Social threshold aggregations*

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Abstract A problem of axiomatic construction of a social decision function is studied for the case when individual opinions of agents are given as m-graded preferences with arbitrary integer $m \geq 3$. It is shown that the only rule satisfying the introduced axioms of Pairwise Compensation, Pareto Domination and Noncompensatory Threshold and Contraction is the threshold rule.

Keywords aggregation · preference · ranking · threshold rule · axiomatics

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1 Introduction

The aim of this paper is to investigate the following problem of construction of a social decision function. Given a set of n agents, each agent evaluates alternatives from a finite set X using complete and transitive preferences (rankings), and we look for a complete and transitive social preference over the alternatives. This kind of aggregation has been considered in many publications, beginning with the seminal work by Arrow [10]. In order to solve the problem, two ways have been proposed. Arrow's kind of axiomatics can be described as the local aggregation, cf. Aleskerov [2]; in other words, the aggregation is done

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on the basis of pairwise comparisons of alternatives. Another way is to use certain non-local procedures, e.g., positional rules, for which only a few works with very well constructed axiomatics exist, cf. Austen-Smith and Banks [11], May [16], Moulin [17], Smith [21] and Young [22–24].

One of the non-local rules is the Borda voting rule (Young [23]). An application of Borda's rule is often not adequate, since any summation of ranks has a 'compensatory nature': a low evaluation of some alternative by an agent can be compensated by high evaluations of the other agents. Thus, if we would like to take carefully into account low evaluations of alternatives when the quality or perfectness of alternatives is important, the Borda rule or its counterparts cannot be applied.

Let us consider two examples (see also Section 3).

Example 1[1]. Suppose that a committee of four members 1, 2, 3 and 4 evaluates three candidates x, y and z to elect for a position. The commitee's evaluations of candidates are given by the following linear preferences:

1	2	3	4
x	x	z	z
y	y	y	y
z	z	x	x

The summation of ranks of the candidates gives the same number of scores 8 for every candidate, and it is impossible to make a choice. However, very often the compromise choice is the candidate y.

Example 2. It is a common practice for scientific journals to accept or reject manuscripts submitted for publication on the basis of reports of two referees. If at least one the referees evaluates the manuscript as 'bad' in a certain sense, the manuscript is rejected. The manuscript is usually accepted if the two referees provide 'positive' opinions. Clearly, this kind of a situation is of noncompensatory nature, and so, the question is: what rule(s) describe(s) the journal's choice to accept a manuscipt?

In a recent series of three articles by Aleskerov, Yakuba and Yuzbashev [7–9] an axiomatic construction of the new aggregation procedure, called the *threshold rule*, has been presented for three-graded rankings, i.e., when the evaluations of alternatives are made by grades 1, 2 and 3 meaning 'bad', 'average' and 'good', respectively. The axioms used are Pairwise Compensation, Pareto Domination, Noncompensatory Threshold and Contraction.

The Pairwise Compensation axiom means that if all agents, but two, evaluate two alternatives equally, and the two agents put 'mutually inverse' grades, then the two alternatives have the same rank in the social decision (which may also be interpreted as 'anonymity of grades').

The Pareto Domination axiom states that if the grades of all agents for one alternative are not less than for the second alternative and the grade of at least one agent for the first alternative is strictly greater than that of the

[1] The idea of this example was proposed by Professor P. Pattanaik.

second one, then in the social ranking the first alternative has a higher rank than the second alternative.

The Noncompensatory Threshold axiom reveals the main idea of the threshold aggregation: if at least one agent evaluates an alternative as ‘bad’, then, no matter how many ‘good’ grades it admits, in the social ranking this alternative is ranked lower than any alternative evaluated as ‘average’ by all agents. In this context the Contraction means that if for two alternatives the evaluations of some agent are equal, then the agent may be ‘excluded’ from the consideration when the social ranking is constructed, and the social decision is achieved by remaining agents’ evaluations.

It was shown by Aleskerov, Yakuba and Yuzbashev [7–9] that the threshold rule is the only rule satisfying the above axioms. In the context of three-graded rankings the threshold rule aggregates individual preferences in the following way: if the number of ‘bad’ evaluations of the first alternative is greater than that of the second one, then the first alternative has lower rank in the social ranking, and if the numbers of ‘bads’ for both alternatives are equal and the number of ‘average’ evaluations of the first alternative is greater than that of the second alternative, then the second alternative is socially more preferable.

In this paper we extend the notion of the threshold rule to the case when the agents’ evaluations are represented by the m-valued grades with an arbitrary integer $m \geq 3$ and show that the threshold rule is the only rule, which satisfies the abovementioned appropriately interpreted axioms. In this model low evaluations of some agents are of main concern: they cannot be compensated by high grades of the other agents. This concerns the situation when the quality or perfectness of alternatives is of great value and interest. On the other hand, an aggregation procedure can be made taking carefully into account high grades of agents: this is the case when we are interested in at least one good feature of alternatives. It is exactly the dual model, and it has all advantages of the dual model including the axiomatic construction of a social decision function.

Yet, one more remark ought to be made concerning an interpretation of the Noncompensatory property. Under this property, any agent giving a low grade to an alternative puts it down in the social decision as compared to an alternative with average grades. Thus, marginal opinions may strongly influence the social decision.

The main results of this paper have been presented in [3] and part of them is published without proofs in Sections 1–3 from [12].

The paper is organized as follows. In Section 2 we present necessary definitions and the main result, Theorem 1. In Section 3 we compare the threshold rule, the simple majority rule and Borda’s rule and show that they produce in general different social rankings on the same individual profile. In Section 4 we show that the equivalence classes of the weak order P, generated by the threshold rule, and the indifference classes generated by P coincide and establish the key properties of monotone representatives of the indifference classes. In Section 5 we develop the dual threshold aggregation axiomatics. Section 6 contains an analysis of manipulability of the threshold rule and its comparison with several other rules.

2 The main result

Let X be a finite set of alternatives of cardinality $|X| \geq 2$, $N = \{1, 2, \ldots, n\}$ be a set of $n \geq 2$ agents and $M = \{1, 2, \ldots, m\}$ be a set of ordered grades $1 < 2 < \ldots < m$ with $m \geq 3$. An evaluation procedure for alternatives from X is a map of the form $E : X \times N \to M$, which assigns to each alternative $x \in X$ and each agent $i \in N$ a grade $x_i = E(x, i) \in M$. As a result of the evaluation procedure E each alternative $x \in X$ is characterized by a collection of n grades $x_1, \ldots, x_n$, i.e.,

$$X \ni x \longmapsto \widehat{x} = E(x, \cdot) = (x_1, \ldots, x_n) \in M^n,$$

where $M^n = \{(x_1, \ldots, x_n) : x_i \in M \text{ for each } i \in N\}$ is the set of all n-dimensional vectors with components from M. In practice the vector-grades $\widehat{x} = (x_1, \ldots, x_n)$ for the alternative x may represent expert grades, questionnaire data, test data, etc.

The set $\widehat{X} = \{\widehat{x} : x \in X\} \subset M^n$ is an individual profile on X. The problem is to rank the elements of X making use of the individual profile $\widehat{X}$. By a *ranking* of X we mean a complete and transitive binary relation on X. Since $\widehat{X} \subset M^n$ and each alternative $x \in X$ is completely characterized by its profile vector $\widehat{x}$, with no loss of generality throughout the paper we assume that $X = \widehat{X} = M^n$, and so,

$$x \in X \quad \text{iff} \quad x = \widehat{x} = (x_1, \ldots, x_n) \in M^n \quad \text{with} \quad x_i \in M,$$

where 'iff' means as usual 'if and only if'.

The following notation will be used throughout the paper. Given $x, y \in X$, we write $x \succcurlyeq y$ to denote the condition $x_i \geq y_i$ for all $i \in N$, and we write $x \succ y$ to mean that $x \succcurlyeq y$ and there is an $i_0 \in N$ such that $x_{i_0} > y_{i_0}$. Note that the partial order relations $\succcurlyeq$ and $\succ$ on X do not solve the problem of ranking of X, because not all profile vectors from X are comparable using these relations. Also, given $x \in X$ and $j \in M$, we denote by $v_j(x)$ the number of grades j in the vector $x = (x_1, \ldots, x_n)$:

$$v_j(x) = |\{i \in N : x_i = j\}|. \tag{1}$$

Note that $0 \leq v_j(x) \leq n$ for all $x \in X$ and $j \in M$ and

$$\sum_{j=1}^{m} v_j(x) = v_1(x) + v_2(x) + \cdots + v_m(x) = n \quad \text{for all} \quad x \in X. \tag{2}$$

Finally, given $x \in X$, we set

$$V_k(x) = \sum_{j=1}^{k} v_j(x) \quad \text{if} \quad 1 \leq k \leq m \quad \text{and} \quad V_0(x) = 0, \tag{3}$$

so that equality (2) can be simply written as $V_m(x) = n$, $x \in X$.

By a *social decision function* on X we mean a function $\varphi : X \to \mathbb{R}$ satisfying the following properties: given $x, y \in X$, we have: (a) the inequality $\varphi(x) > \varphi(y)$ holds iff the alternative x is socially (strictly) more preferable than the alternative y (in the sense to be made precise below), and (b) $\varphi(x) = \varphi(y)$ iff the alternatives x and y are socially indifferent.

We look for a social decision function $\varphi : X \to \mathbb{R}$, which satisfies the following three axioms (A.1), (A.2) and (A.3).

(A.1) (Pairwise Compensation): if $x, y \in X$ and $v_j(x) = v_j(y)$ for all $1 \le j \le m-1$, then $\varphi(x) = \varphi(y)$.

(A.2) (Pareto Domination): if $x, y \in X$ and $x \succ y$, then $\varphi(x) > \varphi(y)$.

(A.3) (Noncompensatory Threshold and Contraction): for each natural number $3 \le k \le m$ the following condition holds:

(A.3.k) if $x, y \in X$, $v_j(x) = v_j(y)$ for all $1 \le j \le m-k$ (if $k = m$, this condition is omitted), $v_{m-k+1}(x)+1 = v_{m-k+1}(y) \neq n - V_{m-k}(y)$, $V_{m-k+2}(x) = n$ and $V_{m-k+1}(y) + v_m(y) = n$, then $\varphi(x) > \varphi(y)$.

Recall that the binary relation $\angle = \angle_k$ on the set $\mathbb{R}^k$ of all k-dimensional vectors with real components is said to be the *lexicographic ordering* if, given $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ from $\mathbb{R}^k$, we have: $u \angle v$ in $\mathbb{R}^k$ iff there exists an $1 \le i \le k$ such that $u_j = v_j$ for all $1 \le j \le i-1$ (with no condition if $i = 1$) and $u_i < v_i$. It is well known (e. g., [14]) that $\angle$ is a linear order on $\mathbb{R}^k$; more precisely, $\angle$ is transitive (i.e., if $u \angle v$ and $v \angle w$, then $u \angle w$), the negation of $\angle$ is of the form: $\neg(u \angle v)$ iff $v \angle u$ or $v = u$, and $\angle$ is trichotomous (i.e., either $u = v$, or $u \angle v$, or $v \angle u$).

Setting

$$v(x) = (v_1(x), \ldots, v_{m-1}(x)) \in \{0, 1, \ldots, n\}^{m-1} \quad \text{for} \quad x \in X, \tag{4}$$

the property $v(x) \angle v(y)$ in $\mathbb{R}^{m-1}$ will be called the *threshold rule* for the comparison of alternatives x and y (with respect to the number of low grades). We say that a binary relation P on X is *generated* by the threshold rule if $P = \{(x, y) \in X \times X : v(x) \angle v(y)\}$. In other words, given $x, y \in X$, we have $(x, y) \in P$ iff $v(x) \angle v(y)$, which can be interpreted in the sense that the alternative x is socially (strictly) more preferable than the alternative y.

The main properties of P are straightforward consequences of the properties of the lexicographic ordering: given $x, y, z \in X$, we have:

(P.1) if $(x, y) \in P$ and $(y, z) \in P$, then $(x, z) \in P$ (transitivity of P);

(P.2) $(x, y) \notin P$ is equivalent to $(y, x) \in P$ or $v(y) = v(x)$ (negation of P);

(P.3) either $v(x) = v(y)$, or $(x, y) \in P$, or $(y, x) \in P$ (generalized "connectedness" of P);

(P.4) $(x, x) \notin P$ (irreflexivity of P);

(P.5) if $(x, y) \notin P$ and $(y, z) \notin P$, then $(x, z) \notin P$ (negative transitivity of P);

(P.6) $(x, y) \notin P$ or $(y, x) \notin P$ (completeness of $P^c = X^2 \setminus P$).

A binary relation P satisfying properties (P.1), (P.4) and (P.5) is commonly known as a *weak order* on X. It is also known (cf. Aleskerov [2]) that any weak order P on X is characterized by the family of its equivalence classes, whose construction is recalled now. Set $X_1' = \pi(X)$ where, given nonempty $A \subset X$, $\pi(A) = \{x \in A : (y,x) \notin P \text{ for all } y \in A\}$ is the choice function for P (cf. Aizerman and Aleskerov [1, Section 2.3]). Inductively, if $\ell \geq 2$ and nonempty subsets $X_1', \dots, X_{\ell-1}'$ of X such that $\bigcup_{k=1}^{\ell-1} X_k' \neq X$ are already defined, we put $X_\ell' = \pi\big(X \setminus (\bigcup_{k=1}^{\ell-1} X_k')\big)$. Since X is finite, there exists a unique positive integer $s = s(X)$ such that $X = \bigcup_{\ell=1}^{s} X_\ell'$. Now, setting $X_\ell = X_{s-\ell+1}'$ for $\ell = 1, 2, \dots, s$, the disjoint collection $\{X_\ell\}_{\ell=1}^{s}$ is said to be the *family of equivalence classes* of the weak order P, and has the following property: given $x, y \in X$, $(x,y) \in P$ iff there exist two integers k and ℓ with $1 \leq k < \ell \leq s$ such that $x \in X_\ell$ and $y \in X_k$. This property shows that the alternative x is more preferable than the alternative y iff x lies in an equivalence class with a greater ordinal number, and so, this defines the canonical (strict) ranking of X. The value $s = s(X)$ for the relation P generated by the threshold rule will be calculated below in Lemma 1(b).

We say that a function $\varphi : X \to \mathbb{R}$ is *coherent* with the family $\{X_\ell\}_{\ell=1}^{s}$ of equivalence classes of the weak order P on X if, given $x, y \in X$, the inequality $\varphi(x) > \varphi(y)$ holds iff there exist $1 \leq k < \ell \leq s$ such that $x \in X_\ell$ and $y \in X_k$.

The main result of this paper is the following

Theorem 1 *A social decision function $\varphi : X \to \mathbb{R}$ satisfies axioms* (A.1), (A.2) *and* (A.3) *iff it is coherent with the family of equivalence classes of the weak order P on X generated by the threshold rule $v(x) \angle v(y)$ in $\mathbb{R}^{m-1}$.*

This theorem will be proved in Section 6. A certain interpretation of it is in order. Given a binary relation P on X and a function $\varphi : X \to \mathbb{R}$, if for all $x, y \in X$ we have

$$(x,y) \in P \qquad \text{iff} \qquad \varphi(x) > \varphi(y), \tag{5}$$

then P is said to be representable by means of φ or, shortly, *φ-representable*, and φ is said to be a *preference function for P*. Taking this into account as well as the definitions preceding Theorem 1, we can reformulate Theorem 1 as follows: *a social decision function on X satisfies the axioms Pairwise Compensation, Pareto Domination, Noncompensatory Threshold and Contraction iff it is a preference function for the binary relation on X generated by the threshold rule.*

3 A comparison with the known rules

In this Section we construct an example, for which the simple majority rule, the Borda voting rule and the threshold rule produce different social decisions.

Let $X = \{x, y, z\}$ be a set of three different alternatives, $N = \{1, \dots, 13\}$ a set of $n = 13$ voters and $M = \{1, 2, 3\}$ the set of grades (i.e., $m = 3$). Consider the following linear preferences of voters from N:

3 voters	4 voters	6 voters	rank
x	x	y	3
y	z	z	2
z	y	x	1

This means that for the first three voters x is the most preferable alternative, y is the next one and z is the less preferable alternative, and likewise for the other voters. The problem of voting is to construct a (linear) binary relation on X corresponding to the social decision of the society N.

(a) According to the *simple majority rule* the pair of alternatives (x, y) is included into the social decision (relation) if the preference of the form "x is more preferable than y" occurs among the simple majority of voters. In our example we have: for $3+4 = 7$ voters x is more preferable than y, for $3+6 = 9$ voters y is more preferable than z and for $3+4 = 7$ voters x is more preferable than z (and there is no simple majority among the other possibilities). Thus, the social decision is $\{(x, y), (y, z), (x, z)\}$, and so, x is more preferable than y, which in its turn is more preferable than z, and the winner is x.

Let us note that the axioms for the simple majority rule and plurality rule have been laid by May [16].

(b) In the Borda voting procedure to each alternative x from X each voter $i \in N$ associates some rank $\rho_i(x)$ in such a way that the more preferable the alternative the higher the rank. In our example for the first voter among the first three voters we have: $\rho_1(x) = 3$, $\rho_1(y) = 2$ and $\rho_1(z) = 1$, and likewise for the remaining voters. Then we set $\rho(x) = \sum_{i \in N} \rho_i(x)$ for all $x \in X$. According to the *Borda voting rule* an alternative x is socially more preferable than an alternative y if $\rho(x) > \rho(y)$. For the example above we have:

$$\begin{aligned} \rho(y) = 3 \cdot 2 + 4 \cdot 1 + 6 \cdot 3 = 28 > \rho(x) = (3+4) \cdot 3 + 6 \cdot 1 = 27 > \\ > \rho(z) = 3 \cdot 1 + (4+6) \cdot 2 = 23, \end{aligned}$$

and so, y is more preferable than x and x is more preferable than z, and the winner is y.

An axiomatization of Borda's rule was developed by Young [23].

(c) Interpreting the ranks of alternatives from (b) as the grades, for the example above we have (the asterisk denotes the ordered vector grades):

$$v_1(x) = 6,\ v_2(x) = 0,\ v_3(x) = 7, \text{ or } x^* = (\underbrace{1,1,1,1,1,1}_{6}, \underbrace{3,3,3,3,3,3,3}_{7}),$$

$$v_1(y) = 4,\ v_2(y) = 3,\ v_3(y) = 6, \text{ or } y^* = (\underbrace{1,1,1,1}_{4}, \underbrace{2,2,2}_{3}, \underbrace{3,3,3,3,3,3}_{6}),$$

$$v_1(z) = 3,\ v_2(z) = 10,\ v_3(z) = 0, \text{ or } z^* = (\underbrace{1,1,1}_{3}, \underbrace{2,2,2,2,2,2,2,2,2,2}_{10}).$$

Since $v_1(z) = 3 < v_1(y) = 4 < v_1(x) = 6$, then $v(z) \angle_2 v(y) \angle_2 v(x)$, and so, according to the *threshold rule* z is socially more preferable than y and y is more preferable than x, and the winner is z.

4 Monotone representatives and indifference classes

Since the binary relation P on X generated by the threshold rule is a weak order, the *indifference* relation I is defined as

$$I = \{(x, y) \in X \times X : (x, y) \notin P \text{ and } (y, x) \notin P\}. \tag{6}$$

Clearly, I is an equivalence relation on X (i.e., it is reflexive, symmetric and transitive) and, by virtue of (P.2) and (P.3), we have:

$$I = \{(x, y) \in X \times X : v(x) = v(y)\}. \tag{7}$$

Then the *indifference class* of an alternative $x \in X$ is the set

$$I_x = \{y \in X : (y, x) \in I\} = \{y \in X : v(y) = v(x)\}, \tag{8}$$

and, as usual, given $x, y \in X$, we find: $I_x = I_y$ iff $(x, y) \in I$, $I_x \cap I_y = \varnothing$ iff $(x, y) \notin I$, and $X = \bigcup_{x \in X} I_x$ (disjoint union). We denote by X/I the quotient set $\{I_x : x \in X\}$ of all the indifference classes with respect to I.

In this way the binary relation $R = P \cup I$ is a canonical *ranking* of X: R is transitive ($(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$) and complete (given $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$). However, throughout the paper we prefer to deal with the strict preference relation P.

Given $I_x \in X/I$ for some $x \in X$, by virtue of (8), the vector $v(I_x) = v(y) = v(x)$ is well defined for any $y \in I_x$. Then the quotient binary relation P/I given by

$$P/I = \{(I_x, I_y) \in (X/I) \times (X/I) : v(I_x) \angle v(I_y) \text{ in } \mathbb{R}^{m-1}\}$$

is a *linear order* on X/I. In fact, since the transitivity and irreflexivity of P/I are clear, it suffices to verify only the connectedness of P/I, i.e., if $I_x, I_y \in X/I$ and $I_x \neq I_y$, then $(I_x, I_y) \in P/I$ or $(I_y, I_x) \in P/I$. Indeed, $I_x \neq I_y$ implies $I_x \cap I_y = \varnothing$ and $(x, y) \notin I$. Thus, $v(x) \neq v(y)$, which gives $v(I_x) \neq v(I_y)$, and by the completeness of the lexicographic ordering $\angle = \angle_{m-1}$ we obtain $v(I_x) \angle v(I_y)$ or $v(I_y) \angle v(I_x)$.

We note that, by virtue of (4) and (2), the equality $v(y) = v(x)$ in (8) actually means that $v_j(y) = v_j(x)$ for all $j \in M$, that is, the vector y can be obtained from the vector x (and vice versa) by a permutation of its coordinates:

$$I_x = \{y \in X : \exists \text{ a permutation } \sigma \text{ of } N \text{ such that } y = x \circ \sigma\},$$

where the equality $y = x \circ \sigma$ involving the composition $x \circ \sigma$ means as usual that $y_i = x_{\sigma(i)}$ for all $i \in N$.

In order to facilitate the treatment of indifference classes I_x from X/I, in each class I_x we select a 'principal' representative $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in I_x$, whose coordinates are ordered in ascending order: $x_1^* \leq x_2^* \leq \ldots \leq x_n^*$ or

$$x^* = (\overbrace{\underbrace{1, \ldots, 1}_{v_1(x)}, \underbrace{2, \ldots, 2}_{v_2(x)}, \ldots, \underbrace{m-1, \ldots, m-1}_{v_{m-1}(x)}, \underbrace{m, \ldots, m}_{v_m(x)}}^{n}), \tag{9}$$

where the numbers $v_j(x)$ under the braces denote the lengths of the corresponding underbraced subvectors. The alternative x^*, called the *monotone representative* of the class I_x (or simply of the vector x), is uniquely determined, although it can be obtained from x by different permutations of its coordinates. It is clear from the above that $v_j(x^*) = v_j(x)$ for all $j \in M$, or $v(x^*) = v(x)$, and so, $I_x = I_{x^*}$ for all $x \in X$. We denote by $X^* = \{x^* : x \in X\}$ the subset of X of all monotone representatives and by P^*—the restriction of P to $X^* \times X^*$.

Let us note that, given $x, y \in X$, we have:

$$(x, y) \in P \quad \text{iff} \quad (I_x, I_y) \in P/I \quad \text{iff} \quad (x^*, y^*) \in P^*$$

and

$$(x, y) \in I \quad \text{iff} \quad I_x = I_y \quad \text{iff} \quad x^* = y^*. \tag{10}$$

It follows from (P.1), (P.4) and (P.3) that P^* is a linear order on X^* and that the bijection $b : X/I \to X^*$, defined by $b(I_x) = x^*$ for all $x \in X$, is linear order preserving in the sense that $(I_x, I_y) \in P/I$ iff $(b(I_x), b(I_y)) \in P^*$; in other words, the pairs $(X/I, P/I)$ and (X^*, P^*) are *linear order isomorphic.*

Thus, we can work with the set X^* equipped with the linear order P^* instead of the quotient linear order set $(X/I, P/I)$.

Lemma 1 (a) $|X/I| = |X^*| = C^{m-1}_{n+m-1} = C^{n}_{n+m-1}$, *where* $C^k_n = \frac{n!}{k!(n-k)!}$ *is the usual binomial coefficient and* $|A|$ *denotes the number of elements in the set* A *under consideration.*

(b) $\{X_\ell\}_{\ell=1}^s = X/I$, *i.e., the family of all equivalence classes of the weak order* P *coincides with the quotient set of all the indifference classes with respect to* I*; hence,* $s = s(X) = C^{m-1}_{n+m-1}$.

The following two lemmas are of fundamental importance for the whole subsequent material. In Lemma 2 we show that the operation of taking the monotone representative $X \ni x \mapsto x^* \in X^*$ preserves the natural partial order relations $\succcurlyeq$ and $\succ$ on X, and in Lemma 3 we show that the relations $x^* \succcurlyeq y^*$ and $x^* \succ y^*$ can be characterized in terms of quantities from (1) and (3).

Lemma 2 *Given* $x, y \in X$*, we have*:

(a) *if* $x \succcurlyeq y$*, then* $x^* \succcurlyeq y^*$;

(b) *if* $x \succ y$*, then* $x^* \succ y^*$.

Lemma 3 *Given* $x, y \in X$*, we have*:

(a) $x^* \succcurlyeq y^*$ *iff* $V_k(x) \le V_k(y)$ *for all* $1 \le k \le m-1$;

(b) $x^* \succ y^*$ *iff there exists a* $1 \le k \le m-1$ *such that* $v_j(x) = v_j(y)$ *for all* $1 \le j \le k-1$ *(no condition if* $k = 1$*),* $v_k(x) < v_k(y)$ *and* $V_p(x) \le V_p(y)$ *for all* $k+1 \le p \le m-1$ *(with no last condition if* $k = m-1$*).*

5 The dual threshold aggregation

If the utmost perfection (quality) of alternatives is of main concern, we can apply the threshold rule to rank the set of alternatives. However, if we are interested in at least one good feature of alternatives, we should employ a different, but related, aggregation procedure, which will be called the *dual threshold aggregation*. Such a dual model for three-graded rankings had already been mentioned by Aleskerov and Yakuba [8]. In this Section we develop an axiomatic theory of the dual threshold aggregation in the general case.

Given an alternative $x \in X = M^n$, we set

$$\overline{v}(x) = (v_m(x), v_{m-1}(x), \dots, v_2(x)) \in \{0, 1, \dots, n\}^{m-1}.$$

The property $\overline{v}(y) \angle \overline{v}(x)$ in $\mathbb{R}^{m-1}$ will be called the *dual threshold rule* for the comparison of alternatives $x, y \in X$ (with respect to the number of high grades), and a binary relation on X of the form

$$\overline{P} = \{(x, y) \in X \times X : \overline{v}(y) \angle \overline{v}(x) \text{ in } \mathbb{R}^{m-1}\}$$

is said to be *generated* by the dual threshold rule. In other words, given two alternatives $x, y \in X$, we have $(x, y) \in \overline{P}$ iff $\overline{v}(y) \angle \overline{v}(x)$, and we say that x is (dually) strictly more preferable than y.

We are going to reduce the dual aggregation theory to the aggregation theory developed above. In order to do this, we introduce a permutation r of the set M as follows:

$$r(j) = m - j + 1 \qquad \text{for all} \qquad j \in \{1, 2, \dots, m\}.$$

Note that r is a bijection between $\{1, 2, \dots, m-1\}$ and $\{m, m-1, \dots, 2\}$, reversing the order of the numbers, and so, its self composition $r^2 = r \circ r$ is the identity on $\{1, 2, \dots, m-1\}$ and on $\{m, m-1, \dots, 2\}$: $r(r(j)) = j$ for all j. Given $x = (x_1, \dots, x_n) \in X = \{1, 2, \dots, m\}^n$, we set

$$\mathbf{r}(x) = (r(x_1), r(x_2), \dots, r(x_n)) = (m - x_1 + 1, m - x_2 + 1, \dots, m - x_n + 1),$$

and note that $\mathbf{r}(\mathbf{r}(x)) = x$, i.e., $\mathbf{r}(x') = x$ iff $x' = \mathbf{r}(x)$.

The following two properties (11) and (12) of $\mathbf{r}$ will be of significance:

$$v_j(\mathbf{r}(x)) = v_{r(j)}(x) \quad \text{for all} \quad x \in X \quad \text{and} \quad 1 \le j \le m. \tag{11}$$

In fact, we have:

$$\begin{aligned} v_j(\mathbf{r}(x)) &= |\{i \in N : r(x_i) = j\}| = |\{i \in N : m - x_i + 1 = j\}| = \\ &= |\{i \in N : x_i = m - j + 1\}| = |\{i \in N : x_i = r(j)\}| = v_{r(j)}(x). \end{aligned}$$

It follows that $v_j(x) = v_{r(j)}(\mathbf{r}(x))$ and

$$\overline{v}(x) = v(\mathbf{r}(x)) \quad \text{and} \quad \overline{v}(\mathbf{r}(x)) = v(x) \quad \text{for all} \quad x \in X, \tag{12}$$

because

$$\overline{v}(x) = (v_m(x), v_{m-1}(x), \ldots, v_2(x)) = (v_{r(1)}(x), v_{r(2)}(x), \ldots, v_{r(m-1)}(x)) = \\ = (v_1(\mathbf{r}(x)), v_2(\mathbf{r}(x)), \ldots, v_{m-1}(\mathbf{r}(x))) = v(\mathbf{r}(x)).$$

Now, given $x, y \in X$, we have:

$$(x, y) \in \overline{P} \text{ iff } \overline{v}(y) \angle \overline{v}(x) \text{ iff } v(\mathbf{r}(y)) \angle v(\mathbf{r}(x)) \text{ iff } (\mathbf{r}(y), \mathbf{r}(x)) \in P \quad (13)$$

or, equivalently, $(x, y) \in P$ iff $(\mathbf{r}(y), \mathbf{r}(x)) \in \overline{P}$.

By virtue of (13), the relation $\overline{P}$ on X satisfies the properties (P.1)–(P.6) (if we replace P in these properties by $\overline{P}$), and so, $\overline{P}$ is a weak order on X. For instance, the negation of $\overline{P}$ is of the form: given $x, y \in X$, $(x, y) \notin \overline{P}$ iff $(y, x) \in \overline{P}$ or $v(y) = v(x)$; in fact, it follows from (13) that

$$(x, y) \notin \overline{P} \text{ iff } (\mathbf{r}(y), \mathbf{r}(x)) \notin P \text{ iff } [(\mathbf{r}(x), \mathbf{r}(y)) \in P \text{ or } v(\mathbf{r}(x)) = v(\mathbf{r}(y))] \\ \text{iff } [(y, x) \in \overline{P} \text{ or } \overline{v}(x) = \overline{v}(y)],$$

and it remains to note that, in view of (2), the condition "$v_j(x) = v_j(y)$ for all $2 \le j \le m$" is equivalent to the condition "$v_j(x) = v_j(y)$ for all $1 \le j \le m-1$". This observation also shows that the indifference relation $\overline{I}$ on X generated by $\overline{P}$ coincides with the indifference relation I:

$$\overline{I} = \{(x, y) : (x, y) \notin \overline{P} \text{ and } (y, x) \notin \overline{P}\} = \{(x, y) : v(x) = v(y)\} = I.$$

In order to treat the axiomatics of preference functions for the relation $\overline{P}$, we note that if φ is a preference function for P and ψ is a preference function for $\overline{P}$, then, given $x, y \in X$, we have:

$$\psi(x) > \psi(y) \text{ iff } (x, y) \in \overline{P} \text{ iff } (\mathbf{r}(y), \mathbf{r}(x)) \in P \text{ iff } \varphi(\mathbf{r}(y)) > \varphi(\mathbf{r}(x)) \\ \text{iff } [-\varphi(\mathbf{r}(x)) > -\varphi(\mathbf{r}(y))]. \quad (14)$$

We conclude that φ is a preference function for P iff the function $\overline{\varphi}$, defined by $\overline{\varphi}(x) = -\varphi(\mathbf{r}(x))$ for all $x \in X$, is a preference function for $\overline{P}$, and vice versa: $\overline{\varphi}$ is a preference function for $\overline{P}$ iff the function φ, defined for $x \in X$ by $\varphi(x) = -\overline{\varphi}(\mathbf{r}(x))$, is a preference function for P. It follows from Theorem 1 that a function $\overline{\varphi} : X \to \mathbb{R}$ is a preference function for $\overline{P}$ iff the function $\varphi(x) = -\overline{\varphi}(\mathbf{r}(x))$ satisfies axioms (A.1)–(A.3), and by virtue of (14) with ψ replaced by $\overline{\varphi}$, given $x, y \in X$, we have:

$$\overline{\varphi}(x) > \overline{\varphi}(y) \quad \text{iff} \quad \varphi(x') > \varphi(y'), \quad \text{where } x' = \mathbf{r}(y) \text{ and } y' = \mathbf{r}(x).$$

So, replacing x by $\mathbf{r}(y)$ and y by $\mathbf{r}(x)$ in axioms (A.1)–(A.3) and taking into account equalities (11) and (12), we obtain the following (dual) axioms for function $\overline{\varphi}$. Axioms (A.1) and (A.2) remain the same, because conditions "$\overline{v}(x) = \overline{v}(y)$" and "$v(x) = v(y)$" are equivalent, and if $x \succ y$, then $\mathbf{r}(y) \succ \mathbf{r}(x)$, and so, $\varphi(\mathbf{r}(y)) > \varphi(\mathbf{r}(x))$ implying $\overline{\varphi}(x) > \overline{\varphi}(y)$. The third dual axiom assumes the following form:

($\overline{\text{A}}$.3) (Noncompensatory Dual Threshold and Contraction): for each integer $3 \le k \le m$ the following condition holds:

($\overline{\text{A}}$.3.k) if $x, y \in X$, $v_j(x) = v_j(y)$ for all $k+1 \le j \le m$ (if $k = m$, this condition is absent), $v_k(y) + 1 = v_k(x) \neq V_k(x)$, $V_{k-2}(y) = 0$ and $V_{k-1}(x) = v_1(x)$, then $\overline{\varphi}(x) > \overline{\varphi}(y)$.

The observations above lead to the following corollary of Theorem 1.

Theorem 2 *A social decision function* $\overline{\varphi} : X \to \mathbb{R}$ *satisfies axioms* (A.1), (A.2) *and* ($\overline{\text{A}}$.3) *iff it is coherent with the family of equivalence classes of the weak order* $\overline{P}$ *on* X *generated by the dual threshold rule* $\overline{v}(y) \angle \overline{v}(x)$ *in* $\mathbb{R}^{m-1}$.

6 Manipulation of the threshold rule

Following Aleskerov et al. [4] we present here a study of the manipulability of threshold aggregation rule and compare its manipulability with several other rules. The extent to which social choice rules are manipulable is studied in Kelly [15], Aleskerov and Kurbanov [6], Favardin and Lepelley [13] who consider well-known anonymous and neutral (hence multiple-valued) social choice rules and analyse their single-valued versions obtained by assuming a tie-breaking linear order over alternatives. This assumption simplifies the computational difficulties embedded in the problem of estimating the degree of manipulability. On the other hand, by breaking the symmetry between candidates, it risks to distort the computational results. Pritchard and Wilson [20] analyse the manipulability of scoring rules under a random tie-breaking rule which preserves neutrality among alternatives.

We explore the degree of manipulability of several multi-valued social choice rules, by extending manipulability indices defined for single-valued social choice rules to the multi-valued case. Our analysis requires to extend preferences over alternatives to sets of alternatives, which we do by using two alternative methods based on lexicographic comparisons. We reveal the degree of manipulability of seven social choice rules, either by theoretical investigations or by computational experiments. We consider an environment of four and five alternatives, hence extending the findings of Aleskerov et al. [5] derived for an environment restricted to three alternatives.

We analyze the following rules:

- **Plurality Rule**
- **2-Approval**
- **Borda's Rule**
- **Black's Procedure**
- **Threshold rule**
- **Uncovered Set I:** Construct lower contour set $L(x)$ of relation μ and binary relation δ as follows:

$$x \delta y \iff L(x) \supset L(y).$$

  Then undominated alternatives on δ are chosen, i.e. $x \in C(\overrightarrow{P}) \iff \left[\neg \exists\, y \in A \mid y \delta x\right]$.

– **Strongest q-Pareto Simple Majority (Aleskerov 1999)**
  Social choice is defined as $C(\overrightarrow{P}) = \bigcap_{I \in \mathcal{T}} f(\overrightarrow{P}; I, q)$, where $f(\overrightarrow{P}; I, q) = \left\{x \in A : |\bigcap_{i \in I} D_i(x)| \leq q\right\}$ and $\mathcal{T} = \{I \subset \mathbb{N} \mid |I| = \lceil n/2 \rceil\}$. The alternative is to be chosen if it is Pareto optimal in each simple majority coalition with $q = 0$. If there is no such an alternatives, then $q = 1$, $q = 2$, etc., is considered until the choice will not be empty. In other words, for every simple coalition we find all alternatives which are Pareto undominated (there are no alternatives which are better for every voter in coalition). Then we find the intersection of such sets for all simple coalitions. If it is empty it means that there no such alternatives which are Pareto undominated for all coalitions and then we look for alternatives dominated no more than by 1 alternative and so on.

Every agent i is assumed to have an extended preference EP_i over A which is induced by her preference P_i over A. We consider two methods to obtain EP_i from P_i, both of which are based on lexicographic comparisons used by Pattanaik [19]. The methods we consider are the leximax and leximin extensions, as described by Ozyurt and Sanver [18].

Under the leximax extension, two sets are compared according to their best elements. If these are the same, then the ordering is made according to the second best elements, etc. The elements according to which the sets are compared will disagree at some step — except possibly when one set is a subset of the other, in which case the smaller set is preferred[2]. To speak formally, take any $P_i \in L$ and any distinct $X, Y \in A$. Write $X = \{x_1, \ldots, x_{|X|}\}$, $Y = \{y_1, \ldots, y_{|Y|}\}$ and let, without loss of generality, $x_{j+1} P_i x_j \ \forall\, j \in \{1, \ldots, |X|-1\}$ and $y_{j+1} P_i y_j \ \forall\, j \in \{1, \ldots, |Y|-1\}$. The leximax extended preference EP_i is defined as follows:

1. If $|X| = |Y|$, then $X \, EP_i \, Y$ iff $x_h P_i y_h$ for the smallest $h \in \{1, \ldots, k\}$ for which $x_h \neq y_h$.
2. If $|X| \neq |Y|$ and $\exists\, h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$, then $X \, EP_i \, Y$ iff $x_h P_i y_h$ for the smallest $h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$.
3. If $|X| \neq |Y|$ and $x_h = y_h \ \forall\, h \in \{1, \ldots, \min\{|X|, |Y|\}\}$, then $X \, EP_i \, Y$ iff $|X| < |Y|$.

The concept of a leximin extension is similarly defined while it is based on ordering two sets according to a lexicographic comparison of their worst elements. Again the elements according to which the sets are compared will disagree at some step — except possibly when one set is a subset of the other, in which case the larger set is preferred[3]. So given any $P_i \in L$ and any distinct $X, Y \in A$ where $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$ are such that

[2] This is exactly how words are ordered in a dictionary. For example, given three alternatives a, b and c, the leximax extension of the ordering $a\ b\ c$ is $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, c\}$, $\{b\}$, $\{b, c\}$, $\{c\}$.

[3] For example, the leximin extension of the ordering $a\ b\ c$ is $\{a\}$, $\{a, b\}$, $\{b\}$, $\{a, c\}$, $\{a, b, c\}$, $\{b, c\}$, $\{c\}$.

$x_{j+1}P_ix_j$ $\forall\, j \in \{1,\dots,|X|-1\}$ and $y_{j+1}P_iy_j$ $\forall\, j \in \{1,\dots,|Y|-1\}$, the leximin extended preference EP_i is defined as follows:

1. If $|X| = |Y|$, then $X\, EP_i\, Y$ iff $x_hP_iy_h$ for the greatest $h \in \{1,\dots,k\}$ for which $x_h \neq y_h$.
2. If $|X| \neq |Y|$ and $\exists\, h \in \{1,\dots,\min\{|X|,|Y|\}\}$ for which $x_h \neq y_h$, then $X\, EP_i\, Y$ iff $x_hP_iy_h$ for the smallest $h \in \{1,\dots,\min\{|X|,|Y|\}\}$ for which $x_h \neq y_h$.
3. If $|X| \neq |Y|$ and $x_h = y_h$ $\forall\, h \in \{1,\dots,\min\{|X|,|Y|\}\}$, then $X\, EP_i\, Y$ iff $|X| > |Y|$.

We consider the following manipulability indices.

Number of alternatives being m, the total number of possible linear orders is equal to $m!$, and total number of profiles with n agents is equal to $(m!)^n$. Kelly [15] introduces the following index (which we call Kelly's index and denote as K) to measure the degree of manipulability of social choice rules:

$$K = \frac{d_0}{(m!)^n},$$

where d_0 is the number of profiles in which manipulation takes place.

Aleskerov and Kurbanov [6] introduce an index to measure *the freedom of manipulation*. In Aleskerov et al. (2010) we introduced two similar indices: *The degree of nonsensitivity to preference change and the probability of getting worse*. Let us note that for an agent there are $(m!-1)$ linear orders to use instead of her sincere preference. Denote as k^+_{ij} ($i = 1,\dots,n$; $0 \leq k^+_{ij} \leq (m!-1)$) the number of orderings in which voter i is better off in j-th profile. Similarly, k^0_{ij} is the number of orderings in which the result of voting remains the same and k^-_{ij} is the number of orderings in which the voter is worse off. It is obvious that $k^+_{ij} + k^0_{ij} + k^-_{ij} = (m!-1)$. Dividing each k_{ij} to $(m!-1)$ one can find the share of each type of orderings for an agent i in j-th profile. Summing up each share over all agents and dividing it to n one can find the average share in the given profile. Summing the share over all profiles and dividing this sum to $(m!)^n$ we obtain three indices

$$I_1^+ = \frac{\sum_{j=1}^{(m!)^n}\sum_{i=1}^{n} k^+_{ij}}{(m!)^n \cdot n \cdot (m!-1)};\ \ I_1^0 = \frac{\sum_{j=1}^{(m!)^n}\sum_{i=1}^{n} k^0_{ij}}{(m!)^n \cdot n \cdot (m!-1)};\ \ I_1^- = \frac{\sum_{j=1}^{(m!)^n}\sum_{i=1}^{n} k^-_{ij}}{(m!)^n \cdot n \cdot (m!-1)}.$$

It is obvious that $I_1^+ + I_1^0 + I_1^- = 1$.

The indices K and I_1^+, I_1^0, I_1^- are calculated for the rules defined in the next section.

The calculation of indices is performed for 4 and 5 alternatives. For 3, 4 and 5 voters, the respective indices are exhaustively computed (i.e., all possible profiles are checked for manipulability) and for greater number of voters the statistical scheme is used.

In both exhaustive and statistical schemes, for each profile under consideration, all $m!-1$ manipulating orderings for each voter are generated and the respective choice sets of manipulating profiles are compared with the choice of the original profile.

Tables 1 and 2 give the exhaustive computation results for Kelly's index when there are 3 or 4 voters and 4 or 5 alternatives. In order to enable comparisons with the single-valued case, we quote in the TBR column, the results from Aleskerov and Kurbanov [6] where an alphabetical tie-breaking rule is used.

One can see that in most cases, particularly in the case of 4 voters, assuming single-valuedness of the social choice rule, the evaluations underestimate its degree of manipulability. Also note that plurality rule is non-manipulable in the case of 3 voters and leximax extension. This fact expands the same finding of Aleskerov et al. (2010) to the case of 4 and 5 alternatives. The explanaition remains almost the same: The only possible type of profile where a manipulating voter exists is one where every voter has a different best alternative. In this case choice will consist of these best alternatives, for example $\{a, c, e\}$. When we consider voter with preference $a \succ c \succ e$, manipulability of the profile depends on the answer of the question, whether $\{c\}$ is better than $\{a, c, e\}$? For leximax extension the answer is no, and the profile is not manipulable.

In Figures 1, 2 and 3, Kelly's index is shown on the Y-axis and the logarithm of the number of voters is shown on the X-axis. The calculation was made for each number of voters from 3 to 25 and then for 29, 30, 39, 40 and so on up to 100. That explains changes at the figures.

We can make several conclusions from these figures.

1. For Black's Procedure (under Leximin), Strongest q-Pareto Simple Majority and Uncovered Set I, the values of Kelly's index depend on whether an even or odd number of voters are considered. At the same time for rules such as Plurality, q-Approval and Threshold, there is a cycle of length m-'jumps' of the values when the number of voters n is divisible by the number of alternatives m – in Kelly's index. One can see a cycle of length 4 on Figure 1 and of length 5 on Figures 2 and 3. The phenomenon — the presence of the cycle with length of the number of alternatives — is explained by differences in number and cardinality of ties produced by rules. For example, the set $\{a, b, c, d, e\}$ can appear as the result of Plurality voting only in the case when the number of voters is divisible by the number of alternatives. For Plurality rule and 5 alternatives we observe 52%, 71,2%, 67,5%, 58% of single-valued choice as the result of voting for 3, 4, 5 and 6 voters correspondingly.

2. Uncovered Set I is less manipulable for odd number of voters than for even.

On Figures 4 and 5, I_1^+, I_1^0, I_1^- are given on the same graph for 5 alternatives, leximin extension method, 3 and 100 voters, correspondingly.

Note that, increasing the total number of voters decreases the influence of a single voter. For example, when there are 100 voters, about 90 percent of the insincere profiles makes no effect on the voting outcome.

Figure 6 gives the results of calculation of I_1^+ index for 5 alternatives, under the leximin extension. One can see that I_1^+ tends to decrease with the number of voters. The Black's procedure is the best rule from a freedom of manipulation point of view.

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