Theorems for Exchangeable Binary Random Variables with Applications

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WIFO

Overview

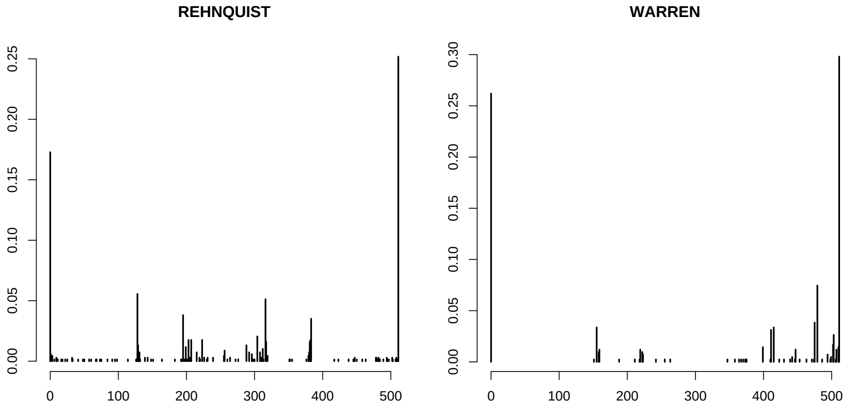
1. Examples of expertise and power computations based on a non-trivial probability distribution over the set of all voting outcomes
2. The property of exchangeability as a stochastic model of a representative agent (voter)
3. Known parameterizations of the joint probability distribution of n correlated binary random variables
4. The probability of at least k successes in n correlated binary trials. Some results for exchangeable random variables with vanishing higher-order correlations
5. The bounds on this probability when the correlations are unknown
6. Application to the Condorcet Jury Theorem and voting power in the sense of Penrose - Banzhaf
7. Concluding remarks
8. Where to find my work

Examples of voting under simple majority rule ($n = 3$)

$\mathbf{v} = (v_1, v_2, v_3)$			$p = 0.5$ $c = 0$	$p = 0.75$ $c = 0$	$p = 0.75$ $c = 0.2$	$p_1 = 0.75$ $p_{2,3} = 0.6$ $c = 0.2$
1	1	1	0.125	0.422	0.506	0.357
1	1	0	0.125	0.141	0.094	0.136
1	0	1	0.125	0.141	0.094	0.136
1	0	0	0.125	0.047	0.056	0.122
0	1	1	0.125	0.141	0.094	0.051
0	1	0	0.125	0.047	0.056	0.056
0	0	1	0.125	0.047	0.056	0.056
0	0	0	0.125	0.016	0.044	0.086
Condorcet probability			0.5	0.844	0.788	0.679
Banzhaf probability 1			0.5	0.376	0.3	0.384
Banzhaf probability 2			0.5	0.376	0.3	0.365
Banzhaf probability 3			0.5	0.376	0.3	0.365

Computing the probability of a correct verdict, or the voting power as the probability of casting a decisive vote, requires a joint probability distribution on the set of all voting profiles $\mathbf{v} \in \mathbb{R}^{2^n}$. The influence of voting weights and decision rule is separate from that of the distribution. Exchangeability leads to a representative agent model, in which the independence assumption is relaxed

Voting in the U.S. Supreme Court



Empirical evidence overwhelmingly refutes the assumption of independent votes required in the classic versions of the Condorcet Jury Theorem and the Penrose - Banzhaf measure of voting power

The joint probability distribution of n binary r.v.

The Bahadur parametrization

$$
\begin{array}{lll}
Z_i = (V_i - p_i)/\sqrt{p_i(1-p_i)} & \text{for all } i = 1, 2, \ldots, n, & p_i = p \\
c_{i,j} = E(Z_i Z_j) & \text{for all } 1 \le i < j \le n, & c_{i,j} = c \\
c_{i,j,k} = E(Z_i Z_j Z_k) & \text{for all } 1 \le i < j < k \le n, & c_{i,j,k} = c_3 \\
\ldots & & \\
c_{1,2,\ldots,n} = E(Z_1 Z_2 \ldots Z_n), & & c_{1,2,\ldots,n} = c_n
\end{array}
$$

$$\pi_{\mathbf{v}} = \bar{\pi}_{\mathbf{v}} \left(1 + \sum_{i<j} c_{i,j} z_i z_j + \sum_{i<j<k} c_{i,j,k} z_i z_j z_k + \cdots + c_{1,2,\ldots,n} z_1 z_2 \ldots z_n \right)$$

where $\bar{\pi}_{\mathbf{v}} = \prod_{i=1}^{n} p_i^{v_i} (1-p_i)^{(1-v_i)}$ is the probability under the independence

The George - Bowman parametrization for exchangeable binary r.v.

$$\pi_i = \sum_{j=0}^{i} (-1)^j C_i^j \lambda_{n-i+j}, \text{ where } \lambda_i = P(X_1 = 1, X_2 = 1, \ldots, X_i = 1), \quad \lambda_0 = 1$$

The probability of at least k successes in n trials

This probability finds wide application in reliability and decision theory

For an odd n, $c_3 = c_4 = \cdots = c_n = 0$ and $(p, c) \in \mathcal{B}_n$ (Bahadur set)

$$P_n^k(p, 0) = \sum_{t=k}^{n} C_n^t p^t (1-p)^{n-t} = I_p(k, n-k+1)$$

$$P_n^k(p, c) = I_p(k, n-k+1) + 0.5c(n-1)\left(\frac{k-1}{n-1} - p\right)\frac{\partial I_p(k, n-k+1)}{\partial p}$$

where $I_x(a, b)$ is the regularized incomplete beta function

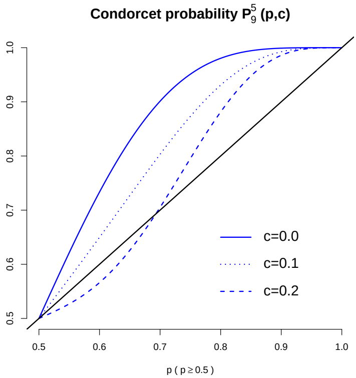
Bounds on $P_{n,p}^k$ for given n and p can be found by linear programming. Di Cecco provides bounds for given n, p and c such that $(p, c) \in \mathcal{B}_n$

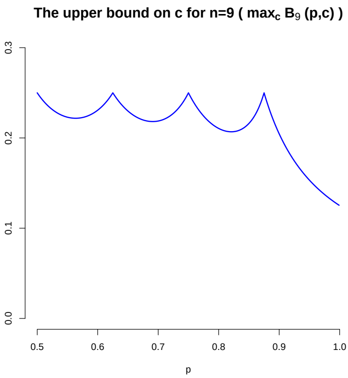
Bounds on $P_{n,p}^k$ when all correlation coefficients are unknown

$$\max\left\{\frac{np-k+1}{n-k+1}, 0\right\} \le P_{n,p}^k(c, c_3, \ldots, c_n) \le \min\left\{\frac{np}{k}, 1\right\}$$

The Condorcet probability and the Bahadur set

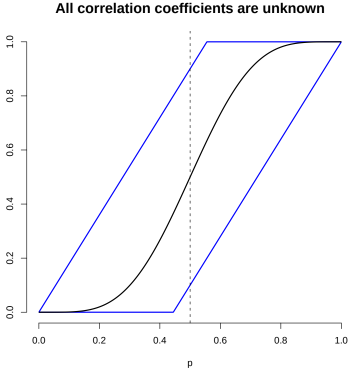
The set $\mathcal{B}_n$ contains all admissible values of c for given n and p, provided $c_3 = c_4 = \cdots = c_n = 0$

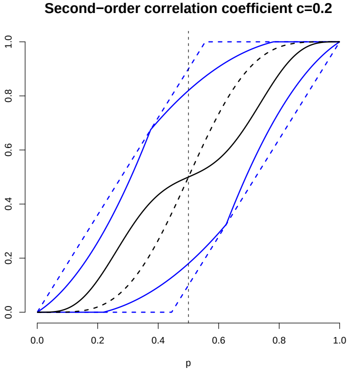




$\mathcal{B}_n$ is such that $0 < c < \frac{1}{n-1}$ for $p \approx 1$ and $0 < c < \frac{2}{n-1}$ for $p \approx 0.5$

Bounds on the probability of at least 5 successes in 9 trials





Voting power

In a 'one person, one vote' election with two alternatives a vote is decisive if it breaks a tie. With $n+1$ voters, the probability of a tie equals $C_n^{\frac{n}{2}} \pi_{\frac{n}{2}}$

For an odd n, $c_3 = c_4 = \cdots = c_n = 0$ and $(p, c) \in \mathcal{B}_n$

$$V_n^k(p, 0) = C_n^{\frac{n}{2}} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}}$$

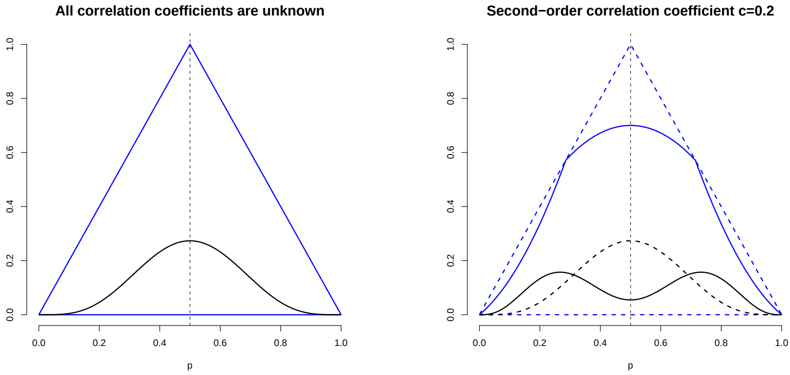
$$V_n^k(p, c) = V_n^k(p, 0) + \frac{nc}{4} C_n^{\frac{n}{2}} p^{\frac{n-2}{2}} (1-p)^{\frac{n-2}{2}} \left(\frac{n(2p-1)^2}{2} + 2p(1-p) - 1 \right)$$

Bounds on $V_n^k(c, c_3, \ldots, c_n)$ when all correlation coefficients are unknown

$$0 \le V_n^k(c, c_3, \ldots, c_n) \le 2 \min\{p, 1-p\}$$

For given n, p and c, bounds can be found by linear programming

Bounds on voting power



Generating a joint probability distribution

This is helpful in developing probabilistic voting scenarios

The following regularization yields a non-exchangeable distribution

$$\pi_{\mathbf{v}} \geq 0, \quad \sum_{\mathbf{v} \in \mathbf{V}} \pi_{\mathbf{v}} = 1 \qquad \text{for all } \mathbf{v} \in \mathbf{V}$$

$$\sum_{\mathbf{v} \in \mathbf{V}(i)} \pi_{\mathbf{v}} = p_i \qquad \text{for all } i = 1, 2, \ldots, n$$

$$\sum_{\mathbf{v} \in \mathbf{V}(i,j)} \pi_{\mathbf{v}} = p_i p_j + c_{i,j} \sqrt{p_i(1-p_i)p_j(1-p_j)} \qquad \text{for all } 1 \leq i < j \leq n$$

$$\min_{\pi_{\mathbf{v}}} 0.5 \sum_{\mathbf{v}} (\pi_{\mathbf{v}} - \bar{\pi}_{\mathbf{v}})^2 \qquad \text{The objective function}$$

$\mathbf{V}(i)$ is the set of all $\mathbf{v}$ such that $v_i = 1$, and $\mathbf{V}(i,j) = \mathbf{V}(i) \cap \mathbf{V}(j)$ the set of all $\mathbf{v}$ such that $v_i = v_j = 1$. The quadratic optimization problem has 2^n variables and $1 + 2n + C_n^2$ constraints. But the Condorcet probability does not depend on $c_{i,j}$ for this distribution (invariance)

Conclusions: Condorcet Jury

- The effect of correlation on the jury's competence is negative for voting rules close to simple majority and positive for voting rules close to unanimity
- If the individual competence is low, it may be better to hire one expert rather than several. In all other cases simple majority rule is the optimal decision rule. A jury operating under simple majority rule will not necessarily benefit from an enlargement, unless the enlargement is substantial. The higher the individual competence, the sooner an enlargement will be beneficial
- Correlation-robust voting rules minimizes the effect of correlation on collective competence by making it as close as possible to that of a jury of independent jurors. The optimal correlation-robust voting rule should be preferred to simple majority rule if mitigating the effect of correlation is more important than maximizing the accuracy of the collective decision
- For a given competence, compute the bounds to a jurys competence as the minimum and maximum probability of a jury being correct

Conclusions: Voting power

- We can assess the magnitude of numerical error or bias in the Penrose - Banzhaf measure that occurs when equiprobability and independence assumptions are not met. The probability bias is more severe than the correlation bias. Common positive correlation biases the measure upwards, common negative correlation downwards
- Despite the Banzhaf measure being a valid measure of *a priori* voting power and thus useful for evaluating the rules at the constitutional stage of a voting body, it is a poor measure of the actual probability of being decisive at any time past that stage
- Derive a modified square-root rule for the representation in two-tier voting systems that takes into account the sizes of the constituencies and the heterogeneity of their electorates. Since in a homogeneous electorate the votes are positively correlated, the larger and the more homogeneous the electorate, the less power a vote has
- Develop realistic voting scenarios that reflect the preferences of the voters via a correlation matrix. Then generate a consistent joint probability distribution and compute the probabilities of interest
- Compute the bounds to voting power as the minimum and maximum probability of the voter being decisive

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Voting power / Square root rule / U.S. Supreme Court

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