of debt makes demand even weaker and worsens the extent of the rationing, as will be shown in more details below. In order to analyse policy interventions before the crisis, it is necessary to add a time period: $t = 0$.

Date 0: *Before the crisis.*
At date 0, both types of entrepreneurs are endowed with a certain amount of capital; they can

[28] As a result, if the rationing on the demand side is sufficiently small compared to the negative externalities generated, a financial authority might continue to prefer lower investment from low-productivity firms.

[29] Korinek and Simsek (2016) show in a setting without capital investment that adding aggregate uncertainty moderates some of the results, without changing the main intuition.

produce and consume in this period, and their choices are unconstrained. As a result, both types of producers are able to perfectly smooth their consumption between date 0 and 1. Only the high productivity entrepreneurs choose to become firms, investing positive amounts for time 1, while the less efficient entrepreneurs prefer to be financiers. By providing the high productivity producers with the necessary financial resources for production, the low productivity entrepreneurs can earn a higher rate of return than their production technology, and equal to the return of more productive firms. The equilibrium interest rate is in fact equal to the return on investment of the high productivity entrepreneurs. This in turn implies that the choice of employment is optimal, as the interest rate is not constrained at the lower bound.

5.1 Interventions Before the Crisis without an ELB

I start from analysing whether there is potential for interventions before the crisis. While the borrowing constraint depends on prices at time 2, the planner can still affect the borrowing constraint through interventions at time 0, given that prices depend on past choices when capital is a production input. In net, the cost of distorting the laissez-faire allocation before the credit crunch is smaller than the benefit of partially undoing the borrowing limit when the shock takes place.

Proposition 5. *The allocation before a financial crisis is not constrained efficient. Active firms in the decentralised economy borrow too much while they can over- or under-invest; the constrained efficient economy features higher output than the laissez-faire economy at date 1 in case of under-investment, vice versa in case of over-investment.*

Proof. See appendix D.1. □

First, note that the planner would never choose to let low-productivity firms produce at time 1, given the absence of financial friction and the superior production technology of productive firms. The choice of debt of the planner at time 0 to be repaid in time 1 can be summarised with the following optimality condition:

$$\Delta \text{MRS}(h,\ell)\left[\frac{B'}{(R)^2}\underbrace{\frac{\partial R}{\partial B}}_{>0}+1\right]=\beta\theta\tilde{\mu}\underbrace{\frac{\partial D_h'}{\partial B}}_{<0} \tag{5.1}$$

In the decentralised economy, the distance in marginal rates of substitution of entrepreneurs with high and low productivity is zero, as no financial friction affects the economy at date 0. There are two elements that the planner internalises which private individuals do not: 1) how a larger stock of debt increases the interest rate, redistributing resources from the borrower to the lender; 2) how a larger stock of debt at time 0 affects future profits of productive firms negatively, tightening their borrowing constraint at time 1. Future investment by high productivity entrepreneurs is negatively affected by current larger borrowings, while the opposite is true for financiers, who can invest more in the future if they save more currently. Lower investment by productive firms at time 1 directly reduces their profits, while higher investment by low productivity firms contributes to increasing the equilibrium wage, thus indirectly decreasing the profits of productive firms. As a result, the value of manufacturing profits of high productivity firms decreases with larger amounts of debt. As current large stocks of debt have a negative impact on the future borrowing constraint, the marginal rate of substitution from date 0 to 1 of high productivity firms is below that of low productivity firms: $\Delta\mathrm{MRS}(h,\ell) < 0$. Crucially, the first effect on the interest rate washes out in the aggregate, unless the second effect on the borrowing limit is at play. The second best allocation at time 0 differs from the laissez-faire allocation, as the planner chooses a lower level of aggregate debt to help with the future borrowing constraint.

The choice of capital investment at date 0 for date 1 of the planner is:

$$1 - \mathrm{MRS}(h)\mathrm{MPK}(h) = \underbrace{\Delta\mathrm{MRS}(\ell,h)}_{>0} \frac{B'}{(R)^2} \underbrace{\frac{\partial R}{\partial K}}_{<0} + \beta\theta\tilde{\mu} \underbrace{\frac{\partial D'_h}{\partial K}}_{>0} \tag{5.2}$$

More investment at time 0 increases the net worth of all entrepreneurs in the following period: both types of firm can invest more at date 1. In the aggregate, this increases the amount of resources that productive firms can use to obtain borrowing. On the other hand, (5.1) shows how the MRS of high-productivity entrepreneurs is lower than that of low-productivity firms. Then, the planner also considers how a larger stock of installed capital at date 1 induces a lower interest rate, which hurts financiers and is negative in the aggregate.

These result would not go through if capital were the only factor of production, as prices would not depend on choice variables in that case, and there would be little that a social planner could do to alter the efficiency of the allocation.[30] Proposition 5 would also not hold if production factors could be chosen in the same period that output is produced. Ottonello, Perez, and Varraso (2019) make the point that the timing of borrowing constraints is crucial for justifying macroprudential interventions, as only current-price constraints involve pecuniary externalities that a planner can intervene to internalise. While valid in model with only labour employed within the period, this does not hold when capital is one of the production factors that

[30]See appendix B.3 for a proof that the allocation at time 1 would be constrained efficient if only capital were used in production. Analogous arguments hold at time 0.

is invested the period before. The proposition therefore demonstrates that there can be scope for macroprudential interventions, even when the borrowing constraint emerges from borrowers' misbehaviour at the time of repayment. However, it remains true that the particular type of required intervention crucially depends on the timing of borrowing limits.

5.2 Conditions That Can Push the Interest Rate Towards the Lower Bound

The economy is at the ELB if the interest rate necessary to ensure that labour demand is optimal (R^*) is too low to satisfy the non-negativity constraint (4.1), so that $R = 1$.[31]

$$R^* = \Omega' \hat{a}_\ell \left(\frac{1}{\hat{a}_h K_h' + \hat{a}_\ell K_\ell'} \right)^{\frac{\psi}{\psi+\alpha}} < 1 \tag{5.3}$$

Lemma 2. *The interest rate is less likely to be at the effective lower bound if the initial stock of debt is large.*[32] *There is a minimum level of aggregate debt $\underline{B}$ above which the effective lower bound in never binding.*

Proof. See Appendix D.2 □

Expression (5.3) shows that the interest rate consistent with efficient labour demand is a decreasing function of future productivity-weighted capital investments. Expectations of reduced TFP in the productive sector mean that goods can no longer be produced as efficiently as before. Labour costs fall, which induces a higher return to investment. A higher stock of debt influences the future efficiency of production in presence of a credit crunch, by inducing less investment from productive firms, who have to repay the debt, and more investment from low-productivity firms, who have a larger net worth when B is large. This brings about a lower TFP and increases the equilibrium interest rate.

The macroprudential literature on aggregate demand externalities has shed light on how larger stock of debt is likely to push the economy towards the effective lower bound.[33] The equilibrium interest rate is in fact *decreasing* in the stock of debt in environments without a supply side. Intuitively, when debt only finances consumption, a larger amount of initial debt before a deleveraging shock requires a larger fall in the interest rate so as to induce savers to demand less bonds and consume more. Vice-versa, in presence of corporate debt backing

[31] Ω' is a time-varying function of parameters defined in the appendix.

[32] While the setting features no uncertainty, the term *likely* is used here to refer to the fact that it can happen for a larger set of initial conditions and parameter values.

[33] See e.g. Korinek and Simsek (2016).

physical capital, a larger stock of initial debt corresponds to lower average productivity in the following period, inducing a lower wage rate which pushes the return to investment up.

5.3 Interventions Before the Crisis with an ELB

Lemma 2 hints at the fact that the choice of debt before the crisis could help reduce the chances of an effective lower bound being binding, hence affecting the outcome of the crisis when the interest rate is subject to a lower bound. A social planner that can choose a sufficiently large level of debt in period 0 can ensure that the market clearing interest rate never becomes constrained. Whether this is beneficial or not, depends on the importance of other spillovers at play away from the ELB.

Proposition 6. *In presence of a lower bound on the real rate, the allocation before a financial crisis is not constrained efficient: it features under-borrowing if the effects of the demand rationing are sufficiently strong. It can feature either under- or over-investment depending on parameter values.*

Proof. See appendix D.3. □

The planner's optimal choice of debt is:

$$\Delta\text{MRS}(h,\ell) = \text{MRS}(h)\underbrace{\frac{\partial\mathcal{D}(\cdot)}{\partial B}}_{>0} + \beta\theta\tilde{\mu}\underbrace{\frac{\partial D_h'}{\partial B}}_{<0} \tag{5.4}$$

Without an ELB, larger debt holdings do not affect current profits before interest payments in any way: only K_1 has an influence on firms' earnings. Introducing a lower bound on the interest rate implies that production is demand-determined: larger quantities of debt directly affect the aggregate level of production.

$$\begin{aligned} C_h + C_\ell + C_w + K_h' + K_\ell' &= Y \\ (1-\hat{\beta})(Y - \omega L - B) + \frac{1}{\beta}\left(\theta\tilde{a}K_h' + K_\ell'\right) + K_h' + K_\ell' &= Y - \omega L \\ \Longrightarrow \qquad \frac{\hat{a}_h - \hat{a}_\ell}{(1-\theta)\hat{a}_h}B + \frac{\mathtt{w}(\omega')}{\hat{\beta}A_h} &= Y - \omega L \end{aligned}$$

where the Euler equation for the low-productivity entrepreneur was used together with $K_h' = \frac{\hat{\beta}\hat{a}_\ell(Y-\omega L-B)}{\hat{a}_\ell-\theta\hat{a}_h}$, $\sum\hat{a}_i K_i' = \mathtt{w}(\omega')$, and where $\omega' = (\hat{\alpha}a_\ell)^{\frac{1}{1-\alpha}}$ from (**??**). From these expressions, it is apparent that B affects aggregate demand positively, which in turn determines output at the lower bound. First, the consumption and investment function of productive firms is not affected by the lower bound. Second, consumption of low productivity firms is constraint by the Euler equation, and depends on investment from both high and low-productivity firms. Finally, investment of low productivity firms can be found as a residual from the future labour market clearing condition.

A larger stock of debt has two effects. First, it assigns more resources to the low productivity firms, which, being unconstrained, are affected by an interest rate that is inefficiently high. This implies that more debt corresponds to more rationing. Second, higher debt means that the high productivity entrepreneurs have access to less resources, which induces a bigger need for investment of low-productivity firms to clear the market. From this point of view, larger debt reduces the incidence of rationing. This latter force always tends to dominate in the aggregate, because high-productivity firm's choices have a larger weight, given their higher productivity. Therefore, a larger stock of debt has an overall positive effect on boosting aggregate demand, and in particular, it contributes to increasing profits, so that $\partial\mathcal{D}(\cdot)/\partial B > 0$.

While subdued due to the interest rate being stuck at one, it is still the case that increased debt negatively affects future cashflows, and likewise increased investment increases them. However, debt now also directly affects profits in the current period, which in turn affect demand and production positively. The social planner now faces a tradeoff between increasing demand, hence reducing the effect of the liquidity trap, and correcting the borrowing externality. If the efficient interest rate is not much below one, the demand rationing is likely to be mild, and the overborrowing component is likely to dominate. Vice versa, if the efficient interest rate is negative and substantial, then undoing the liquidity trap by increasing aggregate demand becomes the main concern and the economy features underborrowing.

Concerning the planner's optimal choice of investment, there is no response of the interest rate to changes in investment demand, as the interest rate is fixed at 1.[34] The constrained efficient choice of capital investment with a ELB is similar to the choice without a ELB, but the benefit of increasing investment is now larger, as it stimulates demand.[35]

Proposition 5 showed how even away from the lower bound, the allocation is not constrained efficient at date 0 as it features overborrowing and inefficient investment connected to the future binding constraint. Proposition 6 on the other hand illustrated how more debt can benefit the economy by reducing the demand rationing. As a result, it is not necessarily the case that a pecuniary externality connected to a borrowing limit always reinforces potential interventions connected to aggregate demand externalities: in this case, there can be a conflict between

[34] The same also holds true for the choice of debt.

[35] For this reason, the expression is not reported here.

optimal interventions aimed at addressing these two sources of inefficiencies.

6 Extensions

Sections 3 and 4 presented the most minimal model suited to analyse the presence of zombie firms at the effective lower bound. Some of its distinctive features were the presence of only one sector, workers as hand-to-mouth consumers, and the type of financial friction considered linking firms' borrowing to their cashflows. In this section, each of these assumptions will be relaxed in turn. I show that the presence of zombie firms in a financial crisis is not altered by the presence of another, less capital-intensive sector. Nevertheless, the presence of a labour-intensive sector introduces a further tradeoff between inefficient resource allocation within and across sectors. I then analyse the case where workers are free to borrow proportionally to their wage earnings. Provided that the fraction of future earnings that they can borrow against is sufficiently large, this overturns the result of zombie firms in a crisis. Finally, the cashflow constraint for firms is replaced with a collateral constraint to show that a financial crisis would not feature zombie firms, but on the contrary, larger investment by low-productivity firms would be considered beneficial, when firms can borrow against the value of a collateral asset.

6.1 Introducing a Labour-Intensive Sector

Assume that the economy is composed not just of a capital-intensive sector with heterogeneous producers, but also of a labour intensive sector, where all producers have access to the same level of technology. I will call the capital intensive products *manufacturing* (m) and the labour intensive products *services* (s). The production function in the service sector is linear: $Y_s = N_s$. Combined with perfect competition, this implies that the nominal wage has to equal the marginal product of labour in the service sector, $\omega = 1$. Assume that workers supply labour to all sectors in the economy and are the owners of the service production technology.

Both workers and entrepreneurs demand manufacturing and service products according to a Cobb-Douglas composite function, $\hat{C} = (C_m)^\gamma (C_s)^{1-\gamma}$. This assumption implies that consumer demand is allocated to the two goods depending on relative prices. Normalising the price of service goods to 1, p_m is the price of manufacturing goods in terms of services. As it is standard with this type of demand function, consumers devote a constant fraction of their overall consumption expenditure to each of the goods in the consumption bundle.

$$p_m c_{mi} = \gamma p \hat{c}_i, \qquad c_{si} = (1-\gamma) p \hat{c}_i \tag{6.1}$$

where p is the aggregate price level of the consumption bundle: $p = \frac{(p_m)^\gamma}{\gamma^\gamma(1-\gamma)^{1-\gamma}}$. The equilibrium price of manufacturing depends on relative productivities in the two sectors. While productivity in the service sector is fixed at 1, the manufacturing sector level of productivity is affected by how capital is distributed among producers. In a first best allocation, only high-productivity firms operate and therefore the price is negatively related to $a_h K_h$.

With the introduction of a cashflow-based borrowing limit, the price of manufacturing products serves an important role, as it enters the borrowing constraint: $b'_{mi} \leq \theta \left(p'_m y'_{mi} - \omega' n'_{mi}\right)$. In a credit crunch, low-productivity firms start investing and producing, so TFP in manufacturing is lower than in a first best allocation. The lower efficiency in aggregate production in turn generates a higher price of manufacturing. Correspondingly, equilibrium real wages are lower and less labour is employed, causing both sectors to be smaller than in the first best. Consumers' preferences are such that the reduction in manufacturing is always larger than services, making manufacturing goods more expensive. A binding borrowing limit therefore generates a redistribution of capital within the sector from high to low productivity firms and a sectoral redistribution of output from production in the constrained manufacturing sector to the unconstrained service sector. Both of these effects contribute to reduce the efficiency of the economy.

Lemma 3. *In presence of two sectors the laissez-faire allocation in a financial crisis is not second best, as the economy features:*

- *Over-investment in manufacturing and zombie firms;*
- *In the aftermath of the crisis, a relative size of the manufacturing sector that is too large;*
- *Aggregate output at time 2 that is too large.*

Proof. See appendix E.1. □

In a setting with only one sector, the wage played a crucial role as it introduced a pecuniary externality connected to the borrowing limit. But in presence of another, labour-intensive sector, the nominal wage rate is uniquely pinned down by productivity in that sector: it is fixed at one and it therefore no longer gives rise to any spillover. It is now the relative price of manufacturing, however, to influence the borrowing ability of productive entrepreneurs, as well as entering agents' budget constraints.

Just like in the case of a one sector economy, the borrowing constraint for productive firms can be relaxed by increasing the value of all firms' cashflows, through an appropriate change in price: in this case, an increase in the price of manufacturing. Less investment from low-productivity firms boosts the value of goods produced in the sector and helps alleviating the financial friction. At the same time, a higher price of manufacturing contributes to further shrinking the relative size of the manufacturing sector compared to the service sector: better TFP within the capital-intensive sector comes at the cost of a worse resource allocation across sectors. A higher price of manufacturing is however also helpful in redistributing resources away from workers, who face a higher aggregate consumption price level, at a time when they value resources less. These two effects are strong enough to overtake the benefit of a lower price of manufacturing, consisting in a redistribution of resources away from high productivity firms in a period when they value consumption less.

6.2 Workers Can Borrow

Assume that workers are now free to save and borrow. They will choose between these two options by comparing the return offered on financial markets, R, to their expected increase in wage earnings. In particular, workers choose to be borrowers if:

$$\frac{\omega' L'}{\beta \omega L} > R$$

They choose to be savers otherwise. When a credit crunch hits the economy, it is not just entrepreneurs that are subject to a borrowing limit, but also workers. As they do not earn any income from production, they can use their labour earnings in order to obtain credit:

$$B'_w \leq \theta_w \omega' L' \tag{6.2}$$

For constant real wages and $1 < R < \beta^{-1}$, workers would like to borrow, so for sufficiently low θ_w, condition (6.2) holds with equality. Vice versa, if they expect negative wage growth and have no preinstalled debt, they optimally choose to save. Whether or not the workers are constrained matters, as the second-best level of investment depends on this aspect.

Lemma 4. *If workers are unconstrained, then the laissez-faire economy in a financial crisis does not feature zombie firms, but under-investment as low productivity firms should invest more. If workers are constrained and θ_w is sufficiently low, then results in proposition 2 continue to hold.*

Proof. See Appendix E.2 □

If workers' marginal rate of substitution is the same as low-productivity entrepreneurs, then whatever change in wages that affect workers will be irrelevant. Indeed, whether they benefit or are hurt from a price change does not matter from an aggregate point of view, unless they are constrained. With unconstrained workers, there are only two spillovers arising from a reduction in wages induced by lower investment of low-productivity firms: the positive pecuniary externality of a higher value of cashflows relaxing the borrowing constraint of productive firms, and the negative externality induced from increasing the amount of resources available to productive firms at time 2, when they value resources less at the margin. As explained in section 3, this latter effect on the budget constraint of productive entrepreneurs tends to dominate, as the impact on the borrowing constraint is linked to a borrowing parameter θ which is lower than 1.

This illustrates how the result of zombie firms can be overturned if there is an additional unconstrained party in the economy: when workers optimally choose not to demand any debt, it is no longer the case that a reduction in investment of low productivity firms helps the economy. On the contrary, increasing investment of low-productivity firms boosts the wage rate and helps redistributing resources away from high-productivity firms at time 2, when they have a lower valuation of consumption. In general, it is not the specific details of what workers do in the model that matters for this result, but rather the presence or absence of other constrained agents in the economy.

6.3 The Role of the Type of Financial Friction: A Collateral Constraint

Much of the literature on macroprudential policy and pecuniary externality is not based on a cashflow constraint as in (2.3), but rather on a collateral constraint,[36] such as:

$$b_i' \leq q_h' h_i' \tag{6.3}$$

where q_h' is the price of a fixed asset that can be used as collateral, h_i' the quantity of collateral available to entrepreneur i, and where the full value of collateral is assumed to be recouped by lenders in case of default.[37]

I now proceed to alter the main model to introduce a fixed asset. To this end, I also have to consider an additional time period, $t = 3$, so as to ensure that the borrowing limit at time 1 can feature a price of the asset at date 2 which is well defined.[38] Assume that every period entrepreneurs have to decide how to allocate their investment between two different types of

[36] See e.g. Lorenzoni (2008), Bianchi and Mendoza (2018) etc.

[37] A collateral parameter ϕ could be introduced, if the entire value of the fixed asset cannot be used as collateral. Here, having a fully collateralisable asset is especially convenient, as shown later, and contrarily to the case of an earning-based constraint, it still affects the efficiency of the economy.

[38] In the absence of a date 3 there can be no trade in the asset at date 2, as all entrepreneurs would want to sell, but no one would want to buy.

capital: a fixed asset h_i and physical capital x_i. The fixed asset can be thought of as land; it is in positive fixed supply in the economy, and can be bought and sold at price q_h. As for the physical capital, it represents machines, which are generated by investing a share of the overall output produced, and are assumed to fully depreciate every period. Together, these form the stock of capital necessary to operate a firm: $k'_i \equiv (x'_i)^\delta (h'_i)^{1-\delta}$, with δ and $1-\delta$ the respective share of physical and fixed capital usage in the aggregate bundle. Similarly to the intratemporal allocation of consumption demand, the optimal demands for land and machines are the following:

$$x'_j = \delta q k'_j, \qquad u h'_j = (1-\delta) q k'_j \tag{6.4}$$

with q the price of the aggregate investment bundle: $q = \frac{u^{1-\delta}}{\delta^\delta (1-\delta)^{1-\delta}}$. $u = q_h - \frac{q'_h}{R}$ represents the per-period user cost of the fixed asset, as the future resale value of the asset is netted from its purchasing price. Entrepreneurs choose to employ constant fractions of overall investment in the two forms of capital, where the fractions are pinned down by the respective shares in the capital stock bundle. Even in presence of a collateral constraint, the demands for investment remain the same, so long as the collateral parameter is set to one.

Lemma 5. *A financial crisis away from the ELB induces no change in aggregate output but changes in consumption at time 1; aggregate productivity and production fall at date 2.*

Proof. See Appendix E.3. □

The presence of a fixed asset influences the net worth of entrepreneurs, thereby affecting equilibrium consumption. Nevertheless, production only depends on pre-installed capital and employment, and the choice of employment is not altered by the presence of a fixed asset. Even though production remains constant in presence of a credit crunch, the value of land changes to reflect 1) the future recession; and 2) the additional benefit of land holding, which now helps relaxing borrowing conditions in presence of a collateral constraint. As these two forces move the asset price in opposite directions, it is in principle not possible to state clearly what happens to it the price.In what follows, a financial crisis at time 1 is redefined as a situation where the collateral constraint (6.3) becomes binding.

6.3.1 Interventions During a Collateral-Constraint-Type of Financial Crisis

Proposition 7. *A credit crunch induced by a collateral constraint, away from the ELB, features a laissez-faire allocation that is not constrained efficient:*

- *There is under-investment and too few low productivity firms;*
- *Output at date 2 is too low.*

Proof. See appendix E.4. □

A collateral constraint involves the price of land, which depends on the aggregate net worth. Increasing investment by low-productivity firms boosts the demand for capital and hence its value, which implies that high-productivity firms have access to more borrowing. In this sense, less efficient entrepreneurs benefit more efficient ones with their production if the value of a firm's asset influences its borrowing capacity. Therefore, the financial authority sees the number of firms operating in the laissez-faire economy as too low, as the additional boost to the asset price that can be obtained from higher investment is not taken into account by individual entrepreneurs. In this case, not only does the financial authority think that no zombie firms are active in a financial crisis, but also that more investment by low-productivity firms should be taking place.

To show this, consider the choice of investment of low productivity firms for the planner:

$$q - \text{MRS}'(\ell)\text{MPK}'(\ell) = \tilde{\mu} H_h' \frac{\partial q_h'}{\partial K_\ell'} + \Delta MRS'(\ell, h) \left(\Delta H_h'' \frac{\partial q_h'}{\partial K_\ell'} + N_h' \frac{\partial \omega'}{\partial K_\ell'} \right)$$

With $\frac{\partial q_h'}{\partial K_\ell'} > 0$ and $\Delta H_h'' \equiv H_{h3} - H_{h2}$. The left hand side corresponds to the optimal choice in the decentralised equilibrium. The planner, however, also internalises the effect of larger aggregate capital investment from low-productivity firms on the collateral value, which can relax the borrowing constraint. Moreover, there are effects connected to the distance in MRSs. The change in collateral price affects purchases or sales of land, which in principle can be either positive or negative depending on whether the financial friction eases, stays the same or becomes more stringent in the following period. Because this effect cannot be signed in general, it is useful to think of a situation where no trade in the asset takes place at time 2 and this effect is shut down. Just like with a cashflow constraint, larger investment by low-productivity firms also increases the wage rate, redistributing resources away from productive firms at time 2, which helps the economy at time 1.

Notice that for this result, the presence of not just a fixed asset but also of physical capital is a very important aspect. Intuitively, with only a fixed asset, any additional investment taken on by less productive firms comes at the cost of smaller investment from productive firms. In that case, no redistribution could help the economy, and the laissez-faire allocation would be constrained efficient. With physical capital used in production, instead, it is possible for the low-productivity firms to increase production without restricting that of high-productivity firms, by simply changing the ratio at which the two types of capital are used in production (u). Finally, whether or not workers are constrained may weaken this result, but it will not in general be sufficient to overturn it.

In presence of a lower bound on the interest rate, the asset price is depressed as demand for investment is low when the interest rate is at the ELB. This gives productive firms access to little funding, worsening the economy TFP and underinvestment problem. Demand-side concerns are therefore even stronger with a lower bound and a collateral constraint, calling for more investment from low-productivity firms, both because it can help relaxing the demand rationing and because it can boost the value of the collateral.

7 Conclusions

This paper presented a model of a recession generating both low aggregate productivity and a demand rationing. This induces a policy conflict between the objectives of efficient demand and capital allocation. From an ex-ante perspective, reducing the amount of debt available to firms can help financial stability, but it can make a liquidity trap more likely. Depending on the amount of flexibility available to the monetary authority, this is something that policy makers should consider when drafting policy. From an ex-post point of view, there can be a trade-off between boosting demand by letting less efficient but unconstrained firms operate and avoiding spillovers from these low-efficiency firms. Crucially, in a liquidity trap, the negative spillovers are likely to be dwarfed by demand-side concerns. For this result, two aspects are particularly important: the type of borrowing constraint, whether cashflow or collateral types; and whether it is only one or multiple sectors of the economy to be subject to financial frictions.

There are various aspects of this work that could be further extended and explored, in order to look at other related questions of interest. As an example, the model could be altered to include nominal frictions, hence introducing a monetary authority facing a different objective function than a social planner. A conflict between these two authorities could potentially arise when their incentives are in contrast with each other. Coordination or lack thereof in a game between the two authorities would then play a role. Another interesting avenue of research could be the exploration of the long run consequences of an imperfect distribution of firms. In the long run, low-productivity firms are likely to generate additional and potentially larger costs for society, in the form of lower aggregate growth and stifled innovation. Finally, one could consider how to best implement the constrained efficient allocation in a decentralised economy. Both from a practical and political point of view, implementing ex-ante versus ex-post policies might have very different impacts, which should also be considered.

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A Main Model: Analytical Derivations

Throughout the paper and the appendix, the following definitions will be used:

$$\hat{\beta}_t \equiv 1 - \mathbb{1}_{t<T}\left(\frac{1}{\sum_{s=0}^{T-t}\beta^s}\right), \quad \hat{\alpha} \equiv \alpha^\alpha(1-\alpha)^{1-\alpha}, \quad \hat{a}_i = a_i^{1/\alpha}, \quad \tilde{a} = \frac{\hat{a}_h}{\hat{a}_\ell}, \quad A_{i1} = \frac{(1-\theta_1)\hat{a}_i}{\hat{a}_l - \theta_1\hat{a}_i}$$

A.1 Market clearing

$$\text{Aggregate employment:} \quad N_t = \pi_h n_{ht} + \pi_\ell n_{\ell t} = L_t \tag{A.1.1}$$

$$\text{Net worth:} \quad Z_t = \alpha Y_t \tag{A.1.2}$$

$$\text{Production goods:} \quad \hat{C}_t + \pi_h \hat{c}_{ht} + \pi_\ell \hat{c}_{\ell t} + \pi_h K_{ht+1} + \pi_\ell K_{\ell t+1} = \pi_h y_{ht} + \pi_\ell y_{\ell t}$$

$$\iff \quad Z_t + \omega_t L_t = \frac{\omega_t N_t}{1-\alpha} \tag{A.1.3}$$

A.2 First Best Allocation

Optimal choices for workers and entrepreneurs imply:

$$L_t^\psi = \omega_t \tag{A.2.1}$$

$$\hat{C}_t = \omega_t L_t \tag{A.2.2}$$

$$\hat{c}_{it} = (1-\hat{\beta}_t) z_{it} \tag{A.2.3}$$

$$K_{t+1} = K_{ht+1} = \hat{\beta}_t Z_t \tag{A.2.4}$$

$$\frac{1}{R_t} = \frac{\beta \hat{c}_{it}}{\hat{c}_{jt+1}} \tag{A.2.5}$$

$$R_t = \hat{a}_h \left(\frac{\hat{\alpha}}{\omega_{t+1}^{1-\alpha}}\right)^{1/\alpha} \tag{A.2.6}$$

$$\omega_t = (1-\alpha)\left(\frac{\hat{a}_h k_{ht}}{n_{ht}}\right)^\alpha \tag{A.2.7}$$

$$Z_{t+1} = \hat{\beta}_t R_t Z_t \tag{A.2.8}$$

Plugging (A.2.7) and (A.2.1) in (A.1.1), one can solve for the equilibrium wage and solve for all prices and quantities:

$$\omega_t = \left[(1-\alpha)\left(\hat{a}_h K_{ht}\right)^\alpha\right]^{\frac{\psi}{\psi+\alpha}} \tag{A.2.9}$$

$$L_t = \left[(1-\alpha)(\hat{a}_h K_{ht})^\alpha\right]^{\frac{1}{\psi+\alpha}} \tag{A.2.10}$$

$$Z_t = \frac{\alpha}{1-\alpha}\left[(1-\alpha)(\hat{a}_h K_{ht})^\alpha\right]^{\frac{1+\psi}{\psi+\alpha}} \tag{A.2.11}$$

$$Z_{ht} = Z_t - B_t, \quad Z_{\ell t} = B_t \tag{A.2.12}$$

$$R_t = \alpha \hat{a}_h \left[\frac{1-\alpha}{(\hat{a}_h K_{ht+1})^\psi}\right]^{\frac{1-\alpha}{\psi+\alpha}} \tag{A.2.13}$$

With consumption of workers and entrepreneurs set according to (A.2.2), (A.2.3) and capital as in (A.2.4).

A.3 Financially Constrained Allocation

Conditions (A.2.1) to (A.2.3) remain valid. However, the other optimal choices are now replaced by the following conditions:

$$\omega_t = (1-\alpha)\left(\frac{\hat{a}_h k_{ht}}{n_{ht}}\right)^{\alpha} = (1-\alpha)\left(\frac{\hat{a}_\ell k_{\ell t}}{n_{\ell t}}\right)^{\alpha} \tag{A.3.1}$$

$$K_{ht+1} = \frac{\hat{\beta}_t Z_{ht}}{1-\theta\tilde{a}}, \quad K_{\ell t+1} = \hat{\beta}_t Z_t - K_{ht+1} \tag{A.3.2}$$

$$Z_{t+1} = \hat{\beta}_t R_t \left(A_h Z_{ht} + Z_{\ell t}\right) \tag{A.3.3}$$

$$\frac{1}{R_1} = \frac{\beta A_i \hat{c}_{i1}}{\hat{c}_{i2}} \tag{A.3.4}$$

$$R_1 = \hat{a}_\ell \left(\frac{\hat{\alpha}}{\omega_2^{1-\alpha}}\right)^{1/\alpha} \tag{A.3.5}$$

$$\omega_t = \left[(1-\alpha)\left(\sum \hat{a}_i K_{it}\right)^{\alpha}\right]^{\frac{\psi}{\psi+\alpha}} \qquad R_t = \alpha\hat{a}_l\left[\frac{1-\alpha}{(\sum \hat{a}_i K_{it+1})^{\psi}}\right]^{\frac{1-\alpha}{\alpha+\psi}} \tag{A.3.6}$$

$$Z_t = \frac{\alpha}{1-\alpha}\left[(1-\alpha)\left(\sum \hat{a}_i K_{it}\right)^{\alpha}\right]^{\frac{1+\psi}{\psi+\alpha}} \tag{A.3.7}$$

$$Z_{ht} = Z_t\left[\frac{(1-\theta_t)\hat{a}_h K_{ht}}{\sum \hat{a}_i K_{it}}\right] \qquad Z_{\ell t} = Z_t\left[\theta_t + \frac{(1-\theta_t)\hat{a}_\ell K_{\ell t}}{\sum \hat{a}_i K_{it}}\right] \tag{A.3.8}$$

B A Financial Crisis

B.1 Proof of lemma 1

When $\theta_1 = 1$ the allocation is first best.

Assume $\theta_1 = 1$. First order conditions for capital and savings for entrepreneurs are:

$$\frac{1}{R_1} = \beta\frac{\hat{c}_{i1}}{\hat{c}_{i2}} + \hat{c}_{i1}\mu_{i1}$$

$$1 = \left(\frac{\beta\hat{c}_{i1}}{\hat{c}_{i2}} + \hat{c}_{i1}\mu_{i1}\right)\frac{\partial y_{i2}}{\partial k_{i2}}$$

Assume by contradiction that the borrowing limit is binding and both agents engage in production. It must then be that:

$$R_1 = \alpha a_h \left(\frac{n_{h2}}{k_{h2}}\right)^{1-\alpha} = \alpha a_\ell \left(\frac{n_{\ell 2}}{k_{\ell 2}}\right)^{1-\alpha} \implies \hat{k}_{h2} = \left(\frac{a_h}{a_\ell}\right)^{\frac{1}{1-\alpha}} \hat{k}_{\ell 2}$$

But from the optimal choice of labour:

$$\omega_2 = (1-\alpha) a_h \left(\frac{k_{h2}}{n_{h2}}\right)^{\alpha} = (1-\alpha) a_\ell \left(\frac{k_{\ell 2}}{n_{\ell 2}}\right)^{\alpha} \implies \hat{k}_{h2} = \left(\frac{a_\ell}{a_h}\right)^{\frac{1}{\alpha}} \hat{k}_{\ell 2}$$

This leads to a contradiction as the capital-labour ratio implied by the optimal choice of labour differs from the one implied by the optimal choice of capital. It must be that the borrowing constraint is not binding when $\theta = 1$ and that high productivity entrepreneurs are the only active firms. □

B.2 Proof of Proposition 1

A financial crisis induces no changes in aggregate output at date 1 but lower productivity and production at date 2.

Unchanged output at time 1. Both demand and supply of labour when the shock hits the economy continue to be set according to (A.2.7) and (A.2.1). Labour market clearing then implies that both wages and aggregate employment are unchanged. For given level of capital, the level of output is therefore unchanged. This also implies that entrepreneurs' net worth in the period stays the same. Therefore, from consumption function (A.2.3), we know consumption not just for workers but also for entrepreneurs stays constant. The only effect of a borrowing limit at time 1 is then on capital demand, where (A.2.4) is replaced with (A.3.2), while total capital demanded stays constant.

Lower TFP and output at time 2. In a first best allocation, TFP is equal to a_h. Investment is entirely carried out by h firms: $K_{ht} = K_t$. When a borrowing constraint binds, the economy's TFP is:[39]

$$\text{TFP}_2^c = \left(\frac{\hat{a}_h K_{h2}^c + \hat{a}_\ell K_{\ell 2}^c}{K_2^c}\right)^{\alpha} < a_h$$

[39] An asterisk superscript is used to indicate the first best allocation, while c is used to indicate the constrained allocation.

given $K_{h2} \leq K_2$. Aggregate output can be rewritten as a function of just capital:

$$Y_t^* = \left[(1-\alpha)^{1-\alpha}(\hat{a}_h K_t^*)^{\alpha(1+\psi)}\right]^{\epsilon} \qquad\qquad Y_t^c = \left[(1-\alpha)^{1-\alpha}(\text{TFP}_t^c (K_t^c)^{\alpha})^{1+\psi}\right]^{\epsilon}$$

with $\epsilon = \frac{1}{\psi+\alpha}$. Given $a_h > \text{TFP}_2^c$, a sufficient condition for $Y_2^* > Y_2^c$ is that $K_2^* \geq K_2^c$. As the aggregate quantity of investment in the constrained and unconstrained case is the same at $K_2^* = \hat{\beta}_1 Z_1^* = K_2^c$, this shows that output and TFP are both lower in the aftermath of the crisis. □

B.3 Only capital used in production

When physical capital is the only input in production, the allocation at time 1 is constrained efficient.

$$\begin{aligned} & V_{i2}(z_{i2}; B_2, K_{i2}, K_{j2}) = \max\ \log \hat{c}_{i2} \\ \text{s.to} \quad & \hat{c}_{i2} = d_{i2} - b_{i2} \end{aligned}$$

where $d_{i2} = a_i k_{i2}$.

$$\frac{\partial V_{i2}}{\partial K_{i2}} = \lambda_{i2}\frac{\partial D_{i2}}{\partial K_{i2}}, \qquad \frac{\partial V_{i2}}{\partial K_{j2}} = 0, \qquad \frac{\partial V_{i2}}{\partial B_2} = -\lambda_{i2}$$

$$\text{with:} \qquad \lambda_{i2} = \frac{1}{\hat{C}_{i2}}$$

Planner's problem

$$\begin{aligned} V_1^P(Z_1, S_1) = & \max_{\hat{C}_1, \hat{c}_{i1} K_{i2}, B_2} \sum_{i \in h,l} \chi_i \pi_i \left[\log \hat{c}_{i1} + \beta V_{i2}(z_{i2}; B_2, K_{h2}, K_{\ell 2})\right] \\ \text{s.to} \quad & \sum_{i \in h,l} \pi_i \hat{c}_{i1} + K_{h2} + K_{\ell 2} = Y_1 \qquad [\tilde{\lambda}_1] \\ & B_2 \leq \theta_1 D_{h2} \qquad [\tilde{\mu}_1] \end{aligned}$$

$$\begin{aligned} \hat{c}_{i1}: \quad & \frac{\chi_i}{\hat{c}_{i1}} = \lambda_1 \\ B_2: \quad & \mu_1 = \beta \sum_i \chi_i \pi_i \frac{\partial V_{i2}}{\partial B_2} \\ K_{i2}: \quad & \lambda_1 = \beta \sum_j \chi_j \pi_j \frac{\partial V_{j2}}{\partial K_{i2}} + \phi_1 \mu_1 \frac{\partial D_{h2}}{\partial K_{i2}} \end{aligned}$$

Combining these with the expressions obtained above, one can show the allocation chosen by the planner corresponds to the decentralised allocation. □

B.4 Proof of proposition 2

Compared to a constrained efficient allocation, the laissez-faire economy features over-invesment an zombie firms; output at time 2 is too high.

$$
\begin{aligned}
& V_{i2}(z_{i2}; B_2, K_{i2}, K_{j2}) = \max \ \log \hat{c}_{i2} \\
\text{s.to} \quad & \hat{c}_{i2} = d_{i2} - b_{i2} \\
& W_2(B_2, K_{h2}, K_{\ell 2}) = \max \ \log \tilde{C}_2 \\
\text{s.to} \quad & \hat{C}_2 = \omega_2 L_2 - b_2^w, \qquad \tilde{C}_2 \equiv C_2 - v(L_2)
\end{aligned}
$$

$$
\text{where:} \qquad d_{i2} = \hat{a}_i k_{i2}^{\alpha} (n_{i2}^m)^{1-\alpha} - \omega_2 n_{i2}, \qquad \omega_2 = \left[(1-\alpha)\left(\sum \hat{a}_i K_{i2}\right)^{\alpha}\right]^{\frac{\psi}{\psi+\alpha}}.
$$

$$
\begin{aligned}
\frac{\partial V_{i2}}{\partial K_{j2}} &= \lambda_{i2}\left[\mathbb{1}_{i=j}\left(\frac{\partial Y_{i2}}{\partial K_{j2}}\right) - N_{i2}\frac{\partial \omega_2}{\partial K_{j2}}\right], & \frac{\partial V_{i2}}{\partial B_{j2}} &= -\mathbb{1}_{i=j}\lambda_{i2} \\
\frac{\partial W_2}{\partial K_{j2}} &= \lambda_2^w L_2 \frac{\partial \omega_2}{\partial K_{j2}}, & \frac{\partial W_2}{\partial B_2^w} &= -\lambda_2^w
\end{aligned}
$$

$$
\text{with:} \qquad \frac{\partial \omega_2}{\partial K_{j2}} = -\frac{\alpha\psi}{\psi+\alpha}\frac{\omega_2 \hat{a}_j}{\hat{a}_h K_{h2} + \hat{a}_\ell K_{\ell 2}}
$$

Planner's problem

$$
V_1^P(Z_1, S_1) = \max_{\hat{C}_1, \hat{c}_{i1} K_{i2}, B_2} \left\{ \log\left(\tilde{C}_1\right) + \beta W_2(K_{h2}, K_{\ell 2}) + \sum_{i\in h,l} \chi_i \pi_i \left[\log \hat{c}_{i1} + \beta V_{i2}(z_{i2}; B_2, K_{h2}, K_{\ell 2})\right] \right\}
$$

$$
\begin{aligned}
\text{s.to} \quad & \sum_{i\in h,l} \pi_i \left(\hat{c}_{i1} + k_{i2}\right) + \tilde{C}_1 = Y_1 - v(L_1) & [\tilde{\lambda}_1] \\
& B_{h2} + B_{\ell 2} + B_2^w = 0 \\
& B_{h2} + B_2^w \le \theta_1 D_{h2} & [\tilde{\lambda}_1 \tilde{\mu}_1] \\
& B_2^w = 0 & [\tilde{\lambda}_1 \tilde{\nu}_1]
\end{aligned}
$$

$$
B_{i2}, B_2^w: \qquad \tilde{\mu}_1 = \beta\left(\frac{\hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\hat{C}_{h1}}{\hat{C}_{h2}}\right), \qquad \tilde{\mu}_1 + \tilde{\nu}_1 = \beta\left(\frac{\hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\tilde{C}_1}{\tilde{C}_2}\right)
$$

$$K_{i2}: \qquad \tilde{\lambda}_1 = \beta\left(\sum_j \chi_j \pi_j \frac{\partial V_{j2}}{\partial K_{i2}} + \frac{\partial W_2}{\partial K_{i2}}\right) + \theta_1 \tilde{\mu}_1 \frac{\partial D_{h2}}{\partial K_{i2}}$$

Focusing on unconstrained choice of investment for low-productivity entrepreneurs:

$$K_{\ell 2}: \quad 1 = \frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} \frac{\partial Y_{\ell 2}}{\partial K_{\ell 2}} - \tilde{\mu}_1 \theta_1 N_{h2} \frac{\partial \omega_2}{\partial K_{\ell 2}} + \beta\left(\frac{\hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\hat{C}_{h1}}{\hat{C}_{h2}}\right) N_{h2} \frac{\partial \omega_2}{\partial K_{\ell 2}} - \beta\left(\frac{\hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\tilde{C}_1}{\tilde{C}_2}\right) L_2 \frac{\partial \omega_2}{\partial K_{\ell 2}}$$
$$1 = \frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} \frac{\partial Y_{\ell 2}}{\partial K_{\ell 2}} - \tilde{\mu}_1 \left[L_2 - (1-\theta_1) N_{h2}\right] \frac{\partial \omega_2}{\partial K_{\ell 2}} - \tilde{\nu}_1 L_2 \frac{\partial \omega_2}{\partial K_{\ell 2}}$$

One can show that $L_2 - (1-\theta_1)N_{h2} > 0$ by noticing that $L_2 - N_{h2} = N_{\ell 2} > 0$. Therefore:

$$1 = \frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} \frac{\partial Y_{\ell 2}}{\partial K_{\ell 2}} - \tau_{\ell 2}$$
$$(1+\tau_{\ell 2})\left(\frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}}\right)^{-1} = \frac{\partial Y_{\ell 2}}{\partial K_{\ell 2}}$$
$$\text{where} \qquad \tau_{\ell 2} \equiv \left\{\tilde{\mu}_1 \left[L_2 - (1-\theta_1)N_{h2}\right] + \tilde{\nu}_1 L_2\right\} \frac{\partial \omega_2}{\partial K_{\ell 2}} > 0$$

This shows that $K^{\text{sp}}_{\ell 2} < K^{\text{c}}_{\ell 2}$, which implies a lower level of production at time 2. In turn, the lower capital investment reduces wages and allows productive firms to invest and produce more in the future. □

C The Effective Lower Bound

C.1 Proof of proposition 3

A financial crisis where the interest rate is at the ELB generates lower employment and output at date 1.

$$d_{h1} = \max_{n_{h1}} \; y_{h1} - w_1 n_{h1} \qquad \text{s.to} \quad y_{h1} \leq \frac{\hat{C}_{h1} + \hat{C}_{\ell 1} + \hat{C}_1 + K_{h2} + K_{\ell 2}}{\pi_h} \qquad \text{if } R_1 = 1$$

At the lower bound, investment demand of unconstrained entrepreneurs is lower:

$$K_{\ell 2}: \qquad \rho = \alpha a_\ell \left(\frac{1}{\hat{K}_{\ell 2}}\right)^{1-\alpha}$$
$$N_{\ell 2}: \qquad a_\ell \left(\hat{K}_{\ell 2}\right)^{\alpha} = \frac{\omega_2}{1-\alpha} = a_h \left(\hat{K}_{h2}\right)^{\alpha}$$

Combining the two one obtains $\omega_2 = \left(\frac{\hat{\alpha} a_\ell}{\rho}\right)^{\frac{1}{1-\alpha}}$. But from labour market clearing, we have:

$$\omega_2 = \left[(1-\alpha)\left(\sum_i \hat{a}_i K_{i2}\right)^{\alpha}\right]^{\frac{\psi}{\alpha+\psi}}$$

The choice of capital of the productive entrepreneurs continues to be constrained, and equal to $K_{h2} = \frac{\hat{\beta}_1 Z_{h1}}{1-\theta\tilde{a}}$. The level of investment of low productivity entrepreneurs can be found by combining the expressions above to obtain:

$$K_{\ell 2} = \Omega(\rho) - \frac{\hat{\beta}_1 \tilde{a}\,(Z_1 - B_1)}{1-\theta\tilde{a}}$$

with $\Omega(\rho) = \left[\frac{\hat{\alpha} a_\ell}{\rho}\right]^{\frac{\alpha+\psi}{\psi\alpha(1-\alpha)}} \frac{1}{\hat{a}_\ell (1-\alpha)^{1/\alpha}}$ a function of parameters decreasing in ρ, the ELB. If ρ could fall to the equilibrium level of interest rate R^*, then the above expression for investment would be higher, such that aggregate capital investment corresponds to the efficient level. Because of the ELB, however, $K_{\ell 2}$ is lower. Furthermore, as a consequence of this lower investment demand, consumption of the low-productivity entrepreneurs is also lower:

$$C_{\ell 1} = \frac{C_{\ell 2}}{\beta\rho} = \frac{Z_{\ell 2}}{\beta\rho} = \frac{\alpha\,(Y_{\ell 2} + \theta Y_{h2})}{\beta\rho} = \frac{K_{\ell 2} + \theta\tilde{a}K_{h2}}{\beta}$$

As a result of lower capital and lower consumption demanded, the level of employment at date 1 needs to be lower to ensure that all output produced corresponds to the aggregate amount of resources demanded:

$$\begin{aligned} & Y_h - \omega N_h = C_h + C_\ell + K_h' + K_\ell' \\ N_h: \quad & \hat{a}_h K_h^{\alpha} N_h^{1-\alpha} - \omega N_h = \frac{\Omega(\rho)}{\hat{\beta} A_h} + \frac{\tilde{a}-1}{(1-\theta)\tilde{a}} B \end{aligned}$$

Due to $\rho > R^*$, the level of employment chosen is lower than away from the lower bound. The level of wage is also depressed, as it is set to ensure that labour supplied $L^{\psi} = \omega$ equals labour demanded in the expression above.

A financial crisis where the interest rate is at the ELB generates better TFP but lower capital and production in manufacturing at $t = 2$

First note that aggregate capital at the ELB is lower, due to both lower K_ℓ, and lower Z_1 inducing lower K_h. As for aggregate productivity:

$$\text{TFP} = \left(\frac{\hat{a}_h K_h + \hat{a}_\ell K_\ell}{K}\right)^\alpha = \left[\hat{a}_h - (\hat{a}_h - \hat{a}_\ell)\frac{K_\ell}{K}\right]^\alpha$$

One can show that $\frac{K_\ell}{K_h + K_\ell}$ is lower in a liquidity trap due to fall in $K_{\ell 2}$ being larger than fall in K_{h2}; as a result TFP increases. Finally, notice that in equilibrium output is a function of productivity and capital invested:

$$Y = f(\text{TFP} \cdot K^\alpha) = f(\hat{a}_h K_h + \hat{a}_\ell K_\ell)$$

As both K_h and K_ℓ are lower, output at time 2 is lower although aggregate productivity is larger. □

C.2 Proof of proposition 4

During a financial crisis where the economy is at the lower bound, the allocation at time 1 is not constrained efficient. Provided that the demand externalities are sufficiently strong, more low-productivity entrepreneurs should engage in production than in the laissez-faire economy.

First note that derivatives with respect to K_{i2} and B_2 of individuals' problem at time 2 are the same as in appendix B.4. At time 1, the social planner's problem is:

$$V_1^P(K_h, K_\ell; \theta) = \max_{\hat{c}_i, L, K_i', B_i'} \left\{ \sum_{i \in h,l} \chi_i \pi_i \left[\log \hat{c}_i + \beta V_{i2}(z_i'; K_h', K_\ell', B')\right] + \right.$$
$$\left. + \log \tilde{C} + \beta W_{t+1}(K_h', K_\ell', B') \right\}$$

$$\text{subject to} \qquad \sum_{i \in h,l} \pi_i \left(\hat{c}_i + K_i'\right) = \bar{Z}; \qquad (\lambda)$$

$$\bar{Z} \equiv a_h K^\alpha L^{1-\alpha} - L^{1+\psi} \leq \left(\frac{(\omega')^{\frac{\psi+\alpha}{\psi}}}{(1-\alpha)a_\ell}\right)^{1/\alpha} + \frac{\hat{a}_h - \hat{a}_\ell}{(1-\theta)\hat{a}_h} B \qquad (\lambda\gamma)$$

$$c_\ell \leq \frac{K_\ell' + \theta \tilde{a} K_h'}{\beta} \qquad (\lambda\varepsilon)$$

$$B' \leq \theta \left(Y_h' - \omega' N_h'\right) \qquad (\lambda\tilde{\mu})$$

where $\omega' = [(1-\alpha)\left(\sum_i \hat{a}_i K_i'\right)^{\alpha}]^{\frac{\psi}{\alpha+\psi}}$. The optimal choice of debt is unchanged. As for capital:

$$K_{i2}: \quad \lambda_1 = \beta\left(\sum_j \chi_j \pi_j \frac{\partial V_{j2}}{\partial K_{i2}} + \frac{\partial W_2}{\partial K_{i2}}\right) + \theta_1 \lambda_1 \tilde{\mu}_1 \frac{\partial D_{h2}}{\partial K_{i2}} - [1 - \mathbb{1}_{i=h}(1-\theta\tilde{a})]\frac{\lambda_1 \epsilon_1}{\beta} - \lambda_1 \gamma_1 \frac{\psi+\alpha}{\psi\alpha}\frac{\partial \omega_2}{\partial K_{i2}}$$

After rearranging, and setting $\zeta \equiv \frac{\psi+\alpha}{\psi\alpha}$ the main expression in the text (4.4) is obtained:

$$\begin{aligned} 1 &= \text{MRS}'(\ell)\text{MPK}'(\ell) - \tilde{\mu}\left[L' - (1-\theta)N_h'\right]\frac{\partial \omega'}{\partial K_\ell'} + \frac{\varepsilon}{\beta} + \gamma\zeta\frac{\partial \omega'}{\partial K_\ell'} \\ 1 &= \text{MRS}'(\ell)\text{MPK}'(\ell) + \tilde{\tau}' \end{aligned}$$

with $\tilde{\tau}' > 0$ if $\tilde{\mu}\left[L' - (1-\theta)N_h'\right]\frac{\partial \omega'}{\partial K_\ell'} < \frac{\varepsilon}{\beta} + \gamma\zeta\frac{\partial \omega'}{\partial K_\ell'}$. □

D Interventions Before the Crisis

D.1 Proof of proposition 5

The allocation before a financial crisis is not constrained efficient. Active firms in the decentralised economy borrow too much while they can over- or under-invest; the constrained efficient economy features higher output than the laissez-faire economy at date 1 in case of under-investment, vice versa in case of over-investment.

$$\begin{aligned} &W_1(B_1, K_1) = \max\ \log \tilde{C}_1 + \beta W_2(S_2) \\ &\text{s.to} \qquad \tilde{C}_1 = \omega_1 L_1 - v(L_1), \qquad \tilde{C}_1 \equiv \hat{C}_1 - v(L_1) \end{aligned}$$

$$\begin{aligned} &V_{i1}(z_{i1}; B_1, K_1) = \max\ \log \hat{c}_{i1} + \beta V_{i2}(z_{i2}; S_2) \\ \text{s.to} \qquad &\hat{c}_{i1} + k_{i2} - \frac{b_{i2}}{R_1} = d_{i1} - b_{i1} \\ &b_{h2} \le \theta_1 d_{h2} \end{aligned}$$

$$\begin{aligned} \frac{\partial W_1}{\partial B_1} &= -\lambda_1^w L_1 \frac{\partial \omega_1}{\partial B_1}, \qquad \frac{\partial W_1}{\partial K_1} = -\lambda_1^w L_1 \frac{\partial \omega_1}{\partial K_1} \\ \frac{\partial V_{i1}}{\partial B_1} &= \lambda_{i1}\left[\frac{\partial D_{i1}}{\partial B_1} - \left(\frac{B_{i2}}{R_1^2}\frac{\partial R_1}{\partial B_1} + 1\right)\right] + \mathbb{1}_{i=h}\mu_{i1}\theta_1 \frac{\partial D_{h2}}{\partial B_1} \\ \frac{\partial V_{i1}}{\partial K_1} &= \lambda_{i1}\left[\frac{\partial D_{i1}}{\partial K_1} - \frac{B_{i2}}{R_1^2}\frac{\partial R_1}{\partial K_1}\right] + \mathbb{1}_{i=h}\mu_{i1}\theta_1 \frac{\partial D_{h2}^m}{\partial K_1} \end{aligned}$$

$$\text{with:} \qquad \lambda_{h1} = \frac{1}{\hat{C}_{h1}}, \quad \lambda_{\ell 1} = \frac{1}{\hat{C}_{\ell 1}}, \quad \lambda_1^w = \frac{1}{\hat{C}_1}$$

$$V_0^P(Z_0, S_0) = \max_{\tilde{C}_0, \hat{c}_{i0}, K_1, B_1} \left\{ \log\left(\tilde{C}_0\right) + \beta W_1(B_1, K_1) + \sum_{i \in h,l} \chi_i \pi_i \left[\log \hat{c}_{i0} + \beta V_{i1}(z_{i1}; B_1, K_1)\right] \right\}$$

$$\text{s.to} \qquad \left(\sum_{i \in h,l} \pi_i \hat{c}_{i0} + \tilde{C}_0 \right) + K_1 = Y_0 - v(L_0)$$

$$\tilde{C}_0 : \qquad \frac{1}{\tilde{C}_0} = \lambda_0 \tag{D.1.1}$$

$$\hat{c}_{i0} : \qquad \frac{\chi_i}{\hat{c}_{i0}} = \lambda_0 \tag{D.1.2}$$

$$B_1 : \qquad \beta \left[\frac{\partial W_1}{\partial B_1} + \chi_h \pi_h \frac{\partial V_{h1}}{\partial B_1} + \chi_\ell \pi_\ell \frac{\partial V_{\ell 1}}{\partial B_1} \right] = 0 \tag{D.1.3}$$

$$K_1 : \qquad \lambda_0 = \beta \left[\frac{\partial W_1}{\partial K_1} + \chi_h \pi_h \frac{\partial V_{h1}}{\partial K_1} + \chi_\ell \pi_\ell \frac{\partial V_{\ell 1}}{\partial K_1} \right] \tag{D.1.4}$$

Excessive borrowing and inefficient investment.

$$D_{h1} = Z_1 = Y_1 - \omega_1 N_1 \qquad\qquad K_2 = \hat{\beta}_1 Z_1 \tag{D.1.5}$$

$$\omega_t = \left[(1-\alpha) \left(\sum_i \hat{a}_i K_{it} \right)^{\alpha} \right]^{\frac{\psi}{\alpha+\psi}} \qquad\qquad K_{h2} = \frac{K_2}{(1-\theta_1 \tilde{a})} \left[1 - \frac{B_1}{Z_1} \right] \tag{D.1.6}$$

$$R_1^* = \hat{a}_\ell \left[\frac{\hat{\alpha}}{\omega_2^{1-\alpha}} \right]^{\frac{1}{\alpha}} \qquad\qquad K_{\ell 2} = K_2 \frac{B_1 - \theta_1 \tilde{a} Z_1}{(1-\theta_1 \tilde{a}) Z_1} \tag{D.1.7}$$

$$\frac{\partial D_{h1}}{\partial B_1} = 0, \qquad \frac{\partial D_{h1}}{\partial K_1} = \alpha \frac{\partial Y_1}{\partial K_1} - N_1 \frac{\partial \omega_1}{\partial K_1}$$

$$\frac{\partial \omega_1}{\partial B_1} = 0, \qquad \frac{\partial \omega_1}{\partial K_1} = \frac{\alpha \psi}{\psi + \alpha} \frac{\omega_1}{K_{h1}}$$

$$\frac{\partial R_1^*}{\partial B_1} = -\frac{\psi}{\psi+\alpha} \frac{R_1^*}{\sum \hat{a}_i K_{i2}} \sum \hat{a}_i \frac{\partial K_{i2}}{\partial B_1}, \qquad \frac{\partial R_1^*}{\partial K_{h1}} = -\frac{\psi}{\psi+\alpha} \frac{R_1^*}{\sum \hat{a}_i K_{i2}} \sum \hat{a}_i \frac{\partial K_{i2}}{\partial K_{h1}}$$

Using the fact that $\partial W_1 / \partial B_1 = 0$ together with (D.1.2), we can rewrite expression (D.1.3) as:

$$\left[\frac{\hat{C}_{h0}}{\hat{C}_{h1}} - \frac{\hat{C}_{\ell 0}}{\hat{C}_{\ell 1}} \right] \left[\frac{B_2}{R_1^2} \frac{\partial R_1}{\partial B_1} + 1 \right] = \theta_1 \mu_1 \frac{\partial D_{h2}}{\partial B_1}$$

where:

$$
\begin{aligned}
&D_{h2} = (1-\theta_1)\left(Y_{h2} - \omega_2 N_{h2}\right)\\
&\frac{\partial D_{h2}}{\partial B_1} = (1-\theta_1)\left[\left(\frac{\partial Y_{h2}}{\partial K_{h2}} - N_{h2}\frac{\partial \omega_2}{\partial K_{h2}}\right)\frac{\partial K_{h2}}{\partial B_1} - N_{h2}\frac{\partial \omega_2}{\partial K_{\ell 2}}\frac{\partial K_{\ell 2}}{\partial B_1}\right]\\
&\frac{\partial \omega_2}{\partial B_1} = -\frac{\alpha\psi}{\alpha+\psi}\frac{\hat{\beta}_1}{1-\theta_1\tilde{a}}\frac{(\hat{a}_h - \hat{a}_\ell)\omega_2}{\sum \hat{a}_i K_{i2}} < 0\\
&\frac{\partial K_{h2}}{\partial B_1} = -\frac{\hat{\beta}_1}{1-\theta_1\tilde{a}} < 0\\
&\implies \quad \frac{\partial D_{h2}}{\partial B_1} = -\frac{\hat{\beta}_1}{1-\theta_1\tilde{a}}\frac{D_{h2}}{K_{h2}}\left[\frac{\psi\left(\hat{a}_\ell K_2 + (\hat{a}_h - \hat{a}_\ell)K_{h2}\right) + \alpha\sum \hat{a}_i K_{i2}}{(\psi+\alpha)\sum \hat{a}_i K_{i2}}\right] < 0
\end{aligned}
$$

So the economy features overborrowing and MRS(H) < MRS(L). For capital:

$$
1 = \beta\frac{\hat{C}_{h0}}{\hat{C}_{h1}}\left[\frac{\partial Y_1}{\partial K_1} - N_{h1}\frac{\partial \omega_1}{\partial K_1}\right] - \beta\left[\frac{\hat{C}_{h0}}{\hat{C}_{h1}} - \frac{\hat{C}_{\ell 0}}{\hat{C}_{\ell 1}}\right]\frac{b_2}{R_1^2}\frac{\partial R_1}{\partial K_1} + \theta_1\mu_1\frac{\partial D_{h2}}{\partial K_1}
$$

where:

$$
\begin{aligned}
&\frac{\partial D_{h2}}{\partial K_1} = (1-\theta_1)\left[\left(\frac{\partial Y_{h2}}{\partial K_{h2}} - N_{h2}\frac{\partial \omega_2}{\partial K_{h2}}\right)\frac{\partial K_{h2}}{\partial K_1} - N_{h2}\frac{\partial \omega_2}{\partial K_{\ell 2}}\frac{\partial K_{\ell 2}}{\partial K_1}\right]\\
&\frac{\partial \omega_2}{\partial K_1} = \frac{\alpha\psi}{\alpha+\psi}\frac{\hat{\beta}_1(1-\theta_1)\hat{a}_h}{1-\theta_1\tilde{a}}\frac{\omega_2}{\sum \hat{a}_i K_{i2}}\frac{\partial Z_1}{\partial K_1} > 0\\
&\frac{\partial K_{h2}}{\partial K_1} = \frac{\hat{\beta}_1}{1-\theta_1\tilde{a}}\frac{\partial Z_1}{\partial K_1} > 0\\
&\frac{\partial D_{h2}}{\partial K_1} = \frac{\hat{\beta}_1}{1-\theta_1\tilde{a}}\frac{D_{h2}}{K_{h2}}\frac{\alpha\sum \hat{a}_i K_{i2} + \psi\left[\hat{a}_\ell K_{\ell 2} + \left(\theta_1 + \alpha(1-\theta_1)\right)\hat{a}_h K_{h2}\right]}{(\alpha+\psi)\sum \hat{a}_i K_{i2}} > 0
\end{aligned}
$$

□

D.2 Proof of Lemma 2

The interest rate is less likely to be at the effective lower bound if the initial stock of debt is large. There is a minimum level of aggregate debt $\underline{B}$ above which the effective lower bound in never binding. Assume $\hat{a}_\ell > \theta_1\hat{a}_h$.

$$
\begin{aligned}
&R_1^* = \Omega_1\hat{a}_\ell\left(\frac{1}{\hat{a}_h K_{h2} + \hat{a}_\ell K_{\ell 2}}\right)^{\frac{\psi}{\psi+\alpha\gamma}}\\
\text{with}\quad &K_{h2} = \hat{\beta}_1\frac{Z_1 - B_1}{1-\theta_1\tilde{a}}, \quad K_{\ell 2} = \hat{\beta}_1\frac{B_1 - \theta_1\tilde{a}Z_1}{1-\theta_1\tilde{a}}
\end{aligned}
$$

$$R_1^* = \Omega_1 \hat{a}_\ell \left(\frac{1 - \theta_1 \tilde{a}}{\hat{\beta}_1 \left[(1-\theta_1)\hat{a}_h Z_1 - (\hat{a}_h - \hat{a}_\ell) B_1 \right]} \right)^{\frac{\psi}{\psi + \alpha\gamma}} \tag{D.2.1}$$

$$\frac{\partial R_1^*}{\partial Z_1} = -\frac{\psi}{\psi + \alpha\gamma} \frac{R_1 (1-\theta_1)\hat{a}_h}{(1-\theta_1)\hat{a}_h Z_1 - (\hat{a}_h - \hat{a}_\ell) B_1} < 0$$

$$\frac{\partial R_1^*}{\partial B_1} = \frac{\psi}{\psi + \alpha\gamma} \frac{R_1 (\hat{a}_h - \hat{a}_\ell)}{(1-\theta_1)\hat{a}_h Z_1 - (\hat{a}_h - \hat{a}_\ell) B_1} = -\frac{\partial R_1^*}{\partial Z_1} \frac{\hat{a}_h - \hat{a}_\ell}{(1-\theta_1)\hat{a}_h} > 0$$

Define $\zeta_1 = (1 - \theta_1 \tilde{a}) \hat{\beta}_1^{-1} (\rho_1 \hat{a}_\ell)^{\frac{\alpha\gamma + \psi}{\psi}}$. We can define the lowest level of debt for which the economy is not in a liquidity trap, and Z_1 does not depend on B_1, because employment is still optimal at the margin. This is the level of debt for which $R_1^* = 1$:

$$\underline{B}_1 = \frac{(1-\theta_1)\hat{a}_h Z_1 - \zeta_1}{\hat{a}_h - \hat{a}_\ell}$$

With Z_1 defined in (A.2.11). A liquidity trap is triggered if:

$$R_1^* < 1 \quad \Longleftrightarrow \quad (1-\theta_1)\hat{a}_h Z_1 - (\hat{a}_h - \hat{a}_\ell) B_1 > \zeta_1 \quad \Longleftrightarrow \quad B_1 < \underline{B}_1$$

□

D.3 Proof of Proposition 6

In presence of a lower bound on the real rate, the allocation before a financial crisis is not constrained efficient: it features under-borrowing if the effects of the demand rationing are sufficiently strong. It can feature either under- or over-investment depending on parameter values.

$$\max_{N_{h1}} D_{h1}^{lt} = Y_{h1} - w_1 N_{h1} \qquad \text{s.t.} \quad \sum_i (c_{i1} + K_{i2}) = Y_{h1}$$

Appendix C.1 shows how the resource constraint in a liquidity trap corresponds to:

$$N^{lt}: \; \mathcal{D}(N^{lt}, K) = \left[a_h K^\alpha (N^{lt})^{1-\alpha} - \omega N^{lt} \right] = \frac{\Omega(\rho)}{\hat{\beta} A_h} + \frac{\tilde{a} - 1}{(1-\theta)\tilde{a}} B \tag{D.3.1}$$

$$\Longrightarrow \qquad N^{lt} = \mathcal{N}(B, K)$$

$$V_{i1}(z_{i1}; B_1, K_1) = \max \; \log \hat{c}_{i1} + \beta V_{i2}(z_{i2}; S_2)$$

$$
\begin{aligned}
\text{s.to} \quad & \hat{c}_{i1} + k_{i2} - \frac{b_{i2}}{R_1} = d_{i1} - b_{i1} \\
& b_{i2} \leq \theta_1 d_{i2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_{i1}}{\partial B_{i1}} =& - \lambda_{i1} \left[\underbrace{\frac{B_{i2}}{R_1^2} \frac{\partial R_1}{\partial B_{i1}}}_{=0} + 1 - \frac{\partial D_{i1}}{\partial B_{i1}} \right] + \theta_1 \mu_{i1} \frac{\partial D_{i2}}{\partial B_1} \\
\frac{\partial V_{i1}}{\partial K_1} =& \lambda_{i1} \frac{\partial D_{i1}}{\partial K_1} + \theta_1 \mu_{i1} \frac{\partial D_{i2}}{\partial K_1}, \qquad \text{with:} \quad \lambda_{i1} = \frac{1}{\hat{C}_{i1}}
\end{aligned}
$$

$$
\begin{aligned}
& W_1(K_1) = \max \ \log\left(\tilde{C}_1\right) + \beta W_2(S_2) \\
& \text{s.to} \qquad \hat{C}_1 = w_1 L_1 - B_1^s \\
& \frac{\partial W_1}{\partial B_1} = \lambda_1^w \left(L_1 \frac{\partial w_1}{\partial B_1} - 1 \right), \qquad \frac{\partial W_1}{\partial K_1} = \lambda_1^w L_1 \frac{\partial w_1}{\partial K_1}, \qquad \text{with:} \qquad \lambda_1^w = \frac{1}{\tilde{C}_1}
\end{aligned}
$$

Besides equation (D.3.1), the following expressions are relevant:

$$
\begin{aligned}
& D_{h2} = Y_{h2} - w_2 N_2, \qquad K_{h2} = \hat{\beta}_1 \frac{D_{h1} - B_1}{1 - \theta_1 \tilde{a}}, \qquad \omega_2 = (\hat{\alpha} a_\ell)^{1/(1-\alpha)} \\
& \omega_1 = L_1^\psi = (N_1^{lt})^\psi, \qquad \omega_2 = \left[(1-\alpha) \left(\hat{a}_h K_{h2} \right)^\alpha \right]^{\frac{\psi}{\psi+\alpha}}.
\end{aligned}
$$

The planner solves the following problem:

$$
\begin{aligned}
V_0^P(Z_0, S_0) = & \max_{\tilde{C}_0, \hat{c}_{i0}, K_1, B_1} \left\{ \log\left(\tilde{C}_0\right) + \beta W_1(B_1, K_1) + \sum_{i \in h,l} \chi_i \pi_i \left[\log \hat{c}_{i0} + \beta V_{i1}(z_{i1}; B_1, K_1) \right] \right\} \\
& \text{s.to} \qquad \sum_{i \in h,l} \pi_i \hat{c}_{i0} + \tilde{C}_0 + K_1 = Z_0 + w_0 L_0 - v(L_0) \\
& \qquad \qquad B_{h1} + B_{\ell 1} + B_1^s = 0
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}_0 : & \qquad \frac{1}{\tilde{C}_0} = \lambda_0, \qquad \hat{c}_{i0} : \qquad \frac{\chi_i}{\hat{c}_{i0}} = \lambda_0 \\
B_1 : & \qquad \beta \frac{\partial W_1}{\partial B_1} = \beta \chi_h \pi_h \frac{\partial V_{h1}}{\partial B_1} = \beta \chi_\ell \pi_\ell \frac{\partial V_{\ell 1}}{\partial B_1} \\
K_1 : & \qquad \lambda_0 = \beta \left[\frac{\partial W_1}{\partial K_1} + \chi_h \pi_h \frac{\partial V_{h1}}{\partial K_1} + \chi_\ell \pi_\ell \frac{\partial V_{\ell 1}}{\partial K_1} \right]
\end{aligned}
$$

After collecting terms and simplifying, one obtains:

$$\frac{\hat{C}_{h0}}{\hat{C}_{h1}} - \frac{\hat{C}_{\ell 0}}{\hat{C}_{\ell 1}} = \frac{\hat{C}_{h0}}{\hat{C}_{h1}}\frac{\partial \mathcal{D}(\cdot)}{\partial B_1} + \theta_1\mu_1\frac{\partial D_{h2}}{\partial B_1}$$

$$1 = \beta\left[\frac{\hat{C}_{h0}}{\hat{C}_{h1}}\frac{\partial \mathcal{D}(\cdot)}{\partial K_1} + \theta_1\mu_1\frac{\partial D_{h2}}{\partial K_1} + \frac{\tilde{C}_0}{\tilde{C}_1}\frac{\partial (w_1 L_1)}{\partial K_1}\right]$$

where:

$$\frac{\partial \mathcal{D}(\cdot)}{\partial B_1} = \frac{\tilde{a}-1}{(1-\theta_1)\tilde{a}} > 0 \tag{D.3.2}$$

$$\frac{\partial \mathcal{D}(\cdot)}{\partial K_1} = \frac{\partial Y_1}{\partial K_1} + \underbrace{\left(\frac{\partial Y_1}{\partial N_1} - w_1\right)}_{>0}\frac{\partial N_1}{\partial K_1} - N\frac{\partial w_1}{\partial K_1} \tag{D.3.3}$$

The effect on the profits of productive firms which affect their future borrowing capacity is present whether the economy is in a liquidity trap or not:

$$\frac{\partial D_{h2}}{\partial B_1} = (1-\theta_1)\left[\left(\frac{\partial Y_{h2}}{\partial K_{h2}} - N_{h2}\frac{\partial \omega_2}{\partial K_{h2}}\right)\frac{\partial K_{h2}}{\partial B_1} - N_{h2}\frac{\partial \omega_2}{\partial K_{\ell 2}}\frac{\partial K_{\ell 2}}{\partial B_1}\right] < 0$$

The effect of larger debt at time 0 is not as strongly negative when the economy is in a liquidity trap. Larger savings increased the amount of investment of low productivity firms, which increased the wage rate, negatively affecting the cashflows of productive firms with unconstrained monetary policy. In a liquidity trap the only effect of larger borrowing is the direct effect on capital demand of productive firms. Similarly:

$$\frac{\partial D_{h2}}{\partial K_1} = (1-\theta_1)\left[\left(\frac{\partial Y_{h2}}{\partial K_{h2}} - N_{h2}\frac{\partial \omega_2}{\partial K_{h2}}\right)\frac{\partial K_{h2}}{\partial K_1} - N_{h2}\frac{\partial \omega_2}{\partial K_{\ell 2}}\frac{\partial K_{\ell 2}}{\partial K_1}\right] > 0$$

From the point of view of time 0, concerning borrowing, there is a trade-off between moderating the credit crunch and boosting demand at time 1. □

E Extensions

E.1 Proof of Lemma 3

In presence of two sectors the laissez-faire allocation in a financial crisis is not second best, as the economy features:

- *Over-investment in manufacturing and zombie firms;*

- *In the aftermath of the crisis, a relative size of the manufacturing sector that is too large;*
- *Aggregate output at time 2 that is too large.*

$$
\begin{aligned}
& V_{i2}(z_{i2}; B_2, K_{i2}, K_{j2}) = \max \ \log \hat{c}_{i2} \\
\text{s.to} \quad & p_2 \hat{c}_{i2} = d^m_{i2} - b_{i2} \\
& W_2(B_2, K_{h2}, K_{\ell 2}) = \max \ \log \tilde{C}_2 \\
\text{s.to} \quad & p_2 \hat{C}_2 = D^s_2 - B^s_2 + BL_2, \qquad \tilde{C}_2 \equiv C_2 - v(L_2)
\end{aligned}
$$

where: $d^m_{i2} = p^m_2 \hat{a}_i k^\alpha_{i2} (n^m_{i2})^{1-\alpha} - Bn^m_{i2}$

$$p^m_2 = \frac{B\Theta^\alpha_2}{1-\alpha} \frac{1}{(\hat{a}_h K_{h2} + \hat{a}_\ell K_{\ell 2})^{\frac{\alpha\psi}{\psi+\gamma\alpha}}}, \quad p_2 = \frac{(p^m_2)^\gamma}{\hat{\gamma}}$$

with $\Theta_t = \left[\frac{\hat{\gamma}B^{1-\gamma}(1-\alpha)^\gamma}{(1+\eta_t)^\psi}\right]^{\frac{1}{\psi+\alpha\gamma}}$ and $\eta_t = \frac{1-\gamma(1-\alpha\hat{\beta}_t)}{(1-\alpha)\gamma}$ time-dependent function of parameters.

$$
\begin{aligned}
\frac{\partial V_{i2}}{\partial K_{j2}} &= \lambda_{i2} \left[\frac{\partial D^m_{i2}}{\partial K_{j2}} - \gamma p_2 \hat{C}_{i2} \frac{\partial p^m_2 / p^m_2}{\partial K_{j2}} \right] \\
\frac{\partial W_2}{\partial K_{j2}} &= -\tilde{\lambda}_2 \hat{C}_2 \frac{\partial p_2}{\partial K_{j2}}
\end{aligned}
$$

with: $$\frac{\partial D^m_{i2}}{\partial K_{j2}} = \mathbb{1}_{i=j} p^m_2 \frac{\partial Y^m_{i2}}{\partial K_{j2}} + Y^m_{i2} \frac{\partial p^m_2}{\partial K_{j2}}, \qquad \frac{\partial p^m_2}{\partial K_{j2}} = -\frac{\psi}{\psi+\gamma} \frac{p^m_2}{\hat{a}_h K_{h2} + \hat{a}_\ell K_{\ell 2}} \hat{a}_j$$

Planner's problem

$$V^P_1(Z_1, S_1) = \max_{\hat{C}_1, \hat{c}_{i1} K_{i2}, B_2} \left\{ \log\left(\tilde{C}_1\right) + \beta W_2(B^s_2, K_{h2}, K_{\ell 2}) + \sum_{i \in h,l} \chi_i \pi_i \left[\log \hat{c}_{i1} + \beta V_{i2}(z_{i2}; B_2, K_{h2}, K_{\ell 2})\right] \right\}$$

$$
\begin{aligned}
\text{s.to} \quad & p_1 \left(\sum_{i \in h,l} \pi_i \hat{c}_{i1} + \tilde{C}_1 \right) + X_{h2} + X_{\ell 2} = p^m_1 Y^m_1 + Y^s_1 - p_1 v(L_1) && [\tilde{\lambda}_1] \\
& B_{h2} + B_{\ell 2} + B^s_2 = 0 && [\tilde{\nu}_1] \\
& B^m_2 + B^s_2 \leq \theta_1 D^m_{h2} && [\tilde{\mu}_1]
\end{aligned}
$$

Where the borrowing constraint of the service sector making zero profits is also considered and added to the manufacturing sector one.

$$B_2: \quad \frac{\mu_1}{\lambda_1} = \beta \left(\frac{p_1 \hat{C}_{\ell 1}}{p_2 \hat{C}_{\ell 2}} - \frac{p_1 \hat{C}_{h1}}{p_2 \hat{C}_{h2}} \right), \quad \frac{\mu_1}{\lambda_1} = \beta \left(\frac{p_1 \hat{C}_{\ell 1}}{p_2 \hat{C}_{\ell 2}} - \frac{p_1 \tilde{C}_1}{p_2 \tilde{C}_2} \right)$$

$$K_2: \quad 1 = \mathbb{1}_{i=h}\frac{\theta_1\mu_1}{\lambda_1}p_2^m\frac{\partial Y_{i2}^m}{\partial K_{i2}} + \frac{\beta p_1\hat{C}_{i1}}{p_2\hat{C}_{i2}}p_2^m\frac{\partial Y_{i2}^m}{\partial K_{i2}} - \frac{\mu_1}{\lambda_1}\left[(1-\theta_1)\alpha p_2^m Y_{h2}^m - \gamma p_2\left(\hat{C}_{h2}+\hat{C}_2\right)\right]\frac{\partial p_2^m/p_2^m}{\partial K_{i2}}$$

Constrained Service Sector.
We know:

$$p_2\hat{C}_{h2} = (1-\hat{\beta}_2)Z_{h2} = (1-\hat{\beta}_2)(1-\theta_1)\alpha p_2^m Y_{h2}^m;$$
$$p_2\hat{C}_2 = w_2 L_2 = w_2(1+\eta_2)N_2^m = (1-\alpha)(1+\eta_2)p_2^m Y_{h2}^m$$

Using these expressions in the last term in parenthesis in the FOC for capital we obtain:

$$(1-\theta_1)\alpha p_2^m Y_{h2}^m - \gamma p_2\left(\hat{C}_{h2}+\hat{C}_2\right) = -p_2^m\left\{(1-\gamma\alpha)Y_{\ell 2}^m + \left[1-\alpha\left(1-\theta_1(1-\gamma)\right)\right]Y_{h2}^m\right\} < 0$$

□

E.2 Proof of Lemma 4

If workers are unconstrained, then the laissez-faire economy in a financial crisis does not feature zombie firms, but under-investment as low productivity firms should invest more.

Unconstrained Service Sector.
When the borrowing constraint of the service sector is not binding,

$$1 = \mathbb{1}_{i=h}\frac{\theta_1\mu_1}{\lambda_1}p_2^m\frac{\partial Y_{i2}^m}{\partial K_{i2}} + \frac{\beta p_1\hat{C}_{i1}}{p_2\hat{C}_{i2}}p_2^m\frac{\partial Y_{i2}^m}{\partial K_{i2}} - \frac{\mu_1}{\lambda_1}\left[(1-\theta_1)\alpha p_2^m Y_{h2}^m - \gamma p_2\hat{C}_{h2}\right]\frac{\partial p_2^m/p_2^m}{\partial K_{i2}}$$

We can show that the last element in parenthesis is positive by using the budget constraint of high-productivity entrepreneurs, where borrowing and saving is set to 0 for consistency:

$$(1-\theta_1)\alpha p_2^m Y_{h2}^m - \gamma p_2\hat{C}_{h2} = \hat{C}_{h2}^s > 0$$

□

E.3 Proof of Lemma 5

A financial crisis away from the ELB induces no change in aggregate output but changes in consumption at time 1; aggregate productivity and production fall at date 2.

Unchanged output at date 1. Without a service sector, $p_t^m = 1$, $L_t = N_t^m$. Because the choice of employment is optimal when $R_1 = R_1^*$, the wage rate and level of employment continues to be as in (??). Output is unchanged compared to an unconstrained setting, for given level of pre-installed capital. However, consumption depends directly on net worth, which depends on the asset price. The price of land will not remain unchanged, as it is now set according to (??) rather than (??).

Lower manufacturing TFP and output at time 2.
A similar argument to the previous proof applies. Because the unconstrained equilibrium is first best, consumption and output are maximised. The equilibrium in which the borrowing limit is binding therefore cannot feature a larger level of output then the first best equilibrium.

□

E.4 Proof of Proposition 7

In a financial crisis away from the ELB the allocation is not constrained efficient. The decentralised economy features:

- *Under-investment and too few low productivity firms;*
- *Low output at date 2.*

$$
\begin{aligned}
& W_2(B_2, K_{h2}, K_{\ell 2}) = \max \ \log \tilde{C}_2 + \beta W_3(S_3) \\
& \text{s.to} \qquad \hat{C}_2 = w_2 L_2 - B_2^w, \qquad \tilde{C}_2 \equiv C_2 - v(L_2) \\
& V_{i2}(z_{i2}; B_2, K_{i2}, K_{j2}) = \max \ \log \hat{c}_{i2} + \beta V_{i3}(z_{i3}; S_3) \\
\text{s.to} \qquad & \hat{c}_{i2} + q_2^h h_{i3} + x_{i3} - \frac{b_{i3}}{R_2} = d_{i2}^m - b_{i2} + q_2^h h_{i2} \\
& b_{h3} \leq 0 \\
& \frac{\partial W_2}{\partial K_{i2}} = \lambda_2^w \frac{\partial w_2}{\partial K_{i2}} \\
& \frac{\partial V_{i2}}{\partial H_{j2}} = \lambda_{i2} \left[\frac{\partial D_{i2}^m}{\partial H_{j2}} + \mathbb{1}_{i=j} q_2^h - \left(\Delta H_{i3} + \frac{B_{i3}}{R_2^2} \frac{\partial R_2}{\partial q_2^h} \right) \frac{\partial q_2^h}{\partial H_{j2}} \right] \\
& \frac{\partial V_{j2}}{\partial X_{i2}} = \lambda_{j2} \left[\frac{\partial D_{j2}}{\partial X_{i2}} - \left(\Delta H_{j3} + \frac{B_3}{R_2^2} \frac{\partial R_2}{\partial q_2^h} \right) \frac{\partial q_2^h}{\partial X_{i2}} \right] \\
& \frac{\partial W_2}{\partial B_2^w} = -\lambda_2^w, \qquad \text{with:} \qquad \lambda_2^w = \frac{1}{\tilde{C}_2} \\
& \frac{\partial V_{i2}}{\partial B_{i2}} = -\lambda_{i2}, \qquad \text{with:} \qquad \lambda_{i2} = \frac{1}{\hat{C}_{i2}}
\end{aligned}
$$

where: $\quad d^m_{i2} = a_i k^\alpha_{i2} n^{1-\alpha}_{i2} - w_2 n_{i2}, \quad k_{i2} = x^\delta_{i2} h^{1-\delta}_{i2}$

$$u_2 = q^h_2 = \frac{\widetilde{\beta\delta_2}}{1-\widetilde{\beta\delta_2}} \alpha Y_2, \qquad R_2 = \frac{\hat{a}_\ell}{q_2}\left(\frac{\hat{\alpha}}{w_3^{1-\alpha}}\right)^{1/\alpha}, \qquad w_t = \left[(1-\alpha)\left(\sum \hat{a}_i K_{it}\right)^\alpha\right]^{\frac{\psi}{\psi+\alpha}}$$

Planner's problem

$$V^P_1(Z_1, S_1) = \max_{\hat{c}_{i1}, X_{i2}, H_{i2}, B_2} \left\{ \log \tilde{C}_1 + \beta W_2(B_2, K_{h2}, K_{\ell 2}) + \sum_{i \in h,l} \chi_{mi} \pi_i \left[\log \hat{c}_{i1} + \beta V_{i2}(z_{i2}; B_2, K_{h2}, K_{\ell 2})\right] \right\}$$

s.to $\quad \sum_{i\in h,l} \pi_i \hat{c}_{i1} + X_{h2} + X_{\ell 2} = Y_1$

$$H_{h2} + H_{\ell 2} = 1$$

$$B_2 + B^w_2 \le q^h_2 H_{h2}$$

$$\hat{c}_{i1}: \quad \frac{\chi_i}{\hat{c}_{i1}} = \lambda_1, \qquad B_2: \quad \mu_1 = \beta \sum_i \chi_i \pi_i \frac{\partial V_{i2}}{\partial B_2}$$

$$\implies \frac{\mu_1}{\lambda_1} = \beta\left(\frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\beta \hat{C}_{h1}}{\hat{C}_{h2}}\right) = \beta\left(\frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\beta \tilde{C}_1}{\tilde{C}_2}\right)$$

$$H_{i2}: \quad \gamma_1 = \beta \frac{\partial W_2}{\partial X_{i2}} + \beta \sum_j \chi_j \pi_j \frac{\partial V_{j2}}{\partial H_{i2}} + \mu_1 \left(H_{h2} \frac{\partial q^h_2}{\partial H_{i2}} + \mathbb{1}_{i=h} q^h_2\right)$$

$$\implies \frac{\gamma_1}{\lambda_1} = \frac{\beta \hat{C}_{i1}}{\hat{C}_{i2}}\left(\frac{\partial Y_{i2}}{\partial H_{i2}} + q^h_2\right) + \mathbb{1}_{i=h} \frac{\mu_1}{\lambda_1} q^h_2 + \frac{\mu_1}{\lambda_1} H_{h2} \frac{\partial q^h_2}{\partial H_{i2}} - \beta\left(\frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\beta \tilde{C}_1}{\tilde{C}_2}\right) L_2 \frac{\partial w_2}{\partial H_{i2}} +$$
$$\beta\left(\frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\beta \hat{C}_{h1}}{\hat{C}_{h2}}\right)\left(\Delta H_{h3} \frac{\partial q^h_2}{\partial H_{i2}} + N_{h2} \frac{\partial w_2}{\partial H_{i2}}\right)$$

$$X_{i2}: \quad \lambda_1 = \beta \frac{\partial W_2}{\partial X_{i2}} + \beta \sum_i \chi_i \pi_i \frac{\partial V_{i2}}{\partial X_{i2}} + \mu_1 H_{h2} \frac{\partial q^h_2}{\partial X_{i2}}$$

$$\implies 1 = \frac{\beta \hat{C}_{i1}}{\hat{C}_{i2}} \frac{\partial Y_{i2}}{\partial X_{i2}} + \frac{\mu_1}{\lambda_1} H_{h2} \frac{\partial q^h_2}{X_{i2}} + \beta\left(\frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\beta \hat{C}_{h1}}{\hat{C}_{h2}}\right)\left(\Delta H_{h3} \frac{\partial q^h_2}{\partial X_{i2}} + N_{h2} \frac{\partial w_2}{\partial X_{i2}}\right)$$
$$- \beta\left(\frac{\beta \hat{C}_{\ell 1}}{\hat{C}_{\ell 2}} - \frac{\beta \tilde{C}_1}{\tilde{C}_2}\right) L_2 \frac{\partial w_2}{\partial X_{i2}}$$

where the fact that $b_{i3} = q^h_3 h_{h3} = 0$ was used. Even if workers are constrained, it is not clear that the underinvestment result would be overturned. □