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Banzhaf-Coleman and Shapley-Shubik indices in games with a coalition structure – a special case study

Abstract In the paper we deal with the comparison of the Shapley-Shubik index and Banzhaf-Coleman index in games with a coalition structure. We analyze two possible approaches in both cases - we calculate voters' power in a composite game or we apply the modification of original indices proposed by Owen for games with a priori unions. The behavior of both indices is compared basing on the voting game with 100 voters and different coalition structures. We analyze changes of power (measured by means of BC index and SS index) implied by changes of the size and composition of coalition structures as well as by different methodology of measuring the voters' power (composite game versus game with a priori unions).

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Introduction

In the paper we investigate how to measure the power of individuals in a voting body possibly divided into some blocks (parties). We are modeling such situation in two different ways – by applying the framework of games with a priori unions (Owen [10]) and by applying composite games (Felsenthal, Machover [6]). In both cases we measure the power of individual voters using Shapley-Shubik and Banzhaf-Coleman indices. We make simulations for a specific voting body composed of 100 members and we compare both approaches. The aim of the paper is to compare the behavior of both indices in those frameworks and to find similarities and differences between them, implied by changes of the size and composition of coalition structures as well as by different methodology of measuring the voters' power (composite game versus game with a priori unions).

We begin with describing the formal model. In the sequel we present the results of simulations for a voting body composed of 100 members with various divisions into parties (blocs).

Model

Let $N = \{1, 2 \dots n\}$ denote the set of voters (or seats). We consider a decision-making situation in which the voting body is supposed to make a decision (to pass or to reject a proposal) by means of a voting rule. We assume that voters who do not vote for a proposal (do not vote "yes") vote against it and there is no possibility of abstention. The voting rule specifies whether the set of voters who accepted the proposal forms a winning coalition or not. Formally, we have 2^n possible coalitions (vote configurations) $S \subseteq N$. The voting rule if then defined by the set of winning coalitions *W*. Usually it is assumed that

- $\emptyset \notin W$,
- $N \in W$,
- if $S \in W$ then $N - S \notin W$,
- if $S \in W$ and $S \subset T$ then $T \in W$.

The voting rule is equivalently given by a simple voting game v_W as follows

$$v_W(S) = \begin{cases} 1 & \text{if } S \in W \\ 0 & \text{if } S \notin W \end{cases}$$

for each $S \subseteq N$.

We say that a voter *i* is *critical* for a coalition *S* if $v_W(S) = 0$ and $v_W(S \cup \{i\}) = 1$ or $v_W(S) = 1$ and $v_W(S \setminus \{i\}) = 0$.

The Banzhaf-Coleman index of a voter *j* in this framework is the probability of a voter to be critical assuming that all voting configurations are equally probable, that is

$$\beta_j(W) = \frac{\#\{S \subset N : (j \in S \in W \wedge S - \{j\} \notin W) \vee (j \notin S \notin W \wedge S \cup \{j\} \in W)\}}{2^n} =$$

$$\frac{1}{2^{n-1}} \sum_{\substack{S \subset N \\ j \in S}} (v_W(S) - v_W(S - \{j\})) .$$

The Shapley-Shubik index of a voter *j* is a truncation of the Shapley value defined for simple games and it is given by the formula

$$Sh_j(W) = \sum_{\substack{S \subset N \\ j \in S}} \frac{(s-1)!(n-s)!}{n!} \left(v_W(S) - v_W(S - \{j\})\right),$$

where $s = |S|$. Shapley-Shubik index has also a probabilistic interpretation – if we assume that all orderings of voters are equally probable, then the Shapley-Shubik index of a voter *j* is the probability, that this voter is pivotal (i.e. changes the already existing coalition from loosing to winning, regardless what happens after his accession). Most important characterizations of BC and SS indices are given in [2], [3], [4], [11], [13], [14], [15].

In real world voting bodies the situation is more complicated since voters are divided into some blocs (parties) ex ante, which may constrain the actual voting behavior. This partition may be the consequence of the political party membership, which is an obvious reason of some constraints in voting in bodies like parliaments. It might also reflect different national interests of citizens of various members of international communities like EU or IMF. This situation can be described by games with a priori unions (precoalitions) introduced in 1977 by Owen in [10]. Let $T = (T_1, T_2, ..., T_m)$ be a partition of the set *N* into subsets which are nonempty, pairwise disjoint and $\bigcup_{i=1}^{m} T_i = N$. The sets T_i are called precoalitions (a priori unions) and they can be interpreted as parties occupying seats in the voting body (note that some of T_i can be singletons). Let *M* denote the set of all precoalitions, that is $M = \{1,2,...,m\}$. Owen proposed the modification of both Shapley value ([10]) and Banzhaf-Coleman index ([12]) for games with a priori unions and we will be dealing in this paper with these modifications. The formulae are as follows:

– modification of Banzhaf-Coleman index – for a voter *j* in a bloc T_i we have

$$O_j^{BC}(W,T)=\frac{1}{2^{m+t_i-2}}\sum_{Q\subset M-\{i\}}\sum_{\substack{K\subset T_i\\ j\in K}}v_w(N(Q)\cup K)-v_w(N(Q)\cup(K-\{j\})),$$

where $N(Q)=\bigcup_{p\in Q}T_p$ and t_i denotes the cardinality of T_i;

– modification of the Shapley value (or Shapley-Shubik index) – for a voter j in a bloc T_i we have

$$O_j^{SS}(W,T)=\sum_{\substack{H\subset M\\ i\notin H}}\sum_{\substack{S\subset T_i\\ j\notin S}}\frac{h!(m-h-1)!s!(t_i-s-1)!}{m!t_i!}(v_W(H\cup S\cup\{j\})-v_W(H\cup S)),$$

where h denotes the cardinality of the set H, t_i – the cardinality of the party T_i and s – the cardinality of the coalition S. This index is often called Owen index. Since we refer here to both modified indices – Banzhaf-Coleman's and Shapley-Shubik's – we shall use the term "coalitional index" in order to avoid misunderstanding.

The coalitional BC index is the ratio of the number of coalitions for which the voter $j\in T_i$ is critical and no bloc different from T_i can be broken to the total number of such coalitions. Laruelle and Valenciano in [8] have given three different probabilistic interpretation of the modified BC index. The coalitional BC index was axiomatized by Albizuri [1].

When calculating coalitional SS index of a player $j\in T_i$ we restrict the number of possible permutations of the set of players. We take into account only those permutations in which all players from each precoalition appear together. To find all such permutations we need to order the parties first and then to order the players in each party. Coalitional SS index can therefore be interpreted as a probability of a player $j\in T_i$ being pivotal provided that all permutations of the set of players, respecting the coalition structure, are equally probable. Coalitional SS index was also axiomatized in various ways, see e.g. Owen [10] or Hart, Kurz [7].

There is also another possibility of measuring the decisiveness of each voter in the context of games with an a priori coalition structure. Suppose that within each party the proposal is accepted or rejected by simple majority voting and then all members of the party vote according to the decision made by inside party voting. This is the case of a composite game (see [6]). In this case the BC index of a member of a party T_i is the product of his index in the simple majority voting game inside the party and the index of the party T_i treated as a player in the top game. In that game the set of players is *M*, that is players are parties and the set of winning coalitions is $W_T = \{Q \subset M : N(Q) \in W\}$, so for a voter $j \in T_i$ we have

$$\beta^c_j(W,T) = \beta_j(W_{t_i}) \cdot \beta_i(W_T),$$

where $W_{t_i} = \{K \subset T_i : \# K \geq \left[\frac{t_i}{2}\right] + 1\}$ and the symbol [*x*] denotes the largest integer not greater than *x*, for any real *x*. In fact this is the BC index in the composite game with the top W_T and the components W_{T_i} for *i* = 1,2,..., *m*.

SS index in a composite game (we will denote it by $Sh^c(W,T)$) does not have such "product" property – we calculate it directly from its definition.

In the sequel we present an example of a voting body composed of 100 voters, who are divided into two parties or vote independently. We calculate the power of voters for both approaches (game with precoalitions and composite game) and using both indices – BC and SS – for all possible configurations of sizes of parties. A part of the results presented here is also examined in Ekes [5].

Description of the special case

We consider the situation where the voting body is composed of 100 voters, who are members of one of two existing parties or who are voting as independent voters. The coalition structure is therefore the following: $T = (T_1, T_2, \{j_1\}, ..., \{j_l\})$, where $2 \leq t_1, t_2$ and $t_1 + t_2 + l = 100$. We assume that the voting rule in our example is the simple majority, which means that any proposal is accepted if it has at least 51 votes for. We are not interested in case where a single party constitutes the winning majority, therefore we assume that $t_1, t_2 \leq 50$. We have calculated values of all four indices indices: $O^{BC}(T)$, $O^{SS}(T)$, $\beta^c(T)$ and $Sh^c(T)$ for all possible configurations of sizes of parties and for all voters (we omit the symbol *W* in the notation of indices since the simple majority rule defines the set of winning coalitions in the game with precoalitions as well as in the composite game). We also note that due to the symmetry of SS and BC indices, the power of all voters belonging to the same party is equal and the power of all independent voters is the same. Therefore we will use the notation $O^{BC}_{T_i}(T), O^{SS}_{T_i}(T), \beta^c_{T_i}(T)$, $Sh^c_{T_i}(T)$ for $i = 1, 2$ and $O^{BC}_{j_k}(T), O^{SS}_{j_k}(T), \beta^c_{j_k}(T)$, $Sh^c_{j_k}(T)$ for *k* = 1,2,…, *l*. Parties are symmetric in our case – if we consider the coalition structures of the form $T^1 = \{T^1_1, T^1_2, \{j^1_1\}, ..., \{j^1_l\}\}$, $T^2 = \{T^2_1, T^2_2, \{j^2_1\}, ..., \{j^2_l\}\}$ such that $t^2_1 = t^1_2 \wedge t^1_1 = t^2_2$, then the value of all considered indices of the first party members given the coalition structure

T^1 is equal to the value of respective indices of the second party members given the coalition structure T^2 while the power of independent voters measured by any of considered indices is the same for both coalition structures T^1 and T^2. This observation allows considering only the value of all indices for members of the first party and for independent voters. The number of elements of our coalition structure is equal to $100 - t_1 - t_2 + 2 = l + 2$.

Let us introduce an additional notation. In order to calculate SS index of a T_1 member in the composite game we have two find the set of all coalitions for which the first party is decisive in the weighted voting game of parties (by parties we mean the two "large" parties and all independent voters). We denote this set of coalitions by $Dec(T_1)$. For a coalition $C \in Dec(T_1)$ we find the number of independent players in this coalition and we denote it by $l(C)$. And finally we take $\tau_1 = \left[\frac{t_1}{2}\right]$, $\tau_2 = \left[\frac{t_2}{2}\right]$. In all formulae below we will assume that $\binom{n}{k} = 0$ for n < k. Therefore in a composite game we have:

- the BC index of a member of T_1 is given by:

$$\beta_{T_1}^c(T) = \frac{1}{2^{100-t_2}} \binom{t_1 - 1}{\tau_1} \left(\sum_{s=\max(51-t_1-t_2,0)}^{\min(50-t_2,l)} \binom{l}{s} + \sum_{s=51-t_1}^{\min(l,50)} \binom{l}{s} \right),$$

(we assume that $t_1, t_2 \leq 50$);

- the SS index of a member of T_1 is given by:

$$Sh_{T_1}^c(T) = \overline{P}_1 + \overline{P}_2,$$

where

$$\overline{P}_1 = \frac{1}{100!} \sum_{\substack{C \in Dec(T_1) \\ T_2 \notin C}} \sum_{p_2=0}^{\tau_2} \binom{t_2}{p_2} \binom{t_1 - 1}{\tau_1} \binom{l}{l(C)} (p_2 + l(C) + \tau_1)! (100 - (p_2 + l(C) + \tau_1) - 1)!$$

$$\overline{P}_2 = \frac{1}{100!} \sum_{\substack{C \in Dec(T_1) \\ T_2 \in C}} \sum_{p_2=\tau_2+1}^{t_2} \binom{t_2}{p_2} \binom{t_1 - 1}{\tau_1} \binom{l}{l(C)} (p_2 + l(C) + \tau_1)! (100 - (p_2 + l(C) + \tau_1) - 1)!$$

- the BC index of an independent voter j is equal to:

$$\beta_{j_k}^c(T) = \frac{1}{2^{l+1}} \left(\binom{l-1}{50} + \binom{l-1}{50 - t_1} + \binom{l-1}{50 - t_2} + b \right),$$

where $b = \binom{l-1}{50 - t_1 - t_2}$ if $t_1 + t_2 \leq 50$ and $b = 0$ otherwise;

- the SS index of an independent voter j_k is equal to:

$$Sh_{j_k}^c(T) = \overline{S}_1 + \overline{S}_2 + \overline{S}_3 + \overline{S}_4,$$

where

$\overline{S}_1 = \frac{1}{100!}\binom{l-1}{50}\sum_{p_1=0}^{\tau_1}\sum_{p_2=0}^{\tau_2}\binom{t_1}{p_1}\binom{t_2}{p_2}(50+p_1+p_2)!(100-(50+p_1+p_2)-1)!$, if $l > 50$, otherwise $\overline{S}_1 = 0$,

$\overline{S}_2 = \frac{1}{100!}\binom{l-1}{50-t_1}\sum_{p_1=\tau_1+1}^{t_1}\sum_{p_2=0}^{\tau_2}\binom{t_1}{p_1}\binom{t_2}{p_2}(50-t_1+p_1+p_2)!(100-(50-t_1+p_1+p_2)-1)!$, if $t_2 \neq 50$, otherwise $\overline{S}_2 = 0$,

$\overline{S}_3 = \frac{1}{100!}\binom{l-1}{50-t_2}\sum_{p_1=0}^{\tau_1}\sum_{p_2=\tau_2+1}^{t_2}\binom{t_1}{p_1}\binom{t_2}{p_2}(50-t_2+p_1+p_2)!(100-(50-t_2+p_1+p_2)-1)!$, if $t_1 \neq 50$, otherwise $\overline{S}_3 = 0$,

$\overline{S}_4 = \frac{1}{100!}\binom{l-1}{50-t_1-t_2}\sum_{p_1=\tau_1+1}^{t_1}\sum_{p_2=\tau_2+1}^{t_2}\binom{t_1}{p_1}\binom{t_2}{p_2}(50-t_1-t_2+p_1+p_2)!(100-(50-t_1-t_2+p_1+p_2)-1)!$, if $t_1+t_2 \leq 50$, otherwise $\overline{S}_4 = 0$.

If we consider a game with precoalitions, then Owen modifications of concerned indices are calculated using the following formulae:

- the O^{BC} index of a member of T_1 is equal to:

$$\frac{1}{2^{t_1+l}}\binom{t_1-1}{50-t_2}+\sum_{s=1}^{\min(50,l)}\binom{l}{s}\binom{t_1-1}{50-s}+\sum_{s=1}^{\min(50-t_2,l)}\binom{l}{s}\binom{t_1-1}{50-t_2-s};$$

- the O^{SS} index of a member of T_1 is equal to:

$$O_{T_1}(T) = P_1 + P_2,$$

where

$$P_1 = \sum_{s=51-t_1}^{\min(50,l)}\binom{l}{s}\binom{t_1-1}{50-s}\frac{s!(l+1-s)!(50-s)!(t_1-(50-s)-1)!}{t_1!(l+2)!},$$

if $t_1 + l \geq 51$, otherwise $P_1 = 0$ and

$$P_2 = \sum_{s=\max(51-t_1-t_2,0)}^{50-t_2}\binom{l}{s}\binom{t_1-1}{50-t_2-s}\frac{(1+s)!(l-s)!(50-t_2-s)!(t_1-(50-t_2-s)-1)!}{t_1!(l+2)!}.$$

- the O^{SS} index of an independent voter j_k is equal to:

$$O_{j_k}(T) = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 = \binom{l-1}{50}\frac{50!(l+1-50)!}{(l+2)!}\text{, if } l > 50\text{, otherwise } S_1 = 0$$

$$S_2 = \binom{l-1}{50-t_1}\frac{(50-t_1+1)!(l+1-(50-t_1+1))!}{(l+2)!}\text{, if } t_2 \neq 50\text{, otherwise } S_2 = 0,$$

$$S_3 = \binom{l-1}{50-t_2}\frac{(50-t_2+1)!(l+1-(50-t_2+1))!}{(l+2)!}\text{, if } t_1 \neq 50\text{, otherwise } S_3 = 0,$$

$$S_4 = \binom{l-1}{50-t_1-t_2}\frac{(50-t_1-t_2+2)!(l+1-(50-t_1-t_1+2))!}{(l+2)!}\text{, if } t_1+t_2 \leq 50\text{, otherwise } S_4 = 0.$$

We do not present the formula for coalitional BC index of an independent voter in a game with coalition structure because it is equal to his BC index in a composite game. It follows from the fact, that the internal power of a member of a “singleton party” is equal to 1 so $\beta^c_{j_k}(W,T) = \beta_{j_k}(W_T)$ for an independent voter j_k and it is equal to $O^{BC}_{j_k}(W,T)$ since swings of the player j_k (or a party composed only of the player j_k) are exactly the same in both cases.

Another important remark is that coalitional SS index has a product property which is similar to the property of BC index in a composite game. After simplification of the formula for $O^{SS}_{T_1}(T)$ we obtain that:

$$O^{SS}_{T_1}(T) = \frac{1}{t_1}(\tilde{P}_1 + \tilde{P}_2),$$

where

$$\tilde{P}_1 = \sum_{s=51-t_1}^{\min(50,l)}\binom{l}{s}\frac{s!(l-s+1)!}{(l+2)!}$$

if $t_1 + l \geq 51$, otherwise $\tilde{P}_1 = 0$ and

$$\tilde{P}_2 = \sum_{s=\max(51-t_1-t_2,0)}^{50-t_2}\binom{l}{s}\frac{(1+s)!(l-s)!}{(l+2)!}.$$

This new formula has an interesting interpretation – it is the product of the SS index of a member of a party T_1 in a (arbitrary) majority voting game inside this party and the SS index of this party in a top game among parties, which is the weighted majority voting game with the quota 51.

Composite game – presentation of results

We begin the analysis of our simulations with the case of composite game. First we consider the power of the first party members. Figures 1 and 2 present the power of members of the first party as the function of the size of the second party.

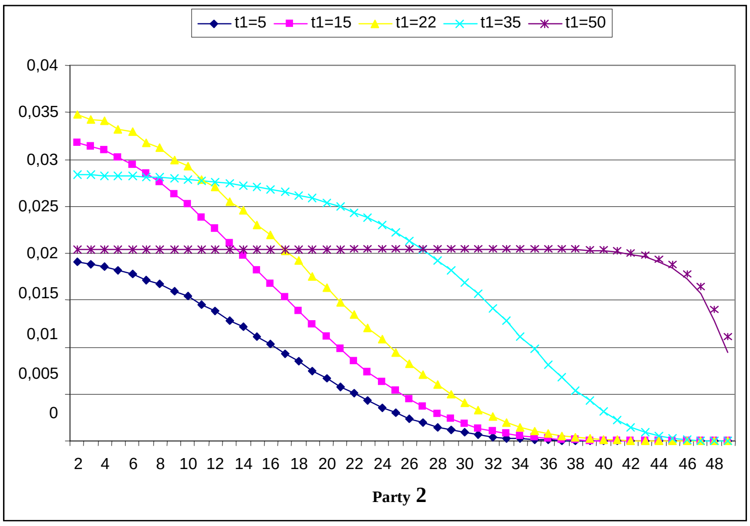

Fig.1 SS index of the member of T_1 (as a function of t_2) for various configurations of sizes of both parties

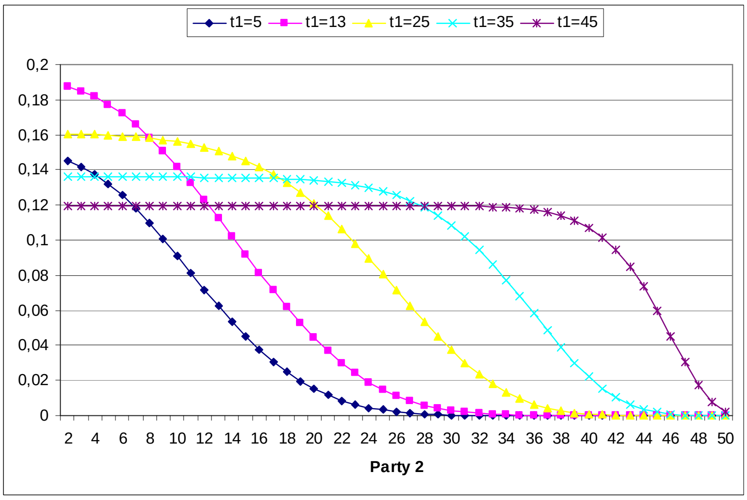

Fig.2 BC of the member of T_1 (as a function of t_2) for various configurations of sizes of both parties

What we can observe from those charts is that values of both indices – BC and SS in a composite game – decrease monotonically with the increasing size of the second party. It means that the power (measured by BC or SS) of the first party's member falls as the size of the opponent grows. We cannot of course compare values of those indices since one of them is normalized and the other – not, but the shape of curves illustrating changes of values of both indices is very similar. The global maximum of the value of SS index in a composite game (for the member of the party T_1) is attained in the configuration $t_1 = 22$, $t_2 = 2$, while the global maximum of the value of the BC index in a composite game is achieved for $t_1 = 13$,

$t_2 = 2$. If the size of the first party increases, the initial value of both indices grows up until t_1 becomes equal to 22 or 13 respectively and then the initial point is coming down. For large values of t_1 the power of the member of the first party measured by both indices is almost constant as a function of t_2 – it decreases slightly only for large, almost maximal, sizes of the second party.

We found it interesting to check in what configurations the power of a member of the first party is maximal while the size of the second party is fixed. Both indices have very similar properties also in this case. The point at which the maximal value of the SS index and BC index in a composite game is attained depends on the fixed size of the second party. For $t_2 = 2$ the maximal power of the member of the first party measured by the SS index is attained for $t_1 = 22$ and maximal power measured by BC index is achieved for $t_1 = 13$. If we increase the fixed t_2 , then the value of t_1 at which the maximum of each index is achieved also increases. For $t_2 > 43$ maximum of power of the party T_1 members is achieved in the situation where the first party is of maximal size for both SS and BC index.

In the sequel we present charts illustrating the behavior of SS and BC indices treated as functions of the own party's size for fixed values of t_2.

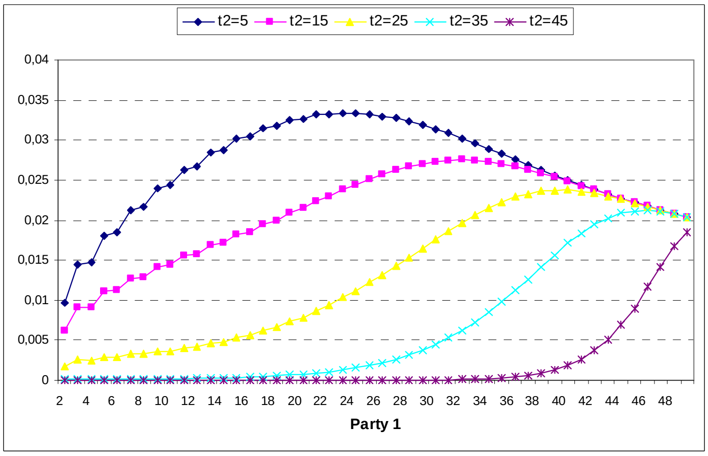

Fig. 3 SS index of the member of T_1 (as a function of t_1) for various configurations of sizes of both parties

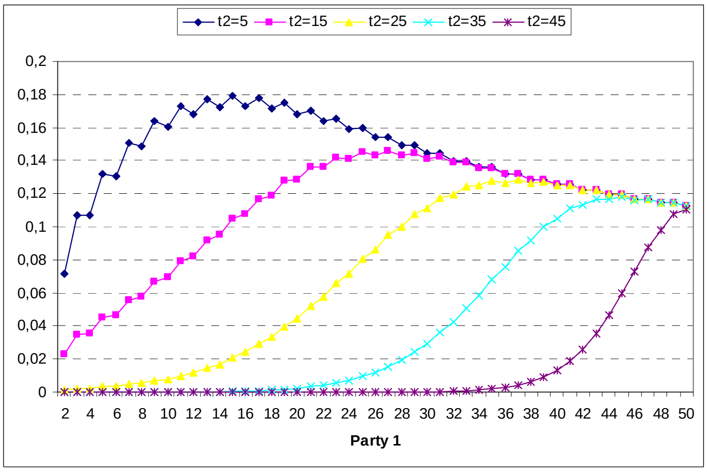

Fig. 4 BC index of the member of T_1 (as a function of t_1) for various configurations of sizes of both parties

The most striking observation is that the power of the member of T_1 measured by both indices – SS and BC – in a composite game is not an increasing function of the own party's size for most values of t_2. For a fixed size of the opponent, the power of the member of the first party increases, attains the maximum and then decreases with an increasing size of the own party. Only for large sizes of the opponent party, the power is an increasing function of the own size. Again the shapes of curves is very similar for both indices. The phenomenon of non-monotonicity of BC index of a party member treated as a function of the own party's size in composite games was also examined in the paper of Leech [9], where it was interpreted as a tradeoff between the increasing power of a bloc (party T_1) and decreasing power of a party member.

Up to this point we have only considered the migration from one party to the set of independent voters – we have fixed the size of one party and increased the size of another party. Now we take a look to the behavior of both indices when the migration appears between parties – we assume that members of the second party are joining the first party. Next figures illustrate the influence of this kind of changes in the configuration of sizes on both indices.

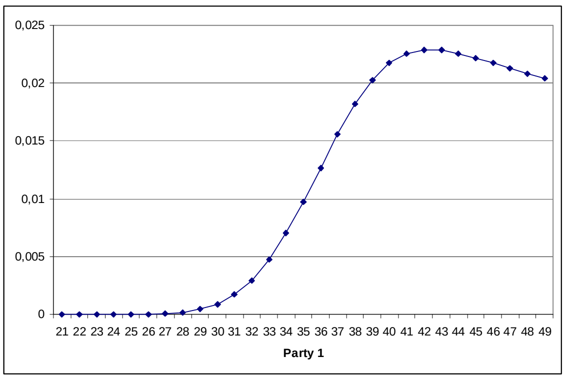

Fig. 5 SS index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 30)

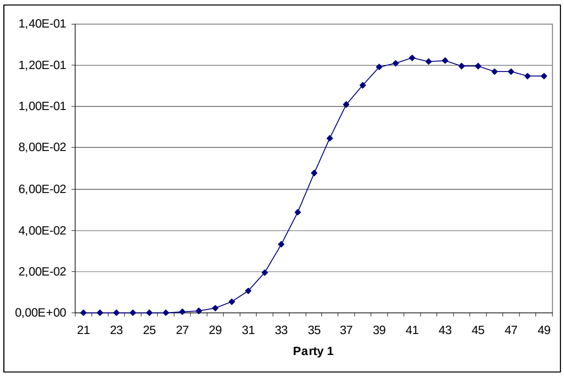

Fig. 6 BC index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 30)

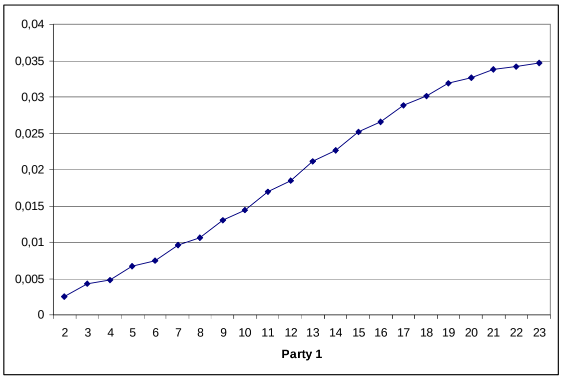

Fig. 7 SS index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 75)

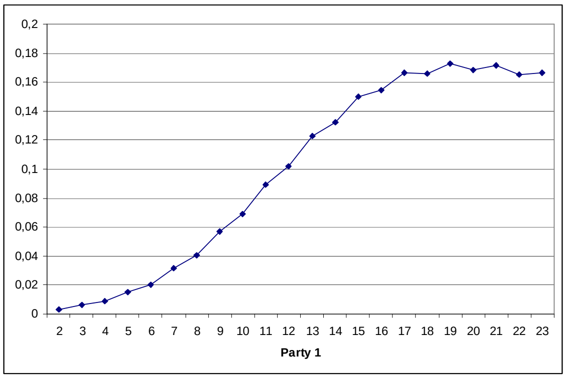

Fig. 8 BC index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 75)

Figures 5 – 8 show that SS index and BC index of a first party member is (in most cases) not monotonic with respect to the own party's size in the situation where members of the opponent party are joining T_1. The power of a voter in T_1 measured by BC index grows up, attains its maximum and then decreases, while if we use SS index the situation is different only for large values of l - then the power of T_1 members is an increasing function of t_1.

If we compare the power of individual voters in a composite game we also obtain similar results for the SS and BC indices. Below we present some figures showing the value of each of two considered indices in a composite game for various configurations of sizes of both parties.

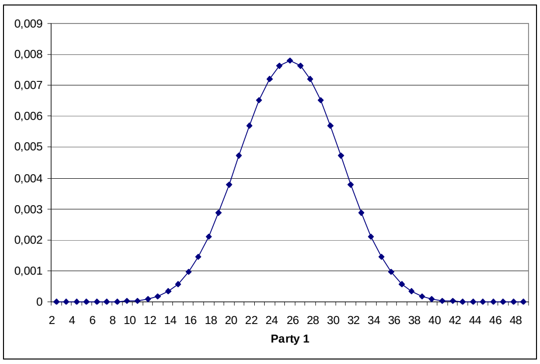

Fig. 9 The SS index of an independent voter as a function of the size of T_1 for l =50

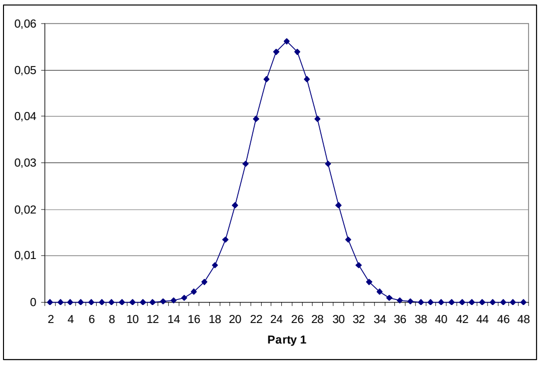

Fig. 10 The BC index of an independent voter as a function of the size of T_1 for l =50

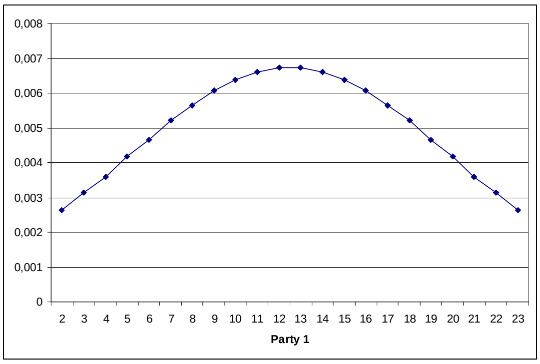

Fig. 11 The SS index of an independent voter as a function of the size of T_1 for l =75

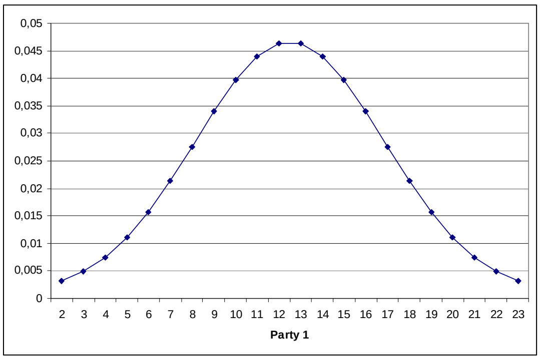

Fig. 12 The BC index of an independent voter as a function of the size of T_1 for l =75

We treat the power of an independent voter as a function of the size of the first party with the number of all independent voters fixed (if we fix l, then choosing t_1 we determine also t_2). Figures 9 – 12 reflect the fact that the SS index and BC index of an independent voter in a composite game is symmetric with respect to t_1 and t_2. This fact is obvious, because if the size of one party grows up then the size of the second one falls down (the sum $t_1 + t_2$ is fixed). What is more interesting is that maximal power of an independent voter measured in both ways is achieved in the situation where both parties are of the same size, or the difference between their sizes is equal to 1 – in this case we have two points with the same maximal value of power. The global maximum of the SS index and BC index of an independent voter is attained when there is only one such voter.

Game with precoalitions – presentation of results

In games with a coalition structure the behavior of coalitional SS index and coalitional BC index is different from the case of composite games. Moreover both indices differ in their behavior much more than in composite games. We will present the results in the same order as in the previous section.

First we consider the behavior of both indices for members of the party T_1, assuming that the size of the own party is fixed (therefore we treat the power of first party members as the function of the size of the second party).

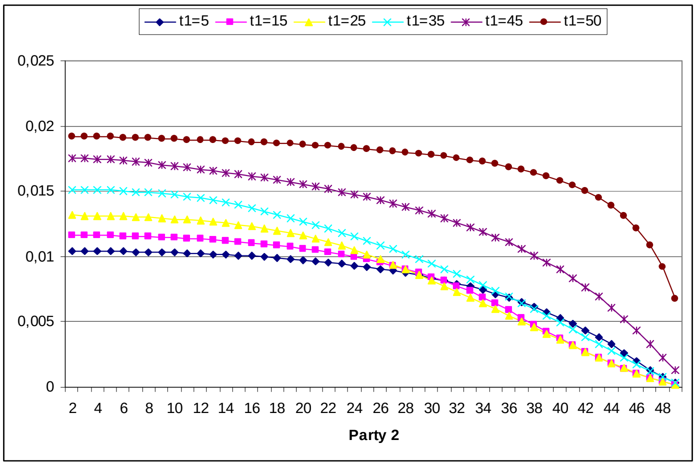

Fig. 13 SS index of a member of T_1 (as a function of t_2) for various configurations of sizes of both parties

Coalitional SS index of a voter from the party T_1 is a decreasing function of t_2 (for all values of t_1) and attains its maximum in the situation where this party has the maximal possible number of members equal to 50 and the opponent has minimal possible number of members equal to 2. Moreover – if we fix the number t_2 , then the maximal value of the coalitional SS index of a voter from the party T_1 is achieved in the situation where the size of the party T_1 is maximal (equal to 50), which means that it does not depend on (fixed) t_2, which was the case in composite games.

The situation appears to be quite different if we consider the coalitional BC index. First note that if we compare the situation where some of the sets T_i are singletons with the situation where singletons join together and form a new party, then the value of the coalitional BC index for members of a new party is the same as it was in the previous partition. Formally, suppose that the partition T is of the form $T = (\{j_1\},...,\{j_k\}, T_{k+1},...,T_m)$, where $\#T_i \geq 2$ for $i = k+1,...,m$ and the new partition is given by $\tilde{T} = (\tilde{T}_1, \tilde{T}_2,...,\tilde{T}_{m-k+1})$, where $\tilde{T}_1 = \{j_1,..., j_k\}$ and $\tilde{T}_i = T_{i+k-1}$ for $i = 2,..., m - k + 1$. Then $O^{BC}_{j_l}(W,T) = O^{BC}_{\tilde{T}_1}(W,\tilde{T})$ for any $l = 1,..., k$.

This equality follows from the observation that when we compute the value of the coalitional BC index in both case swings of players in singletons are the same as swings of players in the new party $\tilde{T}_1$. In the first case the number of swings is divided by $2^{m+1-2} = 2^{m-1}$ because there are m parties and the cardinality of the singleton is 1. In the second case we divide the number of swings by $2^{m-k+1+k-2} = 2^{m-1}$ since the number of parties is equal to $m - k + 1$ and the cardinality of the new party $\tilde{T}_1$ is equal to k.

It means that in our case the value of the coalitional BC index of a voter from the party T_1 depends actually only on the size of the party T_2 and does not depend on the size of the own party (in other words when calculating the power of the first party members using the coalitional BC index we consider the situation where there is only one party of the size t_2 and all remaining voters form singletons). The behavior of this index is shown at the Figure 14.

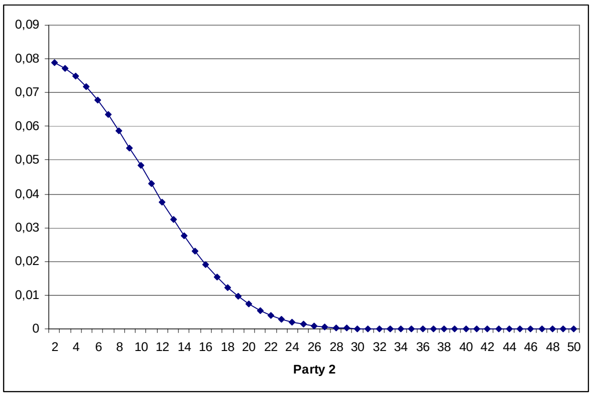

Fig. 14 The coalitional BC index of a member of T_1 (as a function of t_2)

The coalitional BC index of a member of the first party is then the decreasing function of t_2 and it achieves its maximal value in the situation where $t_2 = 2$ (and t_1 is arbitrary). The shape of this curve is rather similar to the shape of curves in case of composite game (and different from the shape of curves illustrating the behavior of coalitional SS index).

If we want to examine the behavior of both indices regarding their dependence on the size of the own party (with t_2 fixed), then the picture is as it can be seen at Figures 15 and 16.

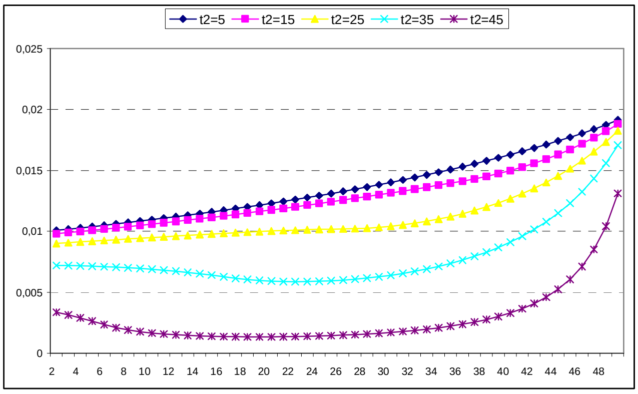

Fig. 15 Coalitional SS index of a member of T_1 (as a function of t_1) for various configurations of sizes of both parties

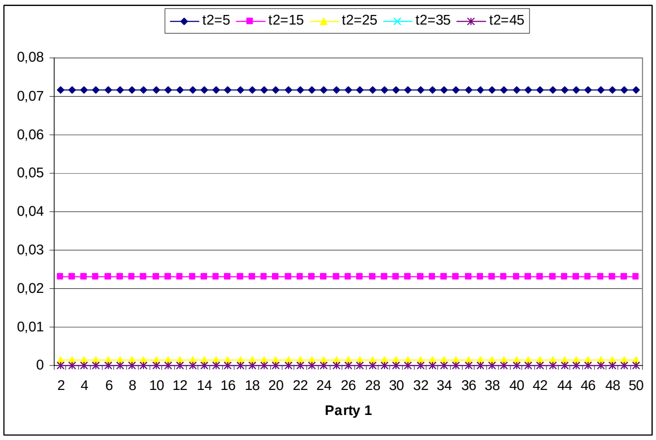

Fig. 16 Coalitional BC index of a member of T_1 (as a function of t_1) for various configurations of sizes of both parties

In case of coalitional SS index we observe that the power of the member of the first party is (almost) monotonic function of the own party's size. For large sizes of the opponent we notice a slight decrease of the power of a member of T_1, but then the power increases monotonically and achieves maximum always for $t_1 = 50$. Besides, we observe that as the opponent party's size increases, corresponding curves are coming down. In case of the coalitional BC index we have horizontal lines, since the power of a voter does not depend on the own party's size, but also lines corresponding to smaller values of t_2 are placed higher.

When considering the migration from the party T_2 to the party T_1 we have the monotonicity result – the larger is the own party's size (and in the mean time the smaller is the size of the opponent), the greater is the power of the first party member measured by both BC and SS coalitional indices. The Figures 17-20 show this result.

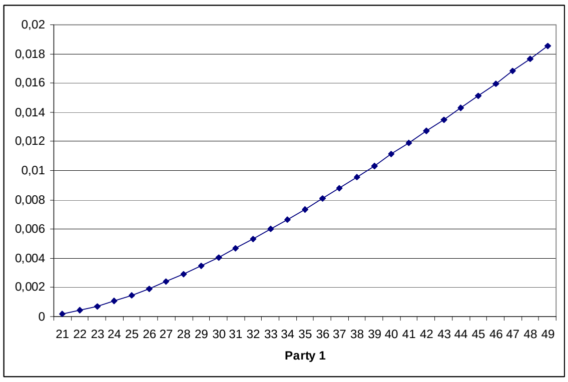

Fig. 17 Coalitional SS index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 30)

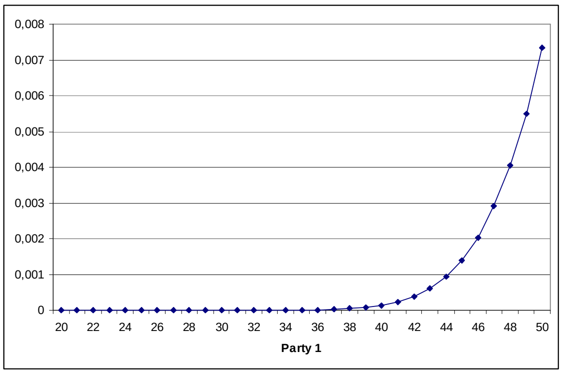

Fig. 18 Coalitional BC index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 30)

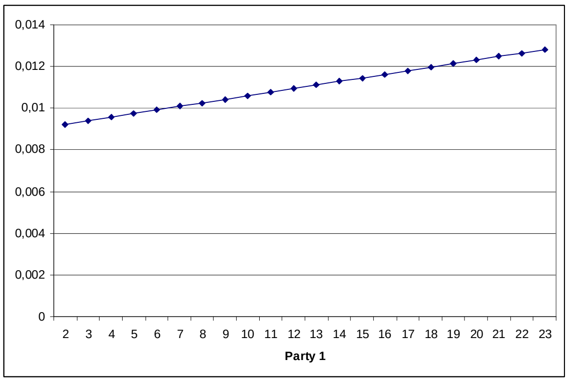

Fig. 19 Coalitional SS index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 75)

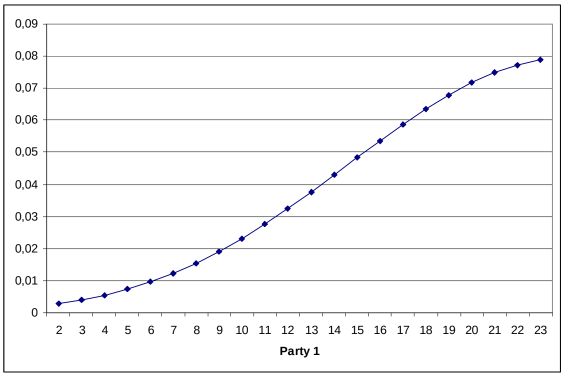

Fig. 20 Coalitional BC index of the member of T_1 (as a function of t_1) - the migration from T_2 to T_1 (l = 75)

And finally we come to the results concerning individual voters, which are similar to the case of composite game. The power of individual voter, measured by coalitional SS index is a symmetric function of the size of one party (keeping the number of individual voters constant) and attains its maximum in the situation where both parties are of the same size (or their sizes differ by 1 member). An example of the behavior of the power of an individual voter measured by coalitional SS index is shown at the Figure 21. We do not consider here the coalitional BC index of an individual voter since we argued that it is the same that his BC index in the composite game.

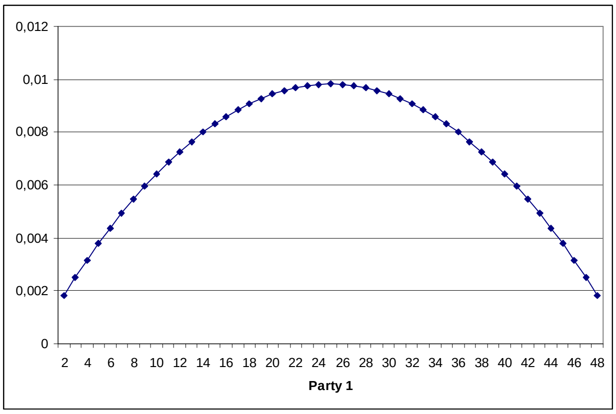

Fig. 21 Coalitional SS index of an independent voter as a function of the size T_1 for l =50

Comparison of the same indices in different games

We though that it could also be interesting to compare the behavior of the SS index and BC index of a voter in two different games (which means using two alternative ways of measuring the power of a voter in a voting body divided into parties). First we show some results concerning the SS index. We compare the range and shape of curves corresponding to the power of party members treated as a function of the size of the opponent.

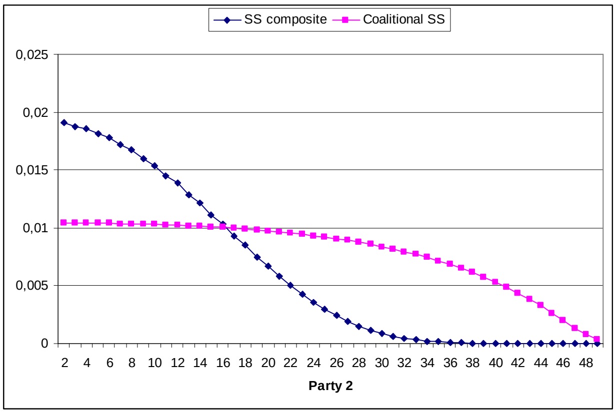

Fig. 22 The power (SS) of the first party's member as a function of t_2 with t_1 =5

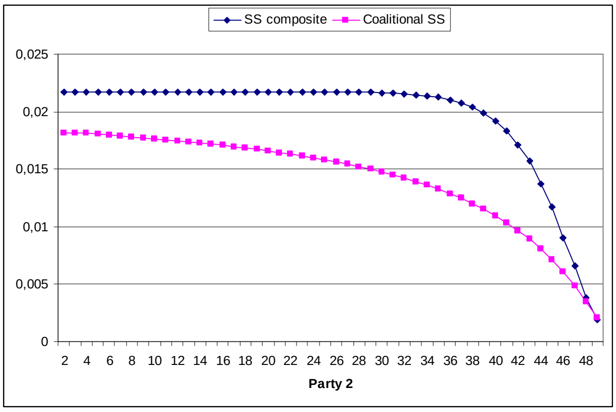

Fig. 23 The power (SS) of the first party's member as a function of t_2 with t_1 =46

We notice that considered indices behave in different way. The range of the coalitional SS index is less than the range of the SS index in a composite game. Coalitional SS index is almost constant for small sizes of the opponent party and then it decreases rather slowly. SS index in a composite game is considerably greater than coalitional SS index for small sizes of the opponent party (for t_1 greater than 46 SS index in a composite game is greater than coalitional SS index for all possible sizes of the second party). For small t_1 the index Sh^c decreases quickly with the increase of the size of the second party, it achieves the level of the O^{SS} index, then it has an inflection point and it decreases slowly to the values close to zero. For larger t_1 the behavior of the Sh^c index is different. For small sizes of the opponent party it is almost constant and starts to decrease as the size of the second party is quite large. The point of intersection with the O^{SS} index curve moves to the right (to the larger sizes of the second party) with the increasing size of the first party and eventually Sh^c index is larger than O^{SS} index for all possible values of t_2 .

What is the picture if we compare the behavior of BC index in two different frameworks? It appears that the conclusions are different.

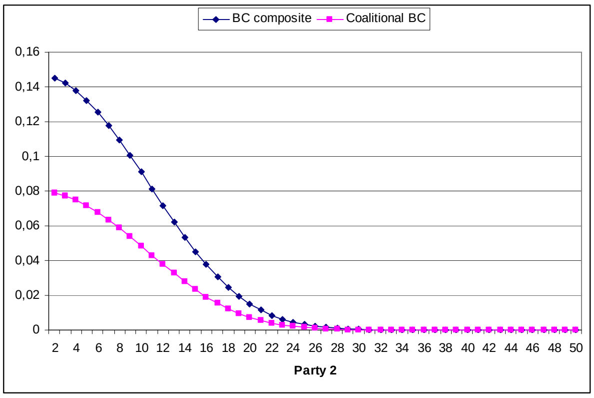

Fig. 24 The power (BC) of the first party's member as a function of t_2 with t_1 = 5

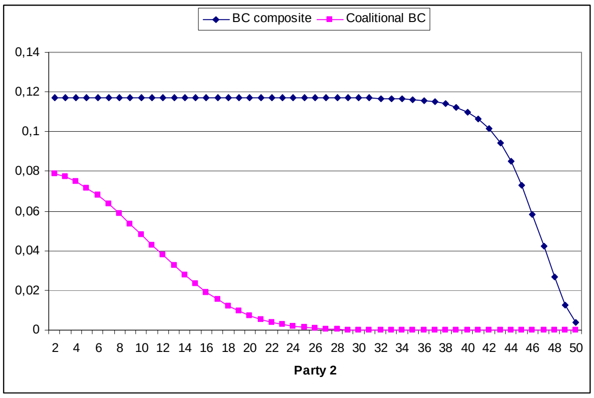

Fig. 25 The power (BC) of the first party's member as a function of t_2 with t_1 = 46

Again the range of the β^c index is greater than the range of the O^{BC} index, but here the value of β^c index is (almost) always greater than the value of the O^{BC} index (the equality occurs in case where t_1 = 2).

If we measure the power of an individual voter by means of the SS index in two different approaches (composite game versus game with a coalition structure), then in turns out that the power of an individual voter is always greater in game with precoalitions than in the composite game (Fig. 26, 27).

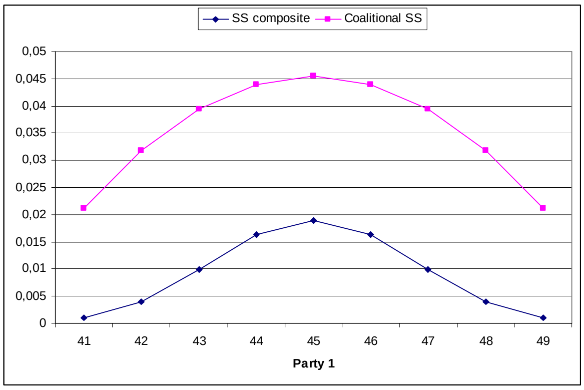

Fig. 26 The power (SS) of an independent voter as a function of the size T_1 for l =10

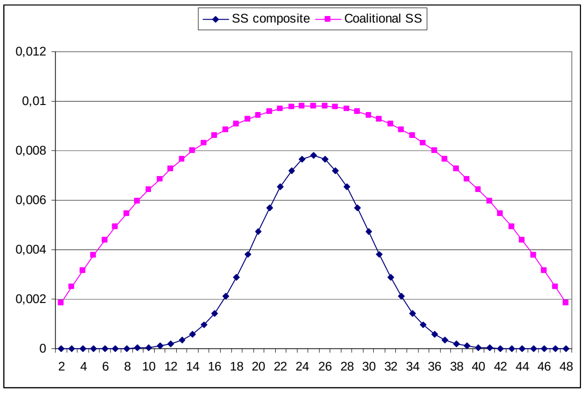

Fig. 27 The power (SS) of an independent voter as a function of the size T_1 for l = 50

In case of BC index we do not have such conclusion, because the value of BC index of an individual voter is the same in both games. What we can conclude is that the relative BC power if an independent voter in a composite game is less than in the game with precoalitions since the BC power of party members in the composite game is greater than in game with precoalitions.

Note that if we do not consider coalition structure in the voting body, then all voters have the same voting power (considering any majority voting rule and any symmetric index). In case of SS index the power of each individual voter in the concerned voting body is equal to 0,01. We can ask the following questions: when the party membership increases the power of a voter or for which coalition structures the power of an independent voter is greater than in the situation when the coalition structure does not exists. The answer to those questions for both indices - Sh^c and O^{SS} - are given in Figures 28-31. At each figure there are shown values of respective index for a member of T_1 or for an independent voter at all possible configurations of sizes of both parties (rows correspond to the size of the first party, and columns – to the size of the second party; the left upper corner corresponds to the case $t_1 = t_2 = 2$ while the right bottom corner describes the case $t_1 = t_2 = 50$). The cells are shaded if the value of respective index is greater than 0,01.

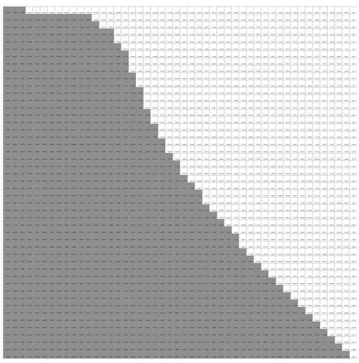

Fig. 28 SS index in a composite game of a member of T_1

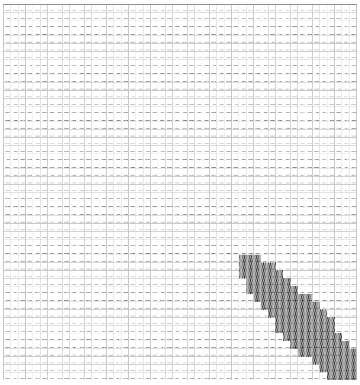

Fig. 29 SS index in a composite game of an independent voter

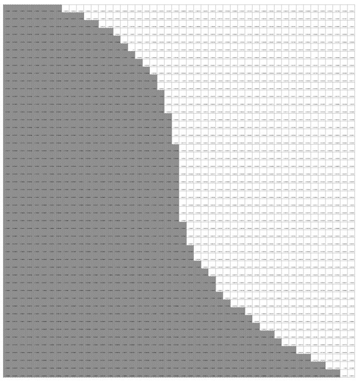

Fig. 30 Coalitional SS index of a member of T_1

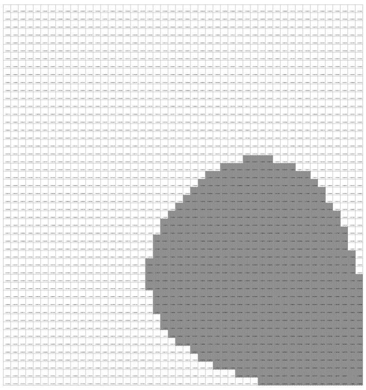

Fig. 31 Coalitional SS index of an independent voter

Looking at those pictures we conclude that for both indices taking into consideration the coalition structure in most cases increases the power of a party member (comparing to the case without any a priori coalition structure). If we take the index Sh^c, then for small t_1 the power of a party member is less than 0,01 in case where t_1 is substantially less than t_2. For larger values of t_1 the power of T_1's member becomes less than 0,01 if the size of the second party is greater than the size the first one. Independent voters are better off when considering the party structure only in cases where both parties are approximately of the same size and both are rather large.

The last observation is such that the coalitional SS index in general promotes independent voters while the SS index in a composite game gives more power to the party members. We do not perform such analysis for BC index because we consider absolute BC index, which is not normalized.

Concluding remarks

The comparison of both indices in the considered case of a voting body leads to some conclusions concerning the properties of both methods of measuring the voting power of individuals in a voting body with a coalition structure. First of all we observe that both indices in a composite game are more sensitive to the changes of coalition structure and have larger range of values than their counterparts in a game with the coalition structure. On the other hand – in games with precoalitions Owen modifications of SS index and BC index are in most cases monotone with respect to the size of ones own party and the size of the opponent. If the size of the opponent is arbitrarily fixed, then maximal power is always achieved while own party's size is maximal (= 50); if ones own party has an arbitrarily settled size, then the power – measured by both indices - of its member is a decreasing function of the opponent's size. Moreover - the larger is own party's size, the larger is maximal possible power of its member, which means, that the global maximum is attained while the own party has 50 members and the opponent has two members. In a composite game indices Sh^c and β^c do not reveal such monotonicity. While the own party's size is fixed, the power of its member is also a decreasing function of the opponent's size. However, with the arbitrarily fixed size of the opponent party, the maximum of power depends on the size of the opponent. The global maximum is achieved in the situation where the own party has 22 members and the opponent - 2 members in case of the Sh^c index and for $t_1 = 13$ and $t_2 = 2$ in case of the β^c index.

Independent voters are better off if we measure their power in game with precoalitions than they are in a composite game.

The conclusion which raises after the analysis of our simulations is that the behavior of SS index and BC index depends much more on the structure of the game considered than on the index itself which implies the fact that the behavior of SS index in a composite game is much more similar to the behavior of the BC index in that game than to the behavior coalitional SS index in a game with precoalitions.

Another issue is the interpretation and – in consequence – the choice of one of described here measures of power (and a proper model) for applications. A criterion which could be helpful is the discipline of voting in parties. If there is a party whip, then the model of composite game should be applied (especially in case of BC index). Notice that the obvious interpretation of the index β^c is that we deal with a situation where all members of each party follow the discipline and vote according to the decision made by internal voting. In case of the β^c index the power of a voter decomposes into two factors – one is the individual power in the internal voting and second is the power of a party as a whole. Relations between these two factors where examined in [9]. On the other hand we can interpret the index O^{BC} as a measure of power of a member of a party where there is no party whip, assuming that in all other parties voters follow the party discipline. We obtained in our simulations an interesting (and not very surprising) result that it is always better for members of the disciplined party when their opponents do not have a party whip. It is worth noting at this point that the choice of the voting model may depend on the subject voted, because discipline of voting inside a given party is usually demanded or not subject to the topic under consideration.

From the numerical point of view the calculation of the BC index is much simpler than of SS index especially in case of composite game. On the other hand the fact that the coalitional SS index can be decomposed into two factors provides quite easy way of obtaining numerical results. We restricted our research to the case of two parties because it allowed for an illustrative presentation of results. Obviously, the methodology presented here can be applied to examine the power of members of actual voting bodies with an arbitrary structure of parties.

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