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One-way monotonicity as a form of strategy-proofness

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Abstract Suppose that a vote consists of a linear ranking of alternatives, and that in a certain profile some single *pivotal voter* v is able to change the outcome of an election from s alone to t alone, by changing her vote from P_v to P'_v. A voting rule $\mathcal{F}$ is *two-way monotonic* if such an effect is only possible when v moves t from below s (according to P_v) to above s (according to P'_v). *One-way monotonicity* is the strictly weaker requirement forbidding this effect when v makes the opposite switch, by moving s from below t to above t. Two-way monotonicity is very strong—equivalent over any domain to *strategy proofness*. One-way monotonicity holds for all *sensible* voting rules, a broad class including the scoring rules, but no Condorcet extension for four or more alternatives is one-way monotonic. These monotonicities have interpretations in terms of strategy-proofness. For a one-way monotonic rule $\mathcal{F}$, each manipulation is paired with a *positive response*, in which $\mathcal{F}$ offers the pivotal voter a strictly better result when she votes sincerely.

Keywords One-way monotonicity · Monotonicity · Participation · No-show paradox · Strategy-proofness · Manipulation · Scoring rule · Sensible virtue · Condorcet extension

JEL Classification D71

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1 Introduction

A real-valued function f is said to be *monotonically increasing* if whenever $x_1 < x_2$ we have $f(x_1) < f(x_2)$ (for *strictly increasing*) or $f(x_1) \leq f(x_2)$ (for *weakly increasing*). Monotonicity properties for voting rules are loosely based on this idea; we say that such a rule $\mathcal{F}$ is monotonic if whenever one or more voters change their votes in a certain "direction," the effect is to move the outcome of the election in a similar direction. A vote, in our context, consists of a strict preference ranking—a linear ordering without ties—of all alternatives, so it is not completely clear what "direction" means; the variety of possible interpretations leaves room for a number of different monotonicity properties. This is the case even for *resolute* rules, which yield a unique winning alternative for each profile. For *irresolute* rules, in which several alternatives may be tied as winners, the possibilities ramify further. In the current paper, we confine ourselves to resolute rules, but take up monotonicity in the irresolute context in the sequel, Sanver and Zwicker (2009).

The monotonicity properties considered in the voting literature are typically of the weakly increasing, or $\leq$ type, because it is unreasonable to expect the outcome of an election to change each time a few voters change their votes—the winner's margin of victory may be too large to be easily overcome. Among these properties, two major classes stand out.

The first class contains monotonicities having a normative appeal independent of any strategic concern. *Simple monotonicity,* which has most often been called *monotonicity*, is certainly the best-known representative of this class. Loosely, it asserts that raising a single alternative s in a voter's preferences (while leaving the ranking otherwise unchanged) is never detrimental to s's prospects for winning.[1] Most voting rules considered in the literature satisfy simple monotonicity, but the following closely related rules are exceptions: all scoring elimination rules (Smith 1973), including Hare (or "alternative vote"), and plurality run-off. Fishburn (1982) discusses other examples. Simple monotonicity is a rather "weak" monotonicity but it does discriminate, albeit to a limited extent, among reasonable voting rules.

This is in contrast to the second class of monotonicities, wherein the normative appeal rests on strategic considerations. These properties arose as a consequence of explorations of strategy-proofness and implementation. For example, Muller and Satterthwaite (1977) prove that for social choice functions that do not admit ties and that are defined over the full domain of preference profiles, strategy-proofness is equivalent to *strong positive association*—a monotonicity condition which Maskin (1977, 1999) showed to be necessary (but not sufficient) for Nash implementability. On other hand, Nash implementability is equivalent to Danilov (1992) monotonicity which, although generally stronger, is equivalent to Maskin monotonicity for social choice rules that do not admit ties and that are defined over the full domain. In this case, a failure of the condition can typically be identified with a situation in which one voter can manipulate the outcome so as to obtain a preferred outcome by misrepresenting

[1] This may be the oldest known monotonicity property. Other names have also been used over its relatively long history, which predates the modern resurrection of social choice theory in Black (1958). See Brams and Fishburn (2002) and comments on page 120 of Fishburn (1982), including footnote 1.

her preference. On other hand, we know from Gibbard (1973) and Satterthwaite (1975) that over the full domain of preference profiles, a strategy-proof social choice function whose range contains at least three alternatives is dictatorial. Thus monotonicity properties in this second class are so strong that they hold for no reasonable (resolute) voting rule. While their theoretical importance is significant, they are less useful as a basis for comparing realistic voting systems in terms of manipulability.[2]

Our investigations arise from the following question. Suppose that when the vote of some particular voter v is the ranking P_v, alternative s is the sole winner of a certain election, but that when v votes instead for the ranking $P'_{v,}$ while all other votes remain unchanged, some different alternative t is the sole winner. Given such a *pivot*, what should "monotonicity" require, in terms of how P_v and P'_v rank s and t, relative to each other?

We introduce here two new monotonicity properties, based on answers to this question. The stronger property, *two-way monotonicity*, requires that the voter v described above must have lifted t from below s in P_v, to above s in P'_v, and falls squarely into the second class, as it is equivalent to strategy-proofness. This property is not of independent interest, but it helps frame the idea for *one-way monotonicity*, its weaker cousin, which requires that v must not have lifted s from below t in P_v, to above t in P'_v.

We may, if we wish, impose one of two possible interpretations on our voter, by identifying one of the rankings with her sincere preferences and the other with an attempt at manipulation:

Interpretation 1 P_v represents v's sincere ranking and P'_v represents an attempt at manipulation

Interpretation 2 P'_v represents v's sincere ranking and P_v represents an attempt at manipulation

Symmetry suggests that we consider both interpretations. Two-way monotonicity is equivalent to the assertion that neither identification ever represents a successful manipulation. One-way monotonicity asserts that whenever one identification represents a successful manipulation, the other represents a failure. But in such a "failure", voter v does strictly better by casting the sincere ballot than by casting the insincere one. Arguably, each such *positive response* represents a disincentive to any attempt at manipulating the social choice rule; for example, we may imagine that the voter does not know the rest of the profile with complete certainty, and is leery of outsmarting herself.

This line of reasoning leads us to the following interpretation of one-way monotonicity for a voting rule $\mathcal{F}$: every example of a manipulation of $\mathcal{F}$ is also an example of a positive response when interpreted in the "opposite order." In this sense, for a one-way monotonic rule any instance of manipulability can be seen as part of the cost of doing business—a payment made in order to respond appropriately to the will of the electorate.

[2] However, it is possible to compare rules in terms of their frequency or probability of vulnerability to manipulation; see Aleskerov and Kurbanov (1999); Smith (1999); Favardin et al. (2002) and Favardin and Lepelley (2006).

One-way monotonicity is of “medium” strength, in that it is satisfied by a number of natural voting rules, yet fails of a number of others, thus discriminating usefully among standard voting rules. We argue that it partakes of some traits from both classes of monotonicities.

The rest of the paper is organized as follows. In Sect. 2, we set the context and present necessary background material. The *sensible* voting rules we introduce in Sect. 3 include all scoring rules and more, and are one-way monotonic. Not every one-way monotonic rule is sensible, however. In Sect. 4, we turn to the *no-show paradox* of Brams and Fishburn (1983), wherein a voter may obtain a preferred outcome by staying home rather than voting. Moulin (1988a,b) shows that the *participation* axiom, which asserts that no no-show paradoxes occur, is satisfied by no Condorcet extension. Campbell and Kelly (2002) provide several rules that satisfy participation but not simple monotonicity; we recycle one of their examples in Sect. 3. We establish that participation implies half-way monotonicity, a weak form of one-way monotonicity, and that the converse holds for voting rules satisfying *homogeneity* and *reversal cancellation*. As stand-alone properties one-way monotonicity and participation are independent, but we show some logical connections for the special case of three or four alternatives.

In Sect. 5 we present our main negative result. By exploiting the parallels between participation and one-way monotonicity, and using some of the ideas from Sect. 4, we are able to elaborate on Moulin's argument and show that no Condorcet extension is one-way monotonic. Hare's rule also fails one-way monotonicity, as does the closely related plurality run-off rule, albeit with a small qualification. In the concluding Sect. 6 we point to future areas of research, including that of further clarifying the relationship among the various monotonicity properties in the context of irresolute voting rules that are both neutral and anonymous. This issue bears directly on the methodology we use throughout the paper, of rendering all voting rules resolute by employing a fixed tie-breaking agenda.

2 Basic notions

Let $N = \{i, j, \ldots\}$ be a finite set of *n voters* and $A = \{s, t, \ldots\}$ be a finite set of $m \geq 3$ *alternatives*. A *profile* $P = \{\mathrm{P}_i\}_{i \in N}$ for N consists of an assignment, to each $i \in N$, of a strict linear ordering P_i of A; $t P_i s$ indicates that a voter i strictly prefers t to s.

A *social choice rule* is a mapping $\mathcal{F}$ that returns, for each profile P for N, a non-empty set of alternatives $\mathcal{F}(P) \subseteq A$. A *variable-electorate social choice rule* is one that is defined for every finite set N of voters.[3] If $\#\mathcal{F}(P) = 1$ for every profile P, $\mathcal{F}$ is a *resolute* social choice rule (of either variety), or a *social choice function (SCF)*, and we write $\mathcal{F}(P) = s$ in place of $\mathcal{F}(P) = \{s\}$.

[3] Any property for SCFs (such as one-way monotonicity) may also be thought of as a property for variable electorate SCFs, simply by asserting the property for all choices of N. However, properties that entail adding or removing voters (such as *participation*, discussed later) make sense only in the variable electorate context. In §6 we discuss a recent result by Doğan and Giritligil that argues for the fixed electorate context in studying one-way monotonicity.

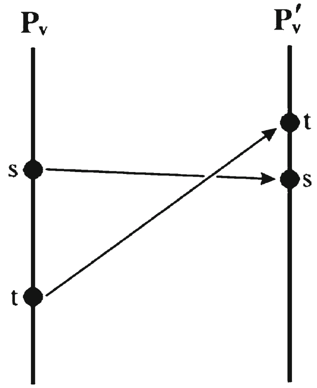

Fig. 1 Two-way monotonicity. If a change in vote from P_v to P'_v switches the winner of the election from s to t, then the pivotal voter must have raised t from below s according to P_v, to above s according to P'_v

Given any profile P, and alternatives s and x, let $Net_P\ (s > x)$ denote the *net pairwise majority* of s over x: the number of voters who rank s over x minus the number of voters who rank x over s. An alternative s is the *Condorcet winner* at the profile P if $Net_P(s > x) > 0$ for each alternative $x \neq s$. The *Copeland score* of an alternative s is the number of alternatives x satisfying $Net_P(s > x) > 0$ while the *Simpson score* of an alternative s for profile P is $\boldsymbol{t}_P{}^*(\boldsymbol{s}) = \text{Min}\{\text{Net}_P\ (\text{s} > \text{x}) | \text{x} \neq \text{s}\}$. The *Copeland rule* and the *Simpson rule* are the social choice rules that select all alternatives with maximal Copeland and Simpson scores, respectively. Note that both rules are *Condorcet extensions*—they select the (unique) Condorcet winner whenever it exists.

Let $v \in N$, Q be a profile for the set $N - \{v\}$, and P_v be any strict ranking of A. Then $Q \wedge P_v$ denotes the profile for N obtained from Q by adding v's vote for P_v: for each $i \in N$

$$(Q \wedge P_v)_i = \begin{cases} Q_i, & \text{if } i \neq v \\ P_v, & \text{if } i = v \end{cases}$$

A *pre-focus* represents some *focal voter*'s choice between two options; formally, it is a vector (Q, v, P_v, P'_v) in which $v \in N$ is a voter, Q is a profile for $N - \{v\}$, and P_v and P'_v are strict rankings of A. A *focus* for a SCF $\mathcal{F}$ records the consequences of such a choice; it is a vector $(Q, v, P_v \to s, P'_v \to t)$ in which (Q, v, P_v, P'_v) is a pre-focus, $\mathcal{F}(Q \wedge P_v) = s$, and $\mathcal{F}(Q \wedge P'_v) = t$. A focus for which $s \neq t$ is a *pivot*; v's choice affects the winner. A social choice function $\mathcal{F}$ is *two-way monotonic* if every pivot $(Q, v, P_v \to s, P'_v \to t)$ for $\mathcal{F}$ satisfies sP_vt and tP'_vs. It is straightforward to see that two-way monotonicity is equivalent to the standard definition of strategy-proofness: a SCF is *strategy-proof* if sP_vt holds for every pivot $(Q, v, P_v \to s, P'_v \to t)$ for $\mathcal{F}$. It would be less strong to require that whenever the change by v switches the election winner from s to t, v must not have dropped t from above s according to P_v, to below s according to P'_v (see Figs. 1 and 2). Thus, we say that a SCF $\mathcal{F}$ is *one-way monotonic* if every pivot $(Q, v, P_v \to s, P'_v \to t)$ for $\mathcal{F}$ satisfies sP_vt or tP'_vs.

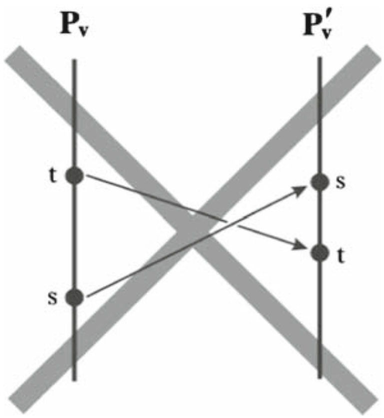

Fig. 2 One-way monotonicity. If a change in vote from P_v to P_v' switches the winner of the election from s to t, then the pivotal voter must not have raised s from below t according to P_v, to above t according to P_v'

We now compare two well-known monotonicities of the literature to our new one. A SCF $\mathcal{F}$ is *Maskin monotonic* if for every pivot $(Q, v, P_v \to s, P_v' \to t)$, P_v' differs from P_v by moving *some* alternative from below s in P_v to above s in P_v'.[4] A SCF $\mathcal{F}$ is *simply monotonic* if for every profile P with winning alternative s, if one voter changes by moving s up in her ranking (while making no changes in the relative order of the other alternatives) then s remains the winner. As Muller and Satterthwaite (1977) show, Maskin monotonicity is equivalent to strategy-proofness—hence to two-way monotonicity. Maskin monotonicity easily implies both one-way monotonicity and simple monotonicity, which are logically independent of each other.[5]

3 Positive results: virtues and sensible rules

A vector $\langle w \rangle = \langle w_1, w_2, \ldots, w_m \rangle$ is a *vector of scoring weights* provided that the w_i are real numbers satisfying $w_1 \geq \cdots \geq w_m$. Such a vector is *proper* if $w_1 > w_m$ and is *strict* if $w_1 > w_2 > \cdots > w_m$. Every vector $\langle w \rangle$ of scoring weights induces a corresponding *scoring rule* as follows: each voter assigns w_1 points to her top-ranked alternative, w_2 to her second-ranked, etc., and the rule chooses the alternatives with maximal score (where the *score of an alternative s* is the sum of all points awarded to s by all voters). We say that the scoring rule is *strict* (resp., *proper*) if it is induced by some strict (resp., proper) vector.[6]

We consider voting rules that can be characterized in terms of certain type of "score" that generalizes the score from a scoring rule. All such rules are one-way monotonic.

[4] An iteration argument shows that this apparently weak definition of Maskin monotonicity implies the standard version on any *path connected* domain. For details, see footnote 13.

[5] Any resolute refinement of Copeland rule obtained through a fixed tie-breaking rule satisfies simple monotonicity but, as shown in Sect. 5, fails one-way monotonicity; see also footnote 12. Proposition 3.7 provides an example that is one-way monotonic but not simply monotonic.

[6] The best-known scoring rules include the *plurality rule*, with scoring vector $\langle 1, 0, \ldots, 0 \rangle$; the *anti-plurality rule*, with $\langle 0, 0, \ldots, 0, -1 \rangle$; and the *Borda count*, with $\langle m, m-1, \ldots, 1 \rangle$.

Given a finite set N of voters, and a set A of three or more alternatives, a *virtue* is a function $\mathcal{V}$ that returns a real number $\mathcal{V}_P(x)$, for each combination of a profile P and an alternative x. Any virtue $\mathcal{V}$ yields an *induced social choice rule* $\mathcal{F}_{\mathcal{V}}$ which declares the social choice for any profile R to be the alternatives x that maximize $\mathcal{V}_P(x)$. Examples of virtues include:

- For each scoring rule with associated vector $\langle w \rangle$ of scoring weights, let $\mathcal{V}_P^{\langle w \rangle}(x)$ denote the total score achieved by alternative x, using the given weights. The induced rule is, of course, the scoring rule for this vector.
- $\mathcal{V}^{OMNINOMINATOR}{}_P(x) = \begin{cases} 1, \text{ if at least one voter top-ranks alternative } x \\ 0, \text{ otherwise.} \end{cases}$
  The induced rule is the omninominator rule (see, for example, Taylor (2005)), wherein the winners are the alternatives that are "nominated" by being top-ranked by at least one voter.
- $\mathcal{V}^{OMNIVETOER}{}_P(x) = \begin{cases} -1, \text{ if at least one voter bottom-ranks alternative } x \\ 0, \text{ otherwise.} \end{cases}$
  The induced rule is the omnivetoer rule, wherein the winners are the alternatives that are not "vetoed" by being bottom-ranked by at least one voter (or are all alternatives, if each alternative is bottom-ranked at least once).
- $\mathcal{V}^{COPELAND}{}_P(x)$= the Copeland score of x. The Copeland rule is the induced system.

In the absence of any further restrictions, the virtue concept is clearly an empty shell, as any voting rule $\mathcal{F}$ is induced by the following trivial virtue:

$$\mathcal{V}^{\mathcal{F}\text{-}TRIVIAL}{}_P(x) = \begin{cases} 1, & \text{if } x \text{ is chosen by } \mathcal{F} \text{ at } P \\ 0, & \text{otherwise.} \end{cases}$$

Thus social choice rules (e.g., scoring rules) can be induced by two quite different virtues.

Suppose some focal voter v changes her vote. We wish our virtue $\mathcal{V}$ to reflect the *sense* of this change. Given any pre-focus $\Pi = (Q, v, P_v, P'_v)$, and any alternative x, let $\Delta x_{\mathcal{V}}$ (or just Δx) denote $\mathcal{V}_{Q \wedge P'_v}(x) - \mathcal{V}_{Q \wedge P_v}(x)$, x's change in virtue. If x and y are distinct alternatives, we say that Π *lifts x over y* if yP_vx and xP'_vy. Then $\mathcal{V}$ is *strictly sensible at* Π if for every pair x, y of distinct alternatives such that Π lifts x over y, $\Delta x_{\mathcal{V}} > \Delta y_{\mathcal{V}}$, and $\mathcal{V}$ is *sensible at* Π if for every pair x, y of distinct alternatives such that Π lifts x over y, $\Delta x_{\mathcal{V}} \geq \Delta y_{\mathcal{V}}$. Finally, $\mathcal{V}$ is *strictly sensible* if it is strictly sensible at every pre-focus, and is *sensible* if it is sensible at every pre-focus. A social choice rule $\mathcal{F}$ is *sensible* (resp. *strictly sensible*) if there exists a sensible (resp. strictly sensible) virtue that induces it. The following result is straightforward; proofs are left to the reader.

Proposition 3.1 (1) *$\mathcal{V}^{\langle w \rangle}$ is sensible for every vector $\langle w \rangle$ of scoring weights and is strictly sensible when $\langle w \rangle$ is strict.*

(2) *Any assignment* $c = \langle c_y \rangle_{y \in A}$ *of real numbers to alternatives gives rise to an initial endowment virtue* $\mathcal{W}$ *defined by* $\mathcal{W}_P(x) = c_x$ *for every profile* P*, and each such initial endowment is sensible.*
(3) $\mathcal{V}^{OMNINOMINATOR}$*, and* $\mathcal{V}^{OMNIVETOER}$ *are sensible.*
(4) *Any positive linear combination* $\lambda_1 \mathcal{V}_1 + \cdots + \lambda_k \mathcal{V}_k$ $(\lambda_i > 0$ *for all* $i)$ *of sensible virtues is sensible.*
(5) $\mathcal{V}^{COPELAND}$ *and* $\mathcal{V}^{BORDA\text{-}TRIVIAL}$ *are not sensible.*

Theorem 3.2 *Every sensible SCF* $\mathcal{F}$ *is one-way monotonic.*

Proof of 3.2 Suppose $\mathcal{F} = \mathcal{F}_{\mathcal{V}}$ where $\mathcal{V}$ is sensible. Let $(Q, v, P_v \to s, P'_v \to t)$ be a pivot for $\mathcal{F}$, and assume by way of contradiction that tP_vs and sP'_vt. As $\mathcal{V}$ is sensible, $\Delta s \geq \Delta t$. As $\mathcal{F}(Q \wedge P_v) = s$, $\mathcal{V}_{Q \wedge P_v}(s) > \mathcal{V}_{Q \wedge P_v}(t)$. Then $\mathcal{V}_{Q \wedge P'_v}(s) = \mathcal{V}_{Q \wedge P_v}(s) + \Delta s > \mathcal{V}_{Q \wedge P_v}(t) + \Delta t = \mathcal{V}_{Q \wedge P'_v}(t)$, which contradicts $\mathcal{F}(\mathrm{Q} \wedge P'_v) = t$. □

At the end of this section we show that the converse to 3.2 fails.

Let $\prec$ be any strict ranking of the alternatives used as a tie-breaking rule and $\mathcal{F}$ be a social choice rule. We denote by $\mathcal{F}^{\prec}$ the SCF obtained from $\mathcal{F}$ by setting $\mathcal{F}^{\prec}(R) = \prec -max[\mathcal{F}(R)]$ where $\prec -max[S]$ is the $\prec$-maximal element of S for any nonempty $S \subseteq A$.

Theorem 3.3 *For every sensible social choice rule* $\mathcal{F}$ *and any tie-breaking rule* $\prec$, $\mathcal{F}^{\prec}$ *is one-way monotonic.*

Proof If $\mathcal{F} = \mathcal{F}_{\mathcal{V}}$ where $\mathcal{V}$ is sensible, then $\mathcal{F}^{\prec} = \mathcal{F}_{\mathcal{V}+\mathcal{W}}$, where is $\mathcal{W}$ is the initial endowment given by any assignment $\langle c_y \rangle_{y \in A}$ of real numbers chosen to satisfy:

- $x \prec y$ iff $c_x < c_y$, for each $x, y \in A$, and
- The c_y are sufficiently small so that $\mathcal{V}_P(x) < \mathcal{V}_P(y) \Rightarrow \mathcal{V}_P(x) + c_x < \mathcal{V}_P(y) + c_y$, for each $x, y \in A$ and each profile P for N.

Then by 3.1 $\mathcal{V} + \mathcal{W}$ is sensible, and so $\mathcal{F}^{\prec}$ is one-way monotonic by 3.2. □

Thus sensible rules include omninominator, omnivetoer, and every scoring rule, whence omninominator$^{\prec}$, omnivetoer$^{\prec}$, and every scoring rule$^{\prec}$ are one-way monotonic. It would be interesting to learn just how large is the class of sensible rules, perhaps by seeking a structure theorem for this class. Any such a theorem should settle whether the omninominator and omnivetoer virtues are isolated, special cases of sensible virtues that do not correspond to scoring rules, or are pieces of some larger picture. If ${\mathcal{V}^{\langle w \rangle}}_R(x)$ represents a scoring virtue, and $K$is a constant, then a *truncated scoring virtue* is any virtue of the form $\mathcal{V}_R(x) = Min(K, {\mathcal{V}^{\langle w \rangle}}_R(x))$for $K \geq 0$, and an *anti-truncated scoring virtue* is any virtue of the form $\mathcal{V}_R(x) = Max(K, {\mathcal{V}^{\langle w \rangle}}_R(x))$for $K \leq 0$. Clearly $\mathcal{V}^{OMNIVETOER}$ and $\mathcal{V}^{OMNIVETOER}$ are truncated versions of plurality score (using $\langle 1, 0, 0, \ldots, 0 \rangle$ with $K = 1$) and anti-truncated versions of anti-plurality score (using $\langle 0, \ldots, 0, 0, -1 \rangle$ with $K = -1$), respectively. Truncated scoring virtues induce scoring rules with a threshold.[7] Yet with 4 or more alternatives, there

[7] A detailed treatment of scoring rules with a threshold can be found in Saari (1990). Moreover, Erdem and Sanver (2005) show that minimal Maskin monotonic extensions of scoring rules can be expressed in terms of scoring rules with a threshold that varies as a function of the preference profile.

are truncated scoring virtues (such as truncated Borda count) whose induced rules are not one-way monotonic.

Some additional insight into when truncation does and does not preserve sensibility can be gained through the following notion. A virtue $\mathcal{V}$ is *absolutely sensible at a pre-focus* Π if for every pair x, y of distinct alternatives such that Π lifts x over y, $\Delta x_{\mathcal{V}} \geq 0$ and $\Delta y_{\mathcal{V}} \leq 0$; $\mathcal{V}$ is *absolutely sensible* if it is absolutely sensible at every pre-focus, and a voting rule $\mathcal{F}$ is *absolutely sensible* if there exists some absolutely sensible virtue that induces $\mathcal{F}$. Clearly, any absolutely sensible virtue or rule is sensible. Now it is straightforward to prove the following analogue to Proposition 3.1:

Proposition 3.4 *The following virtues are absolutely sensible:*

(1) $\mathcal{V}^{PLURALITY}$ *and* $\mathcal{V}^{ANTI\text{-}P}$
(2) Any positive linear combination of absolutely sensible virtues.
(3) Any sum $\mathcal{V} + \mathcal{W}$ *of an absolutely sensible* $\mathcal{V}$ *and an initial endowment* $\mathcal{W}$*.*
(4) Any original and monotonic transform $f o V$ *of an absolutely sensible virtue* $\mathcal{V}$ *(where* $f{:}\mathcal{R} \rightarrow \mathcal{R}$ *is original if* $f(0) = 0$ *and is monotonic if* $a \geq b \Rightarrow f(a) \geq f(b)$*, for all* $a, b \in \mathcal{R}$*).*

Note that truncation and anti-truncation are special cases of original monotonic transforms. Thanks to 3.4 (iv) (which has no analogue in 3.1) we may add, to our list of one-way monotonic voting rules, examples such as the following: the rule induced (after imposition of a tie-breaking agenda) by the virtue $10(\mathcal{V}^{PLURALITY})^3 + \mathcal{V}^{\langle w \rangle} + 11\mathcal{V}^{OMNIVETOER}$, for any vector $\langle w \rangle$ of scoring weights. For four or more alternatives, if $\langle w \rangle$ is the Borda count vector then $\mathcal{V}^{\langle w \rangle}$ is sensible but not absolutely sensible. However, with three alternatives *every* scoring vector is equivalent to[8] some positive linear combination of the plurality and antiplurality vectors, so that every $\mathcal{V}^{\langle w \rangle}$ is absolutely sensible, by 3.4 (ii).

The following example appears in Campbell and Kelly (2002): given a profile P, take each possible ranking and divide the number of voters who chose that ranking by 2, dropping any fractional part, to obtain an *induced* profile ${}^1/_2 P$. Apply plurality rule to ${}^1/_2 P$, and then break ties using any fixed agenda. If we set aside the tie-breaking step, their rule is induced by the following virtue: $\mathcal{C}\text{-}\mathcal{K}_P(x) = \mathcal{V}^{PLURALITY}_{(1/2)P}(x)$. It is easy to see that $\mathcal{C}\text{-}\mathcal{K}$ is absolutely sensible, and it follows that their rule is one-way monotonic. Note that $\mathcal{C}\text{-}\mathcal{K}$ is an original and monotonic transform of the plurality virtue (but quite different from a truncation).

We close the section by showing that the converse to Theorem 3.2 fails. First, we prove that sensibility implies a strong, coalitional version of one-way monotonicity. We then construct an example of an SCF that is one-way monotonic but does not satisfy this stronger property. An immediate consequence is:

Corollary 3.5 *Not every one-way monotonic SCF is sensible.*

A SCF is *weakly coalitional one-way monotonic* if whenever a sub-group of voters having a common preference ranking σ simultaneously all switch their rankings to

[8] Specifically, any scoring vector for three alternatives generates the same voting rule as some shifted version of that vector for which the middle scoring weight is 0, and any vector with middle weight 0 is a nonnegative linear combination of $\langle 1, 0, 0 \rangle$ and $\langle 0, 0, -1 \rangle$.

some common ranking τ,[9] and the effect is to switch the winning alternative from s alone to t alone, then either $s\sigma t$ or $t\tau s$. Such a SCF is *strongly coalitional one-way monotonic* if whenever a group of voters simultaneously all switch their rankings (which need not agree before the switch, and need not agree after), and the effect is to switch the winning alternative from s alone to t alone, then there exists at least one voter v in the group for whom sT_vt or for whom $tT_v's$ (where T_v is v's initial ranking, which she switches to T_v'). We omit the proof of the following proposition, which is quite similar to that of 3.2.

Proposition 3.6 *Every sensible SCF is strongly coalitional one-way monotonic.*

Proposition 3.7 *There exists a resolute voting rule $\mathcal{F}$ satisfying one-way monotonicity, but not weak coalitional one-way monotonicity (and not simple monotonicity). It follows that $\mathcal{F}$ is not sensible.*

Proof of 3.7 We use $n = 2$ voters and $m = 3$ alternatives: a, s, and t. Consider the following rankings:

α	β
a	a
t	s
s	t

Let $\mathcal{G}$ be the voting rule induced by the sensible virtue $\mathcal{V}+\mathcal{W}$, where $\mathcal{V}$ is plurality score and $\mathcal{W}$ is the following initial endowment:

$s \mapsto$ *2.02* points
$t \mapsto$ *2.01* points
$a \mapsto$ *0.00* points

Note that $\mathcal{G}$ is resolute and one-way monotonic, and also satisfies:

1. $\mathcal{G}(\alpha \wedge \beta) = s$,
2. $\mathcal{G}(2\beta) = s$,
3. $\mathcal{G}(2\alpha) = s$,
4. $\mathcal{G}(\delta \wedge \alpha) = x = \mathcal{G}(\delta \wedge \beta)$ whenever $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ (so $x \in \{s, t\}$).

Our desired rule $\mathcal{F}$ is obtained by changing $\mathcal{G}$'s value on exactly two *exceptional* profiles:

1. $\mathcal{F}(\alpha \wedge \beta) = a$
2. $\mathcal{F}(2\beta) = t$.

Note that $\mathcal{F}$ is not weakly coalitional one-way monotonic, for if both voters choose α and they simultaneously change to β then they both lift s from under t to over t, but the winner switches from s to t. Also, $\mathcal{F}$ is not simply monotonic, as $\mathcal{F}(2\alpha) = s$

[9] We will use a lower case Greek letter in place of "P_v" when there is no particular voter v associated with the preference ranking at hand.

and $\mathcal{F}(\alpha \wedge \beta) = a$. To see that $\mathcal{F}$ is one-way monotonic, consider any pivot in which the (single) pivotal voter changes from σ to τ. As there are but two voters, the profile changes from $P = \sigma \wedge \lambda$ to $Q = \tau \wedge \lambda$.

Case 1 Neither P nor Q are exceptional. For the non-exceptional profiles, $\mathcal{F}$ agrees with $\mathcal{G}$, which is one-way monotonic.

Case 2 $P = 2\beta$ and $Q = \alpha \wedge \beta$, or vice versa. Then $\mathcal{F}(2\beta) = t$ and $\mathcal{F}(\alpha \wedge \beta) = a$, which does not violate one-way monotonicity.

Case 3 $P = \beta \wedge \alpha$ and $Q = 2\alpha$, or vice versa. Then $\mathcal{F}(\beta \wedge \alpha) = a$ and $\mathcal{F}(2\alpha) = s$, which does not violate one-way monotonicity.

Case 4 $P = 2\beta$ and $Q = \delta \wedge \beta$ (where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ), or vice versa. Then $\mathcal{F}(2\beta) = t$ and $\mathcal{F}(\delta \wedge \beta) = x$, which does not violate one-way monotonicity.

Case 5 $P = \alpha \wedge \beta$ and $Q = \delta \wedge \beta$ (where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ), or vice versa. Then $\mathcal{F}(\alpha \wedge \beta) = a$ and $\mathcal{F}(\delta \wedge \beta) = x$, which does not violate one-way monotonicity.

Case 6 $P = \beta \wedge \alpha$ and $Q = \delta \wedge \alpha$ (where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ), or vice versa. Then $\mathcal{F}(\beta \wedge \alpha) = a$ and $\mathcal{F}(\delta \wedge \alpha) = x$, which does not violate one-way monotonicity. □

4 Participation and the no-show paradox

Brams and Fishburn (1983) introduced the *no-show paradox*: one additional *participating voter* shows up to cast her vote, and the winner is then an alternative that is strictly inferior (according to the preferences of the participating voter) to the alternative who would have won had she not shown up. Thus, the paradox represents an opportunity to manipulate by abstaining. Moulin (1988a) and Moulin (1988b) expressed the corresponding form of strategy-proofness: A variable electorate SCF $\mathcal{F}$ satisfies *participation* if for each profile Q for a finite set N of voters, and each preference ranking P_v for a *participating voter* $v \notin N$, $\mathcal{F}(Q \wedge P_v) = \mathcal{F}(Q)$ or $\mathcal{F}(Q \wedge P_v) P_v \mathcal{F}(Q)$.

How are one-way monotonicity and participation related? A comparison requires some attention to the difference in context. Certainly one-way monotonicity can be considered to be a property of variable-electorate SCFs, and this is the form of the property we use in this section. But participation enforces some connection between what $\mathcal{F}$ does for profiles with n voters and what it does for profiles with $n + 1$, while one-way monotonicity does not. As we see below, this makes it easy to construct a rule that satisfies one-way monotonicity but not participation. Such a comparison does not seem entirely fair, however—the better question may be whether one-way monotonicity implies participation in the presence of some mild axioms that do forge connections among election outcomes for different size electorates. The theorem that follows provides a positive answer, but with a strong qualification: the second axiom is not exactly "mild." The proof makes use of an additional property, *half-way monotonicity*, as an interpolant between one-way monotonicity and participation. After giving the related definitions, we state the theorem.

If m is any positive integer, and P is any profile, then $\boldsymbol{mP}$ is the profile obtained from P by replacing each single voter v of P with m voters having the same preference as v. An anonymous, variable-electorate SCF $\mathcal{F}$ is *homogeneous* if $\mathcal{F}(mP) = \mathcal{F}(P)$ holds for all choices of P and m. [10] For any (strict) ranking σ, let $rev(\sigma)$ denoting the ranking obtained by reversing σ, so that $x\ \sigma\ y$ iff $y\ rev(\sigma)\ x$. For a profile P and a strict ranking σ let $\boldsymbol{P} \wedge \sigma \wedge rev(\sigma)$ denote the profile obtained by adding two additional voters—one with ranking σ and another with ranking $rev(\sigma)$. An anonymous, variable-electorate SCF satisfies *reversal cancellation* if for all choices of P and σ, $\mathcal{F}(P) = \mathcal{F}(P \wedge \sigma \wedge rev(\sigma))$.[11] A SCF $\mathcal{F}$ is *half-way monotonic* if for every pivot $(Q, v, P_v \to s, rev(P_v) \to t)$ for $\mathcal{F}$, $s P_v t$. Half-way monotonicity, which is implied by one-way monotonicity, has an interesting interpretation in terms of strategy-proofness: a rule that violates half-way monotonicity can be manipulated by some voter who *completely* misrepresents her preferences, in the sense that she announces a preference ranking that misstates every possible pairwise comparison among alternatives.

Our main result is

Theorem 4.1 *Consider properties of variable-electorate SCFs. Then*

(1) *one-way monotonicity and participation are independent,*
(2) *participation ⇒ half-way monotonicity,*
(3) *[half-way monotonicity + homogeneity + reversal-cancellation] ⇒ participation,*
(4) *for the case of exactly three alternatives, participation ⇒ one-way monotonicity, and*
(5) *for the case of exactly four alternatives, [participation + simple monotonicity] ⇒ one-way monotonicity.*

Corollary 4.2 *(immediate) For anonymous and variable-electorate SCFs satisfying both homogeneity and reversal cancellation:*

- *Participation and half-way monotonicity are equivalent. In other words, participation is equivalent to the corresponding weak form of strategy-proofness stating that no one can improve the outcome by completely misrepresenting their preferences.*

[10] Homogeneity is a very weak form of *reinforcement* (also known as *consistency*), discussed in Smith (1973). It is known to hold for almost every social choice rule, though Fishburn (1977) shows that the Dodgson and Young procedures can each fail to be homogeneous, depending on the details in the precise formulation of these systems. It is probably fair to deem homogeneity an "innocuous" assumption.

[11] Reversal cancellation is closely related to work by Saari (1994, 1999), and by Saari and Barney (2003), who consider the effect of reversing an entire profile, and examines the vector component of a profile corresponding to reversal. Every social choice rule that is *pairwise* (depends only on the information in $\{\mathrm{Net}_P(x>y)\}_{x \neq y}$) satisfies reversal cancellation; these include Copeland, Simpson and Borda (see Zwicker (1991)). But the property fails for other scoring rules, such as plurality and antiplurality (and fails for Hare) so it can hardly be called innocuous.

- *Participation, half-way monotonicity, and one-way monotonicity are all equivalent for the special case of three alternatives, or of four alternatives with the additional assumption of simple monotonicity.*

Proof of 4.1 The proof exploits an additional similarity between one-way monotonicity and participation—there is a notion of *variable electorate virtue* $\mathcal{V}$ with some properties analogous to those in Sect. 3. Such a $\mathcal{V}$ is defined for a variable electorate, and is *participation sensible* if each profile Q and ranking P_v satisfies $x P_v y \Rightarrow \Delta^{Part} x_{\mathcal{V}} \geq \Delta^{Part} y_{\mathcal{V}}$, for every two alternatives x and y. Here, $\Delta^{Part} x_{\mathcal{V}}$ denotes $\mathcal{V}_{Q \wedge P_v}(x) - \mathcal{V}_Q(x)$. We leave the reader to verify the following: □

Claim The participation axiom is satisfied by any variable-electorate SCF that is induced by a participation sensible variable electorate virtue.

Proof of 4.1(i) First consider the following variable electorate SCF that satisfies one-way monotonicity but not participation: Let $\mathcal{F}$ act as the plurality rule$^{\prec}$ for profiles with 5 or fewer voters, and as the Borda count$^{\prec}$ with 6 or more. Then $\mathcal{F}$ is one-way monotonic by 3.3. To see that participation fails, consider the 5-voter profile below, and a participating voter with ranking P_v. Note that $\mathcal{F}(Q) = s$, while $\mathcal{F}(Q \wedge P_v) = t$.

		Q			P_v
1	1	1	1	1	
s	s	t	x	y	x
t	t	y	t	t	y
x	y	x	y	x	s
y	x	s	s	s	t

The following rule $\mathcal{G}$ for four alternatives satisfies participation, but not one-way monotonicity. For the record, we note that $\mathcal{G}$ is anonymous, but does not satisfy neutrality, homogeneity, or reversal cancellation. As predicted by part (v), $\mathcal{G}$ also violates simple monotonicity. Certainly it would be interesting to find a similar example with five or more alternatives, in which simple monotonicity holds.
Description of $\mathcal{G}$

(1) Our four alternatives are x, y, s, and t.
(2) Our preliminary version of $\mathcal{G}$ is the scoring rule with scoring weights 3, 2, 2, 1, but this is modified by the remaining clauses.
(3) Each alternative has a fixed endowment of points before the voting begins, which is added to that alternative's point total, as determined by the profile at hand, to determine that alternatives *final score*. The endowments are as follows:
    $x \mapsto$ *2.03 points*
    $y \mapsto$ *0.02 points*
    $s \mapsto$ *2.01 points*
    $t \mapsto$ *2.0 points*
    The effect is the same as giving x, s, and t each 2 points, with 0 points to y, and imposing the tie-breaking agenda $x \succ y \succ s \succ t$ on the penultimate outcome.
(4) There is an *exceptional* profile "β" with a single voter, to which the above rules do not apply. That voter has a preference ranking that we also call β:

β	
y	3.02
s	4.01
t	4.0
x	3.03

The numbers in the right column give the final scores, so that the fractional amounts, in effect, break the tie in favor of s. Instead, we declare that $\mathcal{G}(\beta) = t$.

Proof that $\mathcal{G}$ satisfies participation, but fails both simple monotonicity and one-way monotonicity. Consider the one-voter profile "α" in which the single voter has the following ranking α:

α	
y	3.02
t	4.0
s	4.01
x	3.03

This is not the exceptional profile, and so $\mathcal{G}(\alpha) = s$. Let $\emptyset$ denote the empty profile, with no voters. The pivot $(\emptyset, v, \alpha \to s, \beta \to t)$ shows that $\mathcal{G}$ fails both simple monotonicity and one-way monotonicity. To see that $\mathcal{G}$ satisfies participation consider the transition from some profile Q to $Q \wedge P_v$, where v is the participating voter.

Case 1 Neither Q nor $Q \wedge P_v$ is β. Then participation holds for this transition, as $\mathcal{G}$ is completely given by clauses (1) - (3), which describe a participation-sensible virtue.

Case 2 $Q \wedge P_v = \beta$. Then $Q = \emptyset$, $\mathcal{G}(Q) = x$ and $\mathcal{G}(Q \wedge P_v) = t$. The participating voter has ranking β, which ranks t over x, so participation is satisfied for this transition.

Case 3 $Q = \beta$. We claim that $\mathcal{G}(Q \wedge P_v)$ is equal to the top-ranked alternative of P_v. The claim suffices to show that participation is satisfied for this transition.

Subcase 3.1 s is top-ranked by P_v. By referring to the final scores for β we see that the final score for s in profile $\beta \wedge P_v$ is 7.01, which is greater than any other final score, so that $\mathcal{G}(\beta \wedge P_v) = s$.

Subcase 3.2 t is top-ranked by P_v. The argument is similar to that of the previous subcase.

Subcase 3.3 y is top-ranked by P_v. Then the final score for y in profile $\beta \wedge P_v$ is 6.02 and for x is at most 5.03; s and t each get at most 6.01, and $\mathcal{G}(\beta \wedge P_v) = y$.

Subcase 3.4 x is top-ranked by P_v. Then the final score for x in profile $\beta \wedge P_v$ is 6.03; each other alternative gets at most 6.02, and $\mathcal{G}(\beta \wedge P_v) = x$.

Proof of 4.1 (2), (4) and (5) Our somewhat unorthodox approach is to launch an attempt to prove that participation implies one-way monotonicity. This proof breaks down in one of the cases. We then observe that the obstacle is circumvented under the additional assumption that $\tau = rev(\sigma)$, or that there are only three alternatives.

Alternatively, the obstacle is circumvented for exactly four alternatives if the rule is simply monotonic. This approach suggests some insight into the relationships among the four properties under consideration.

Assume that $\mathcal{F}$ satisfies participation, that $\mathcal{F}(P \wedge \sigma) = s$, and $\mathcal{F}(P \wedge \tau) = t$. We must show that $s\sigma t$ or $t\tau s$.

Case 1 $\mathcal{F}(P) = s$ or $\mathcal{F}(P) = t$. If $\mathcal{F}(P) = s$ then as $\mathcal{F}(P \wedge \sigma) = t, t\tau s$ by participation. If $\mathcal{F}(P) = t$, then as $\mathcal{F}(P \wedge \sigma) = s$, participation implies $s\sigma t$.

Case 2 $\mathcal{F}(P \wedge \sigma \wedge \tau) = s$ or $\mathcal{F}(P \wedge \sigma \wedge \tau) = t$. If $\mathcal{F}(P \wedge \sigma \wedge \tau) = s$, then as $\mathcal{F}(P \wedge \tau) = t$, participation implies $s\ \sigma\ t$. If $\mathcal{F}(P \wedge \sigma \wedge \tau) = t$ then as $\mathcal{F}(P \wedge \sigma) = s$, participation implies $t\ \tau\ s$.

Case 3 $\mathcal{F}(P) = x$ with $x \notin \{s, t\}$ and $\mathcal{F}(P \wedge \sigma \wedge \tau) = y$ with $y \notin \{s, t\}$. As $\mathcal{F}(P) = x$ and $\mathcal{F}(P \wedge \sigma) = s$, participation implies $s\ \sigma\ x$. Also, from $\mathcal{F}(P) = x$ and $F(P \wedge \tau) = t$, participation implies $t\ \tau\ x$. But from $s\ \sigma\ x$ and $t\ \tau\ x$ we can draw no conclusion about how σ or τ rank s versus t. However, if $\tau = rev(\sigma)$ then "$s\ \sigma\ x$ and $t\ \tau\ x$" becomes "$s\ \sigma\ x$ and $t\ rev(\sigma)\ x$," which is "$s\ \sigma\ x$ and "$x\ \sigma\ t$," whence $s\ \sigma\ t$, as desired. Similarly, from $\mathcal{F}(P \wedge \sigma \wedge \tau) = y$ and $F(P \wedge \sigma) = s$, participation implies $y\ \tau\ s$. Also from $\mathcal{F}(P \wedge \sigma \wedge \tau) = y$ and $\mathcal{F}(P \wedge \tau) = t$, participation implies $y\ \sigma\ t$. Again, we can conclude nothing from "$y\ \tau\ s$ and $y\ \sigma\ t$," unless $\tau = rev(\sigma)$, in which case $s\ \sigma\ t$ again follows. Finally, observe that we in fact have *four* facts to work with:

- $s\ \sigma\ x$,
- $t\ \tau\ x$,
- $y\ \tau\ s$, and
- $y\ \sigma\ t$

If we knew $x = y$, then we could conclude both $s\ \sigma\ t$ and $t\ \tau\ s$ with no additional assumption that $\tau = rev(\sigma)$. If there are exactly three alternatives, then the case 3 assumption implies $x = y$, and we conclude that $\mathcal{F}$ is one-way monotonic. If there are exactly four alternatives s, t, x, and y, then the four inequalities just listed, coupled with the assumption that one-way monotonicity fails (in that $t\ \sigma\ s$ and $s\ \tau\ t$), completely determines the orderings σ and τ: $y\ \sigma\ t\ \sigma\ s\ \sigma\ x$, and $y\ \tau\ s\ \tau\ t\ \tau\ x$. As $\mathcal{F}(P \wedge \sigma) = s$, and $\mathcal{F}(P \wedge \tau) = t$, these orderings yield a failure of simple monotonicity. Thus, with exactly four alternatives, if $\mathcal{F}$ satisfies participation, then any failure of one-way monotonicity implies a failure of simple monotonicity.

Proof of 4.1 (3) Consider a failure of participation: a profile P and a single added voter v with ranking σ such that $t = \mathcal{F}(P \wedge \sigma)$ is ranked below $s = \mathcal{F}(\mathrm{P})$ by σ, so that $\mathcal{F}(\mathrm{P})\sigma\ \mathcal{F}(\mathrm{P}\wedge\sigma)$. Then $F((2P)\wedge\sigma\wedge rev(\sigma)) = \mathcal{F}(2P) = \mathcal{F}(P) = s$, and $F((2P)\wedge\sigma \wedge \sigma) = F(2(P\wedge\sigma)) = \mathcal{F}(P\wedge\sigma) = t$. But the profile $(2P) \wedge \sigma \wedge \sigma$ is obtained from the profile $(2P) \wedge \sigma \wedge \mathrm{rev}(\sigma)$ by having the voter v with preference ranking $rev(\sigma)$ flip his ranking upside down so that it becomes σ. This voter raises s from below t in $rev(\sigma)$ to above t in σ, and the effect is that s now loses while t wins ... a failure of half-way monotonicity. □

In the presence of homogeneity + reversal-cancellation, might one-way monotonicity and participation be equivalent? Our proof of part (i) leaves this possibility open, but we conjecture that one-way monotonicity is strictly stronger than participation with these assumptions.

From Theorem 4.1 together with Moulin's result that every Condorcet extension fails participation, we can immediately conclude that every homogeneous and reversal-canceling Condorcet extension fails one-way monotonicity.[12] However, our direct modification of Moulin's proof in the next section avoids the need to assume homogeneity and reversal cancellation.

5 Negative results: Condorcet extensions, Hare, and plurality run-off

Our approach to the main result makes use of the following coalitional version of half-way monotonicity: a SCF is *weakly coalitional half-way monotonic* if whenever a set of voters having *identical* strict ranking σ all simultaneously change their votes to *rev*(σ), and the effect is to switch the winner from s alone to t alone, it must be that $s\ \sigma\ t$.

Proposition 5.1 *Half-way monotonicity implies weak coalitional half-way monotonicity.*[13]

Proof Assume a set of k voters having identical strict ranking σ all change their votes to *rev*(σ), and the effect is to switch the winner from s to t. Let P^j be the profile in which j of these k voters have changed from σ to *rev*(σ), and $k - j$ have not changed, and consider the sequence of profiles $P = P^0, P^1, \ldots, P^k$. Suppose the winners for the profiles $P^0, P^1, \ldots, P^k$ are $s = x^0, x^1, \ldots, x^u = t$ where x^q denotes the (common, unique) winner for profiles $P^{j(q)}, P^{j(q)+1}, \ldots, P^{j(q+1)-1}$, with $0 = j(1) < j(2) < \cdots < j(u) \le t$ (and $x^q \neq x^{q+1}$). Then by ordinary half-way monotonicity applied to each transition from $P^{j(q+1)-1}$ to $P^{j(q+1)}$, it follows that $s\ \sigma\ x^1\sigma, \ldots, \sigma\ x^u = t$, so that $s\ \sigma\ t$, as desired. □

Theorem 5.2 *With four or more alternatives and sufficiently many voters, no Condorcet extension satisfies weak coalitional half-way monotonicity.*

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12 Merlin and Saari (1997) show that the Copeland rule fails a broad variety of monotonicity properties, and one-way monotonicity is within this scope. Similar methods might extend to Condorcet extensions that are *homogeneous* and *pairwise* (see footnote 11), but our results in Sect. 5 apply to *all* Condorcet extensions.

13 The proof that follows applies to any *path connected* domain—any domain D with the property that every pair of profiles in D is linked by some ordered chain of "connecting" profiles such that each profile differs in only one voter from the next in the chain. Note that the path connected domains include all domains obtained as some cartesian product of restricted sets of preference rankings. One can formulate a strong coalitional version of half-way monotonicity, as well as weak and strong coalitional versions of participation, simple monotonicity, and two-way monotonicity (with strong coalitional two-way monotonicity equivalent to coalitional strategy-proofness). It then turns out that there is considerable variation as to whether the simple iteration argument used in the proof of Proposition 5.1 suffices to derive one or both coalitional forms from the individual form. In particular, this argument does show that for every path connected domain the strong coalitional forms of Maskin monotonicity (which is, in fact, the standard form in the literature) and of simple monotonicity follow respectively from the individual forms we have defined here. On the other hand, Proposition 3.7 shows that the weak coalitional form of one-way monotonicity does not follow from the individual form.

Corollary 5.3 *With four or more alternatives and sufficiently many voters, no Condorcet extension satisfies half-way monotonicity, and so no Condorcet extension satisfies one-way monotonicity.*

Proof of 5.2 Our proof is an elaborated version of the argument in Exercise 9.3(c), page 251 of Moulin (1988a). The elaboration uses the ideas behind the homogeneity and reversal cancellation axioms, but does not require any assumption that these axioms hold. (Rather, it uses that these axioms hold, speaking loosely, for Condorcet winners and for Simpson scores.) Let C denote the profile, for $m \geq 4$ alternatives, in which each possible ranking occurs exactly once. Note C has $m!$ voters. For an arbitrary profile P with $n = n(P)$ voters, let $k = k(P)$ be the maximum integer j such that each of the $m!$ rankings occurs at least j times in P. Informally, $k(P)$ represents the "number of copies of C contained in P." Let $n^*(P)$ denote $n(P) - (m!)k(P)$. Informally, $n^*(P)$ is the number of voters remaining once one ignores the copies of C. We will say that a profile P satisfies Condition M if $2k(P) \geq n^*(P) + 2$. Informally, condition M says that P contains "enough" copies of C relative to the number of voters who would remain if all copies of C were removed.

Claim Let $\mathcal{F}$ be a Condorcet extension satisfying weak coalitional half-way monotonicity. Let a and b be any two alternatives, and P be any profile for four or more alternatives that meets the following three conditions:

- condition M,
- the Simpson score $t^*(b)$ is even and $t^*(b) \leq 0$, and
- $Net(b > a) > -t^*(b) + 2$.

Then $\mathcal{F}$ does not elect alternative a at profile P.

Proof of Claim Let a, b, and P be as stated. Note that in general, we know that $t^*(b)$ satisfies $-n \leq t^*(b) \leq n$. However, copies of profile C have no effect on the value of $t^*(b)$, so in fact we know $-n^* \leq t^*(b) \leq n^*$, whence $-t^*(b) + 2 \leq n^* + 2 \leq 2k$.

Let $r = \frac{-t^*(b)+2}{2}$, a strictly positive integer. Choose σ to be any ranking such that a is at the bottom, b is immediately above a, and all other alternatives are ranked above b. By assumption M there exist at least r voters who voted σ. Now let Q be obtained from P by having r voters who voted σ all change their votes to $rev(\sigma)$. For each alternative x other than a or b, the effect of these changes is that

$Net_Q(b > x) = Net_P(b > x) + 2r = Net_P(b > x) + -t^*(b) + 2.$

As $Net_P(b > x) \geq t^*(b)$, this makes $Net_Q(b > x) > 0$. Furthermore

$Net_Q(b > a) = Net_P(b > a) - 2r = Net_P(b > a) + t^*(b) - 2 > 0.$

Hence, b is a Condorcet winner for profile Q (and a is not a winner). If a had been a winner for P, this would be a violation of weak coalitional half-way monotonicity. This completes the proof of the claim.

Next consider profile R:

6	6	10	8	
a	a	d	b	
d	d	b	c	Profile R
c	b	c	a	
b	c	a	d	

Note that $n(R) = n^*(R) = 30$. Let P be obtained from R by adding 28 copies of the profile C. Then $n^*(P) = n^*(R) = 30$.

The claim can now be applied three times to show that b, c, and d cannot be elected at profile P, so that a is the sole winner. (Note that when calculating any value of t^* or $Net(x > y)$, C can be ignored, so the values for P are the same as those for R.) Next suppose that four of the voters from P who have ranking $d\sigma b\sigma a\sigma c$, simultaneously reverse their rankings, yielding some new profile Q. Note that $n^*(Q) = 30 + 8 = 38$ and $k(Q) = k(P) - 8 = 20$, so that Q satisfies condition M. Apply the claim two more times to show that neither a nor c can be elected at Q. This contradicts our assumption that $\mathcal{F}$ is weakly coalitional half-way monotonic. □

The situation painted by Corollary 5.3 seems somewhat odd. On the one hand, it is easy to see that on the domain $\mathcal{D}_{Con}$ of profiles having Condorcet winners, pairwise majority rule satisfies the strong property of two-way monotonicity. But it is impossible to extend pairwise-majority rule over the full domain without violating the weaker property of one-way monotonicity. Meanwhile, there are rules such as scoring rules or the omninominator rule that "do less well" than does pairwise majority rule on $\mathcal{D}_{Con}$, yet do better on the full domain. It is almost as if pairwise majority rule paints itself into a corner by trying too hard on $\mathcal{D}_{Con}$.

Next, we consider two closely related voting rules. In *plurality with run-off*, if no candidate achieves a strict majority of first-place votes, there is a run-off between the two alternatives x and y achieving the greatest number of first-place votes: the winning alternative is whichever of x or y is ranked over the other by a majority of voters. In the *Hare* rule (or "alternative vote," as it termed in Moulin (1988a)) alternatives are eliminated in sequential stages, based on fewest first-place votes. Each stage considers only the relative rankings over surviving alternatives, and the winner is the last alternative (or final group of tied alternatives) remaining.

The following profile S is adapted from one used in Moulin (1988a) to show that neither of these two rules are simply monotonic:

6	5	6	
a	c	b	
b	a	c	Profile S
c	b	a	

We reason about both plurality run-off and Hare together. In the above profile S, c is eliminated, all 5 votes for c then turn to a, who wins the run-off against b. However, if two of group of 6 who ranked b on top change rankings as indicated below

6	5	4	2	
a	c	b	a	
b	a	c	b	Profile S''
c	b	a	c	

Then in S'' it is b who is eliminated and c wins the run-off against a. This represents a failure of both simple monotonicity and weak coalitional one-way monotonicity—but can we obtain a failure of ordinary one-way monotonicity? It makes sense to consider the intermediate profile below, in which only one of the six b-voters has made the switch:

6	5	5	1	
a	c	b	a	
b	a	c	b	Profile S'
c	b	a	c	

The most straightforward interpretation of "Plurality with run-off" seems to be that for this profile both alternatives b and c would be eliminated, leaving a the winner. In that case, the transition from S' to S'' provides the desired failure of one-way monotonicity. This interpretation appears to be a standard one for the Hare (alternative vote), so we conclude that Hare fails one-way monotonicity. What else might plurality run-off actually do in a situation such as S' ? We imagine that the ambiguity may not be of much concern, at least not for large presidential elections in which exact ties are extremely unlikely (or even ill-defined, as truly exact vote counts do not seem to exist in the real world). The (only) other alternative that suggests itself is that plurality run-off might declare a three-way tie among a, b, and c for S'. In that case, if the tie-breaking dictator throws the contest to a, then the transition from S' to S'' again provides the desired failure of one-way monotonicity. This is not an entirely satisfactory solution, so we phrase the corresponding proposition conservatively:

Proposition 5.4 *The Hare rule is not one-way monotonic. Plurality run-off violates weak coalitional one-way monotonicity.*[14]

6 Conclusions

One-way monotonicity stands apart from previously studied monotonicity properties because of its distinct interpretation in terms of strategy-proofness. Our feeling is that one-way monotonicity has some additional normative appeal, apart from this interpretation, and discriminates in a useful way among realistic voting rules, so that it has some features common to each of the classes we described in the introduction.

At the same time, one-way monotonicity shares important qualitative features with the participation axiom, in terms both of common negative results for Condorcet extensions, and of positive results for voting rules induced by certain types of cardinal functions, called here sensible virtues, that respond appropriately to changes in a profile. We do not yet understand the exact relationship between one-way monotonicity and sensible virtues; the example in Proposition 3.7 leaves open the possibility that some strong version[15] of one-way monotonicity implies sensibility. Similar comments apply to participation.

[14] The referee has pointed out that if we employ the fixed agenda to break ties *at each stage* of a scoring run-off rule $\mathcal{F}$ (rather than only at the very end), then a modification of the proof yields a stronger result: for every choice of scoring weights, scoring run-off procedures fail to be one-way monotonic.

[15] In particular, it seems that the strong coalitional version of one-way monotonicity would be involved. Also, it can be shown that any sensible rule $\mathcal{F}$ extends to a *social welfare rule* $\mathcal{F}^*$ that satisfies a form

In terms of logical strength, there seems to be an intricate relationship among one-way monotonicity, half-way monotonicity, participation, simple monotonicity, and some other axioms that bridge the gap between properties for a fixed electorate and those for a variable electorate. In one sense, Theorem 4.1 together with the counterexamples provided in Campbell and Kelly (2002) and in this paper, already tell us a lot about these relationships.

These counterexamples, however, typically fail to be neutral or fail to be anonymous, and so they do not address questions such as the following:

($*$) *Does participation imply one-way monotonicity for neutral and anonymous rules?*

This question may at first seem to be poorly conceived, as neutral and anonymous rules would need to be rendered resolute before the question made sense, and the mechanism employed (such as a tie-breaking agenda) would destroy one or the other of neutrality and anonymity. The question would become

($**$) *Does participation imply one-way monotonicity for neutral and anonymous rules, after they are rendered resolute via a tie-breaking agenda?*

which may not seem to be all that interesting.

On the other hand, one might approach question ($*$) in a different way, by adapting the properties directly so that they make sense when applied to irresolute voting rules:

($***$) *Does the irresolute form of participation imply the irresolute form of one-way monotonicity, for neutral and anonymous rules?*

In Sanver and Zwicker (2009), we suggest some new methods for adapting one-way and two-way monotonicity, simple monotonicity, and participation to the irresolute context, and show that an irresolute voting rule $\mathcal{F}$ is irresolutely one-way monotonic in this sense if and only if for *every* choice of a tie-breaking agenda $\prec$, the resolute rule $\mathcal{F}^{\prec}$ is one-way monotonic according to the definition we have used throughout this paper. One implication of this result is that the method we have used in this paper, of rendering voting rules resolute via a tie-breaking agenda, is less problematic than may first appear. Another is that question ($**$) is more natural than one might think; in effect, it is identical to an instance of question ($* * *$).

The most common approach for adapting *strategy-proofness* to the irresolute context is to extend, in any one of a number of possible ways, a voter's preferences over individual alternatives to preferences over sets of alternatives (see, for example, Gärdenfors (1979), or Taylor (2005)). This method may be applied to monotonicity properties, as well, suggesting an alternative to the approach sketched above. In general, we find (perhaps unsurprisingly) that the study of irresolute monotonicity is rich and complex. However, some results suggest that there may be enough agreement among the various approaches to avoid a devolution into Byzantine intricacy.

Any comparison of participation with one-way monotonicity requires that we consider variable-electorate voting rules. In the fixed-electorate context, however, neutrality and anonymity no longer necessarily force the existence of ties. In fact,

Footnote 15 continued
of one-way monotonicity appropriate to the social welfare context. (A social welfare rule yields, as the election outcome, a *ranking* of all alternatives, with ties allowed.) So any version of one-way monotonicity implying sensibility might need to be phrased in the social welfare context.

Doğan and Giritligil (2007) have recently shown that there exists a neutral, anonymous, one-way monotonic, and simply monotonic (resolute) SCF if and only if the number n of voters and m of alternatives satisfy that m! and n are relatively prime. Their result suggests that the fixed-electorate context is the "natural home" of one-way monotonicity; this home allows us to study properties that are ruled out for participation. Their methods also suggest possible extensions of their results. For example, when m! and n are not relatively prime, is there a good notion of "minimally irresolute" voting rule, and can such a rule be one-way monotonic? What do their methods tell us when the output of a voting rule is a ranking?

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