Optimal Quota in a Mixed Fairness Model[1]

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Abstract

We will analyze indirect weighted voting systems as the Council of the European Union and examine fair distributions of voting weights for the delegates, representing voting bodies of size $n_1, \ldots, n_m$, applying a mixed fairness model. We will see that using a specific quota jointly with mixed voting weights yields a concordance of voting powers and voting weights. We will prove this result and furthermore examine other fairness concepts for the Council of Ministers in the European Union.

1 Motivation

The central question when designing indirect voting systems is to find a fair allocation of the voting weights for all members. Unfortunately the concept of fairness is a rather subjective perception. Therefore, some reasonable and objective definitions are necessary. In addition, the fact that voting weights in general do not represent voting power complicates the characterization of fairness. Therefore, it is indispensable that we foremost introduce the theory of power measurement in section 2. Having introduced these fundamentals, we will give an overview of existing fairness ideas in section 3:

With regard to the "*one man, one vote*" principle, the well-known result of L.S. Penrose states that the a priori voting power of a delegate has to be proportional to $\sqrt{n_i}$ for all $i = 1, \ldots, m$. Following on this theory, W. Słomczyński and K. Życzkowski showed with the *Jagiellonian Compromise*, that assigning the voting weights to $w_i = \sqrt{n_i}/\sum_{j=1}^{m}\sqrt{n_j}$ and using an optimal quota ensures the correspondence of voting weight and voting power. On the other hand the "*one state, one vote*" principle suggests the same treating of each state. To fulfill this objective, all voting weights have

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to be equal, which provides equal voting powers for each state independent from the chosen quota.

In section 4 we will use the ideas of A. Laruelle and M. Widgrén and consider a fairness model which combines these two fairness principles and assigns each delegate the mixed voting weight $w_i(c) = cw_i + (1-c)\frac{1}{m}$ for $c \in (0.5, 1]$. Note that this is a convex combination of the voting weights from above[2]. Furthermore, we will derive and analyze such a mixed model in all details, using the Council of Ministers of the European as our main example. We will see in section 4 and prove in section 5 that a simultaneous application of the quota $q(c) = \frac{1}{2}\left(1 + \sqrt{\sum w_i(c)^2}\right)$ will lead to a concordance of the voting power of each delegate and the mixed voting weight $w_i(c)$.

Our findings correspond perfectly with the existing theory, however provide an even more general result, since for example the *Jagiellonian Compromise* is a special case of our mode with $c = 1$.

2 Measurement of Power

To measure voting power, a game theoretic approach has emerged to be useful to define political or economic bodies as a mathematical model. The theory of *simple (voting) games* was developed by von Neumann and Morgenstern in 1944 [14] and then used by Felsenthal and Machover [5] in the context of measuring power in political and social sciences. In the political bodies that we will consider, we allow different voting weights for the participants, and therefore will use the concept of *weighted voting games* which is a special case of simple voting games.

For a formal definition of this situation, let $N = \{1, \ldots n\}$ be a finite and nonempty set of size n, called the *assembly*. The members of the assembly N are the *voters* or *players* and any set of voters $S \subseteq N$ is called a *coalition*. Each member $i \in N$ is assigned a *voting weight* w_i and furthermore, the *weight of a coalition* S is denoted by $w(S) = \sum_{j \in S} w_j$. A coalition S is said to be *winning* if its weight is at least a given *quota* $q \in [0, \sum_{j \in N} w_j]$, that is $w(S) \geq q$. Otherwise S is said to be *loosing*. We will use the well established notation $[q; w_1, w_2, \ldots, w_n]$ for weighted voting games. The corresponding simple voting game $\mathcal{W} = \{S \subseteq N : w(S) \geq q\}$ is the collection of all winning coalitions.

[2]The necessity to restrict the domain of c follows from the fact that the weight, given to the quota independent principle, is less that 50%, because we want the resulting model to be quota dependant.

Now, having defined this mathematical model, we want to quantify the power of each voter in the game $[q; w_1, w_2, \ldots, w_n]$. There are many existent methods for measuring power, yet many are not applicable to the political sciences. For a substantial overview and discussion of these methods see Felsenthal and Machover [5].

We are interested in the influence of a member's vote on the outcome of a decision. Penrose (1946) [16], Banzhaf (1965) [1], and Coleman (1986) [3] have independently proposed power measures which address this issue using a probabilistic approach, where they interpret voting power as the probability that a voter is decisive in a ballot.

For their approach it is essential to define an appropriate probability space and analyze the corresponding assumptions. Therefore, let Ω be the space of all possible proposals. For the Council of Ministers of the European Union it is reasonable to not consider abstentions, and so each voter has to vote either "yes" or "no" for a proposal $\omega \in \Omega$[3]. We will code a "yes" with the number 1 and a "no" with -1 and thus, a *configuration* or *division of the assembly* for a proposal $\omega \in \Omega$ is an n-tuple from the space $\{-1, 1\}^n$. As our underlying probability space we will consider the *Bernoulli model* B_n, where each bipartition $b \in \{-1, 1\}^n$ has the same probability, namely $\mathbb{P}[b] = \frac{1}{2^n}$.

The interpretation of B_n is that (a) all n voters cast their vote independently from each other, (b) they vote "yes" and "no" with equal probability of $\frac{1}{2}$ and (c) all coalitions are equally likely. While initially these assumptions seem to be rather restrictive, we will now explain, why they are reasonable for our applications. Our goal is to measure the *a priori* power of each voter assigned to him by the structure of the game $\mathcal{W}$. Therefore, for designing a formal voting system, we do not want to take personal interests into account and thus power measurement has to be independent of the voting behavior. Hence, this validates the independence assumption (a). There are also various explanations why we may assume condition (b), that is equal probabilities for voting either "yes" or "no". One such explanation is that for every proposal $\omega \in \Omega$ there is a counter-proposal $\bar{\omega} \in \Omega$ such that a rational participant would vote antipodal for ω and $\bar{\omega}$ and thus the probability of voting "yes" for a random proposal chosen from all possible proposals should be equal to $\frac{1}{2}$. The last assumption (c) follows directly from the assumptions (a) and (b). These assumptions are further discussed by Leech [9] as well as Felsenthal and Machover [5].

[3]To include abstentions one can introduce ternary weighted voting games, see Lindner [11].

Now, we define the random variable $X_i : \Omega \to \mathbb{R}_0^+$ for all $i \in N$ where

$$X_i(\omega) = \begin{cases} w_i & \text{if player } i \text{ votes in favor of the proposal } \omega \\ 0 & \text{if player } i \text{ votes against the proposal } \omega \end{cases}.$$

According to the definition of the underlying probability space and the corresponding assumptions, which were discussed above, these random variables are mutually independent. Furthermore we have for all $i \in N$ that $\mathbb{P}[X_i = w_i] = \mathbb{P}[X_i = 0] = \frac{1}{2}$ and $\mathbb{E}[X_i] = \frac{w_i}{2}$ and $\operatorname{Var}[X_i] = \frac{w_i^2}{4}$.

Next we define for all $i \in N$ and $\omega \in \Omega$ the random variable

$$Y_{\neg i}(\omega) = \sum_{j=1}^{n} X_j(\omega) - X_i(\omega) = \sum_{j=1, j \neq i}^{n} X_j(\omega),$$

where $\mu_i := \mathbb{E}[Y_{\neg i}] = \frac{1-w_i}{2}$ and $\sigma_i^2 := \operatorname{Var}[Y_{\neg i}] = \left(\sum_{j=1}^{n} \frac{w_j^2}{4} \right) - \frac{w_i^2}{4}$. Hence, a voter i can change the outcome of the decision, pertaining a proposal $\omega \in \Omega$ if

$$Y_{\neg i}(\omega) < q \ \text{ but } Y_{\neg i}(\omega) + w_i \geq q.$$

The *absolute (Banzhaf) voting power* $\psi_i[\mathcal{W}]$ of a voter $i \in N$ in the simple voting game $\mathcal{W}$ is defined as the probability that he is decisive and therefore,

$$\psi_i[\mathcal{W}] = \mathbb{P}[q - w_i \leq Y_{\neg i} < q].$$

Various enumerative methods exist to calculate these probabilities. However, the running time grows exponentially with the number of players n and is $\mathcal{O}(n2^n)$ in the worst case. One can reduce this complexity by using generating functions, but as these functions are still highly dependant on n, this method becomes very slow as well (Bilbao et al. [2]). Furthermore, it is not possible to define an explicit function for the calculation of $\psi_i[\mathcal{W}]$, which makes the application circuitous and intransparent. To avoid these disadvantages it is therefore helpful to find approximate formulae for the probabilities of interest.

The key to gain such formulae is a normal approximation of the random variables $Y_{\neg i}$ for $i \in N$. Owen (1975) [15] and Merrill (1982) [13] were the first to use this concept. In 2007, Feix et al. [4] provided a more rigorous study of the normal approximation for the Council of Ministers of the European Union. They stated different general conditions for the application of the normal approximation, pointed out that these are fulfilled for this special case and verified their claim heuristically by comparing the graph of the density of the normal distribution to the histogram of all possible values of $Y_{\neg i}$. The normal approximation is pertinent as long as (a) the number n of all voters is large enough, (b) the voting weights are sufficiently scattered such that all 2^n configurations provide a variety of different weights and (c) $\max_{i=1,\ldots,n} w_i << \sqrt{\sum_{j=1}^{n} w_j^2}$ (Słomczyński, Życzkowski [17] and Lindner, Machover [12]).

Given that the necessary conditions are valid, we have that $\frac{Y_{\neg i}-\mu_i}{\sigma_i}$ approximately follows a standard normal distribution with the cumulative distribution function Φ. Let $x := q - \frac{1}{2}$ be the portion of the threshold q which exceeds the majority quota of 50%, then

$$\begin{aligned}
\psi_i[\mathcal{W}] &= \mathbb{P}\left[\frac{q - w_i - \mu_i}{\sigma_i} \leq \frac{Y_{\neg i} - \mu_i}{\sigma_i} < \frac{q - \mu_i}{\sigma_i}\right] \\
&\approx \Phi\left[\frac{\frac{1}{2} + x - w_i - (\frac{1}{2} - \frac{1}{2}w_i)}{\sigma_i}\right] - \Phi\left[\frac{\frac{1}{2} + x - (\frac{1}{2} - \frac{1}{2}w_i)}{\sigma_i}\right] \\
&= \Phi\left[\frac{x + \frac{1}{2}w_i}{\sigma_i}\right] - \Phi\left[\frac{x - \frac{1}{2}w_i}{\sigma_i}\right] \\
&= \int_{\frac{x-\frac{1}{2}w_i}{\sigma_i}}^{\frac{x+\frac{1}{2}w_i}{\sigma_i}} \varphi(t)\mathrm{d}t \qquad (1)
\end{aligned}$$

Since we have required that $\max_{i=1,\ldots,n} w_i << \sqrt{\sum_{j=1}^{n} w_j^2}$ it follows that $\frac{\frac{1}{2}w_i}{\sigma_i} << 1$ and hence, the integration area in (1) is concentrated around the point $\frac{x}{\sigma_i}$. This allows a further approximation of the absolute Banzhaf power

$$\psi_i[\mathcal{W}] \approx \varphi\left(\frac{x}{\sigma_i}\right)\frac{w_i}{\sigma_i}. \qquad (2)$$

Finally, it is important to note that for a better comparison of voting powers between members of a voting body, it is helpful to consider a normalized or relative measure of power, rather than the absolute values. Therefore, we will refer to

$$\beta_i[\mathcal{W}] = \frac{\psi_i[\mathcal{W}]}{\sum_{j=1}^{n} \psi_j[\mathcal{W}]}$$

as the *share of (Banzhaf) voting power* or the *relative (Banzhaf) voting power* of voter i.

3 Different Fairness Concepts

In this section we will consider two-stage, indirect voting systems. First, the population of each member state elects a delegate who, in a second step, will represent his state in a superordinated voting body. A classic example for such a system is the Council of Ministers of the European Union.

Let $\mathcal{V}$, the *council*, be the second stage of such an indirect voting system which is a weighted voting game among the designated delegates $j \in \{1, \dots, m\}$. The election of these delegates in the first stage is modeled in each member state by the simple voting games $\mathcal{W}_1, \dots, \mathcal{W}_m$. Then the *composite of* $\mathcal{W}_1, \dots, \mathcal{W}_m$ *under* $\mathcal{V}$ is denoted by $\mathcal{W} := \mathcal{V}[\mathcal{W}_1, \dots, \mathcal{W}_m]$ and represents an indirect voting system. The assembly N of this composite game $\mathcal{W}$ is the union of the assemblies $N_1, \dots, N_m$ of the simple voting games $\mathcal{W}_1, \dots, \mathcal{W}_n$ ($N = \bigcup_{i=1}^{m} N_i$).

Since all members of N vote independently from each other, we can guarantee that the delegates vote independently as well[4]. Furthermore, we need the constraints that the assemblies $N_1, \dots, N_m$ are disjoint and that all $\mathcal{W}_1, \dots, \mathcal{W}_m$ are simple majority games. If we denote the size of N_i as n_i this means that for all $i \in \{1, \dots, m\}$ the quota is equal to $\frac{n_i}{2} + \varepsilon$ for a sufficiently small $\varepsilon > 0$. We further require that each delegate $j \in \{1, \dots, m\}$ represents the opinion of the majority in the weighted voting game $\mathcal{V}$. Hence, for the composite game $\mathcal{W}$, all winning coalitions $S \in \mathcal{W}$ have to satisfy that $\{j \in \{1, \dots, m\} : S \cap N_j \in \mathcal{W}_j\} \in \mathcal{V}$.

Next, while analyzing such an indirect voting system, we must address the question of fairness. Although this is a rather subjective task, it is clear that an appropriate choice of voting weights and quota is essential to answer this question.

Penrose's Square-Root Rule (PSQRR)

The most reasonable and widely accepted fairness principal requires that every member $i \in N = \bigcup_{i=j}^{m} N_j$ has the same power to influence a decision. This idea originates from Penrose (1946) [16] and is also known as the "One person, one vote" (OPOV) principle. The theorem of *Penrose's Square-Root Rule* provides necessary and sufficient conditions to fulfill this fairness idea in a two-stage decision-making process.

For all voters $i \in N$, the indirect, absolute Banzhaf powers $\psi_i[\mathcal{W}]$ are

[4] Note that this follows from the assumptions in section 2.

equal (with negligible error) if and only if the absolute Banzhaf powers $\psi_j[\mathcal{V}]$ of all delegates $j \in \{1, \ldots, m\}$ in the council are proportional to $\sqrt{n_j}$. This means that for all delegates $j \in \{1, \ldots, m\}$ we have to require that the relative voting power satisfies

$$\beta_j[\mathcal{V}] = \frac{\sqrt{n_j}}{\sum_{i=1}^{n} \sqrt{n_i}}.$$

Penrose did not provide a rigorous proof of his Square-Root Rule, rather he gave a semi-heuristic argument. For a complete proof see Felsenthal and Machover [5]. This proof utilizes Stirling's approximation for factorials and therefore it is necessary that all n_j, $j \in \{1, \ldots, m\}$, are large enough such that the corresponding approximation error is negligible. This demand is clearly met by the Council of Ministers of the European Union[5].

Unfortunately, Penrose's Square-Root Rule has an essential problem due to its conditions on the voting powers rather than the voting weights. The procedure of finding the corresponding voting weights, given the desired voting powers, is called the *inverse problem* and is to our best knowledge not yet studied thoroughly. Nevertheless, Leech [10] developed an iterative algorithm to find an approximation of the desired voting weights which can help to solve this issue.

Jagiellonian Compromise (JC)

The Jagiellonian Compromise sought to create a transparent, simple and easily extendible voting system that satisfies Penrose's Square-Root Rule. Although Penrose provided instructions on how to designate the voting powers, the authors Słomczyński and Życzkowski ([18, 19]) set all voting weights w_j equal to $\frac{\sqrt{n_j}}{\sum_{i=1}^{m} \sqrt{n_i}}$ and hence proportional to $\sqrt{n_j}$. First, they showed numerically that there exists a quota q^*_{JC} such that all voting weights and their corresponding voting powers coincide. Later they derived mathematically a formula for this *critical point* [17],:

$$q^*_{JC} = \frac{1}{2}\left(1 + \sqrt{\sum_{j=1}^{m} w_j^2}\right) = \frac{1}{2}\left(1 + \frac{\sqrt{\sum_{j=1}^{m} n_j}}{\sum_{j=1}^{m} \sqrt{n_j}}\right). \tag{3}$$

Due to the structure of this quota formula the Jagiellonian Compromise is also called the *double square root voting system*.

[5] The smallest state Malta has a population of about 400.000, which is large enough for an accurate approximation.

One state, one vote (OSOV)

A seemingly naive, yet justified fairness concept is to consider the European Union as an association of totally independent states, where each state is treated equally. Under this model, all states should have the same probability of being decisive and thus, have the same voting power ("One state, one vote", [8]). In terms of relative power this means that $\beta_j[\mathcal{W}] = \frac{1}{m}$ for all $j \in \{1, \ldots, m\}$. This can be easily achieved by assigning the same voting power to each state. Note that such a unitary system is totally independent of the quota q.

Mixed fairness model (MFM)

Laruelle and Widgrén combine the fairness concepts from above: "*If the EU is a federal state, each state must be treated in accordance to a weighted average between the two extreme principles* ("One man, one vote" and "One state, one vote")", [8]. In order to bring these two presented fairness ideas together, it is appropriate to consider a convex combination. In such a corresponding *mixed fairness model* the relative Banzhaf power (4) is allocated to each member in the corresponding game $\mathcal{W}_c$.

$$\beta_i[\mathcal{W}_c] = c \cdot \frac{\sqrt{n_j}}{\sum_{i=1}^{m} \sqrt{n_i}} + (1-c) \cdot \frac{1}{m}, \quad c \in [0,1] \tag{4}$$

The mixture parameter c represents the weight that is given to the "One person, one vote" idea. We will analyze such mixed fairness models more closely in the following two sections.

Minimizing the mean majority deficit (Min. MMD)

Finally, we present a fairness principle developed by Felsenthal and Machover to minimize the *mean majority deficit*. If for a coalition $S \in N$ we have that $|S| - |N \setminus S| \geq 0$ [6], then the majority deficit is equal to 0. Otherwise, we have a positive majority deficit equal to $|N \setminus S| - |S| = |N| - 2|S|$.

A fair voting system should minimize the expected discrepancy which we denote as the *mean majority deficit* $\Delta[\mathcal{W}]$. Felsenthal and Machover [6] proved in [6] that

$$\Delta[\mathcal{W}] = \frac{\sum_{j=1}^{m} \psi_j[\mathcal{W}] - \frac{k}{2^{n-1}} \cdot \binom{n}{k}}{2} \tag{5}$$

[6] $|S|$ is the notation for the size of S and $N \setminus S$ is the set N without the elements of S.

where $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$ is the least integer greater than $\frac{n}{2}$. For given voting weights $w_1, \ldots, w_m$ in the council, $\Delta[\mathcal{W}]$ will be minimal if and only if the quota is chosen as $\frac{\sum_{j=1}^{m} w_j}{2} + \varepsilon$ for a sufficiently small $\varepsilon > 0$. However, this means that for given voting weights the simple majority game leads to a minimum of $\Delta[\mathcal{W}]$ and therefore, it is considered to be the most fair according to the fairness idea of the minimal mean majority deficit. Finally, note that we can gain information about fairness with respect to the presented idea by comparing the given mean majority deficit of a given voting system to its possible minimal value.

4 Council of Ministers and Fairness

The current basis for the European political system is the *Treaty of Nice* of 2001. The treaty assigns the current voting weights for each member state in the Council of the Ministers, as shown in the second column of table 1. Although a "triple majority" must be satisfied in order that a proposal will be passed, a mathematical analysis shows that only one of the three conditions is significant. Except for a very small number of the 2^m possible cases[7], a proposal will be accepted if at least 255 votes (that is 73,9%) will be cast in favor of it [7]. Columns three and four of table 1 show the relative voting weights and the relative Banzhaf voting powers. It can be observed that these values are far from being concordant.

To illustrate the Jagiellonian Compromise for the Council of Ministers we use the current population of all member states and calculate the relative square roots ($\frac{\sqrt{n_j}}{\sum_{i=1}^{m} \sqrt{n_i}}$), which not only yield the relative voting weights but also the corresponding relative Banzhaf powers if the special quota q_{JC}^* is used (see column five and six of table 1). The absolute values of the discrepancies between the voting powers of the current voting system and the Jagiellonian Compromise can be found in the last column of table 1 and are illustrated in figure 1. We can observe that the four largest countries are under-represented, whereas the smallest four countries are over-represented. However, we can not detect any concrete structure for the medium-sized states, as half of them are entitled more power and half of them less. Poland profits the most from the current system, while Germany is the most disadvantaged.

[7] This number is in fact so small, that we will not see any effect of the other two conditions on the voting powers, when using the same rounding as in tables 1 and 2.

Member state of the European Union	Voting weight in the CM	Voting weight in the CM in %	Relative voting power in the CM	Population	Square root of population in %: voting weight and voting power in the JC	Discrepancy of voting powers: Nice-JC
Germany	29	8,41%	7,78%	82.310.995.000	9,44%	-1,6557%
France	29	8,41%	7,78%	63.392.140.000	8,28%	-0,5003%
United Kingdom	29	8,41%	7,78%	60.798.438.000	8,11%	-0,3291%
Italy	29	8,41%	7,78%	59.131.287.000	8,00%	-0,2171%
Spain	27	7,83%	7,42%	44.474.631.000	6,94%	0,4819%
Poland	27	7,83%	7,42%	38.125.479.000	6,42%	0,9962%
Romania	14	4,06%	4,26%	21.565.119.000	4,83%	-0,5719%
Netherlands	13	3,77%	3,97%	16.357.992.000	4,21%	-0,2336%
Greece	12	3,48%	3,68%	11.170.957.000	3,48%	0,2073%
Portugal	12	3,48%	3,68%	10.599.095.000	3,39%	0,2975%
Belgium	12	3,48%	3,68%	10.584.534.000	3,38%	0,2998%
Czech Republic	12	3,48%	3,68%	10.287.189.000	3,34%	0,3477%
Hungary	12	3,48%	3,68%	10.066.158.000	3,30%	0,3837%
Sweden	10	2,90%	3,09%	9.113.257.000	3,14%	-0,0481%
Austria	10	2,90%	3,09%	8.298.923.000	3,00%	0,0955%
Bulgaria	10	2,90%	3,09%	7.679.290.000	2,88%	0,2096%
Denmark	7	2,03%	2,18%	5.447.084.000	2,43%	-0,2472%
Slovakia	7	2,03%	2,18%	5.393.637.000	2,42%	-0,2353%
Finland	7	2,03%	2,18%	5.276.955.000	2,39%	-0,2090%
Ireland	7	2,03%	2,18%	4.319.425.000	2,16%	0,0187%
Lithuania	7	2,03%	2,18%	3.384.879.000	1,91%	0,2668%
Latvia	4	1,16%	1,25%	2.281.305.000	1,57%	-0,3211%
Slovenia	4	1,16%	1,25%	2.010.377.000	1,48%	-0,2249%
Estonia	4	1,16%	1,25%	1.342.409.000	1,21%	0,0449%
Cyprus	4	1,16%	1,25%	778.537.000	0,92%	0,3323%
Luxembourg	4	1,16%	1,25%	476.187.000	0,72%	0,5323%
Malta	3	0,87%	0,94%	406.020.000	0,66%	0,2793%
Column sum	345	100,00%	100,00%	495.072.299.000	100,00%	0,0000%

Table 1: Council of Ministers of the European Union: Current situation with the Treaty of Nice; Population as basis for calculating the weights for the Jagiellonian Compromise; Discrepancy between current voting weights and fair voting weights according to the OPOV principle.

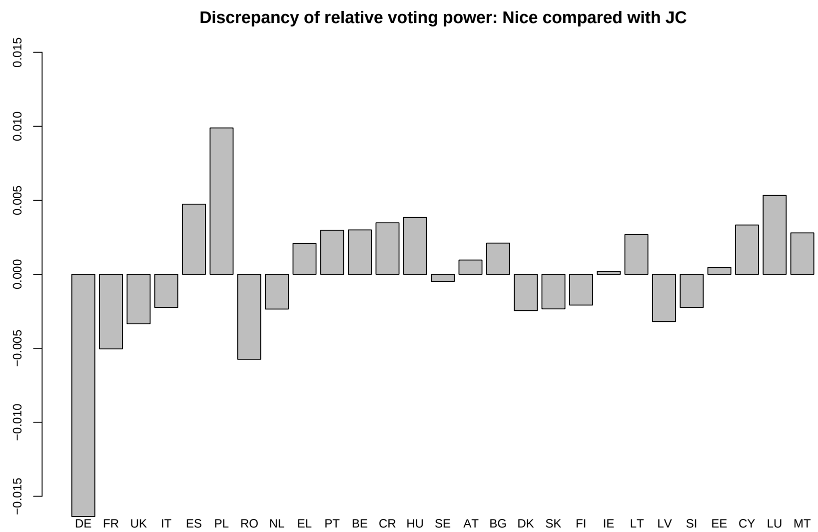

Figure 1: Discrepancies of relative voting powers between the current voting weights and the optimal voting weights according to the Jagiellonian Compromise

From figure 1 we can deduce that the voting system in the Council of Ministers of the European Union according to the Treaty of Nice does not satisfy the fairness principal of "One person, one vote", nor the "One state, one vote" idea, where each state should have equal power. At the end of this section we will additionally show that the current voting system is also unfair according to the principal of minimal mean majority deficit. Hence, it is of interest to determine if the voting system of the Council of Ministers is fair according to a mixed fairness model and determine the corresponding mixture parameter c^*.

Now, we apply the same idea used by Laruelle and Widgrén used to identify a mixed model for the European Union with 15 member states with respect to two different voting rules [8]. We will provide an estimation of c^* for the current situation in the European Union with 27 member states. Therefore, we will use a simple linear regression on the weighted voting game given by the Treaty of Nice $\mathcal{W}_{\text{Nice}}$. The current voting powers according to the Treaty of Nice will serve as the dependent variables, and the square root shares of the population $\frac{\sqrt{n_j}}{\sum_{i=1}^{m} \sqrt{n_i}}$ as the independent variables. Thus, the mixture parameter c^* is the regression parameter. As a result we obtain that c^*=0,91961 and hence:

$$\beta_i[\mathcal{W}_{\text{Nice}}] = 0,91961 \cdot \frac{\sqrt{n_i}}{\sum_{j=1}^{n} \sqrt{n_j}} + 0,08039 \cdot \frac{1}{27} \tag{6}$$

Note that the value of the regression parameter is significantly larger than the values that Laruelle and Widgrén obtained in 1998 for the European Union with only 15 member states[8]. Thus, with the enlargement of the European Union, there has been an apparent shift in favor of the "One person, one vote" principle.

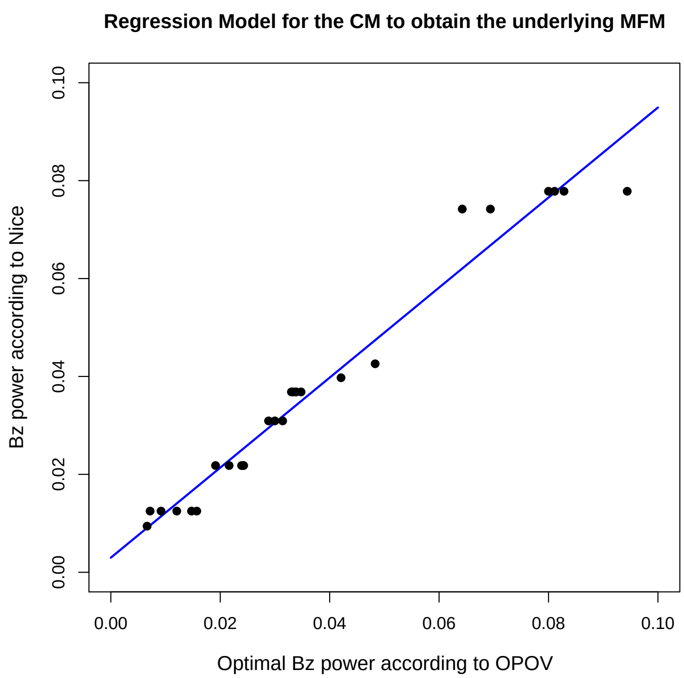

Figure 2: Regression model: The regression line shows the allocation of the voting powers, which correspond to the mixed fairness model with the mixture parameter $c^* = 0,919612$; The dots represent the current situation according the Treaty of Nice.

The standard error from the estimation of c^* is equal to 0,035075, the standard error from estimating the intercept is 0,001563 and we have an

[8]These values are 0,79 for the qualified majority and 0,6 for the qualified majority with at least 10 countries being in favor of a proposal.

overall residual standard error of 0,004511. From examining the regression line in figure 2 and the fact that R^2 = 0,9649, it appears that we have derived a satisfying approximation. Except for the obvious outliers Germany, Spain and Poland, all values are well described by the linear formula (6), and therefore we can argue that from a statistical point of view the current distribution of voting powers follows a mixed fairness model with the weight c^* =0,919612 assigned to the "One person, one vote" principle.

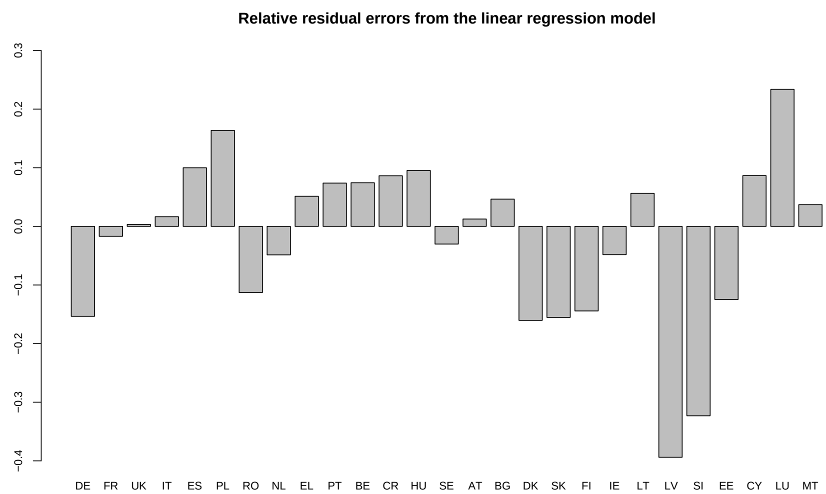

Figure 3: Visualization of the relative residual errors:
(current voting power - fair voting power)/current voting power.

Nevertheless, looking at the relative errors with respect to each member state we can detect significant over- and under-representations, especially for the small member states (compare figure 3). Hence, although statistically there is no evidence that the current distribution of voting weights is unfair according the mixed model (6), the relative error analysis emphasizes that it is still rather inappropriate. Therefore, before performing further analysis, one should correct the obvious discrepancies and choose the voting powers according to the regression model $\beta_i[\mathcal{W}_{c^*}] = 0,91961 \cdot \frac{\sqrt{n_i}}{\sum_{j=1}^{n} \sqrt{n_j}} + 0,08039 \cdot \frac{1}{27}$.

The problem with the mixed fairness model is similar to the problem encountered with Penrose's Square-Root Rule, in which we have conditions on voting powers rather than voting weights. Therefore, we apply the same idea as Słomczyński and Życzkowski [19] and first set all the relative voting weights to $w_i(c^*) = 0,91961 \cdot \frac{\sqrt{n_i}}{\sum_{j=1}^{n} \sqrt{n_j}} + 0,08039 \cdot \frac{1}{27}$, which correspond to the desired values of $\beta_i[\mathcal{W}_{c^*}]$ for all $i \in \{1, \ldots, m\}$. Given these voting weights, we calculate the relative voting powers with respect to a varying quota. We desire to show that for a special quota, voting weights and voting power will coincide. For a visualization of this procedure, we will display the ratios of relative voting powers to voting weights for all 27 member states in figure 4. The fairness idea of the mixed fairness model is met if for all member states we have that this ratio is equal to 1. Fortunately, for the

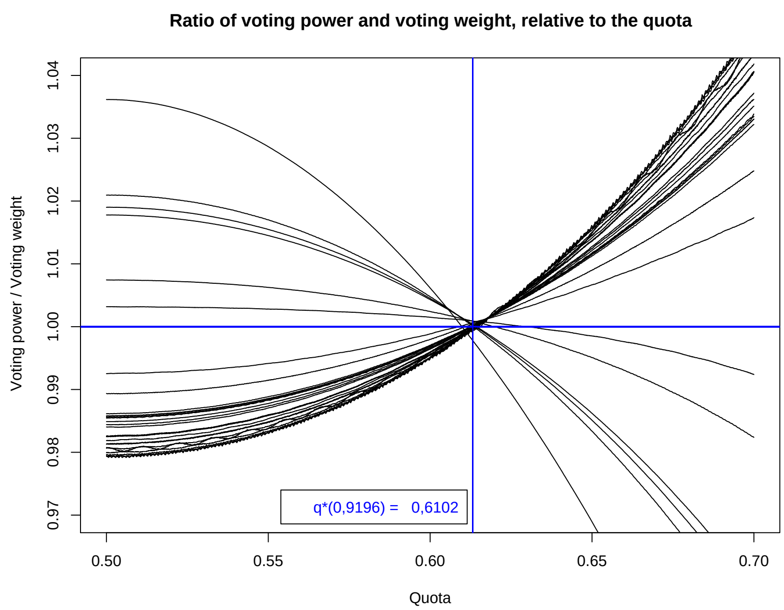

Figure 4: Ratio of relative voting power to relative voting weight with respect to the quota for all 27 member states, where the voting weights are set to $0,91961 \cdot \frac{\sqrt{n_i}}{\sum_{j=1}^{n} \sqrt{n_j}} + 0,08039 \cdot \frac{1}{27}$.

special quota $q^* = 0,6102$, all 27 curves meet and additionally all ratios are approximately equal to 1 (figure 4). Hence, the fairness principle is satisfied simultaneously for all 27 member states.

To support this observation, we define the *squared cumulative error* SCE,

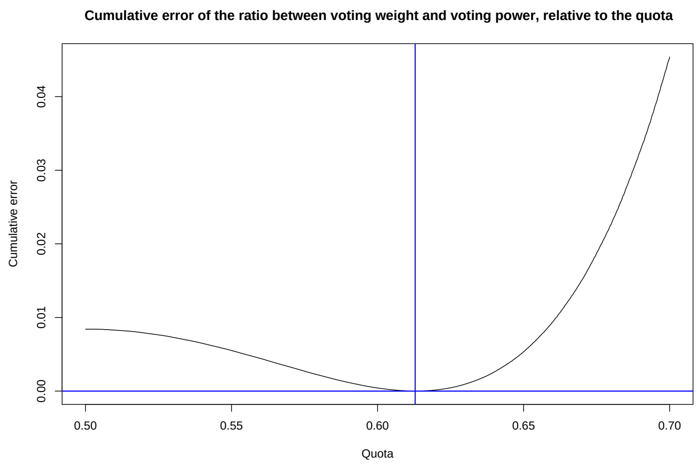

Figure 5: Development of the squared cumulative error SCE with respect to the quota. A minimal error value is attained in $q^* = 0,6102$.

with respect to the quota q:

$$\mathrm{SCE}(q) = \sum_{j=1}^{m} \left(\frac{\beta_i[\mathcal{W}_{c^*}](q)}{w_i(c^*)} - 1 \right)^2 . \tag{7}$$

Figure 5 depicts this error measure. We can detect a minimum, and hence optimal value, $q^* = 0,6102$. In the next section, we will show that this optimal quota q^* is a function of c and that we can calculate this optimum via the formula:

$$q^*(c) = \frac{1}{2}\left(1 + \sqrt{\sum w_i(c)^2}\right) . \tag{8}$$

For a better overview, we now summarize our results in table 2. The second column displays the relative voting weights which we have defined as the fair shares of voting powers in the mixed fairness model with $c^* = 0{,}919612$. Using the quota $q^*(c^*) = 0,6102$, which we gain from (8), we obtain the relative Banzhaf voting powers shown in column three. As column four of table 2 illustrates, the absolute differences between voting powers and voting weights are rather negligible. However, we can observe a slight shift in favor of the small-sized member states to the expense of the large-sized member states. The medium-sized states are well represented by our approach. As a basis for the squared cumulative error, the difference between the ratio

EU member state	$w_i(c^*)$	$\beta_i[\mathcal{W}_{c^*}](q^*)$	$\beta_i[\mathcal{W}_{c^*}](q^*) - w_i(c^*)$	$\frac{\beta_i[\mathcal{W}_{c^*}](q^*)}{w_i(c^*)} - 1$
Germany	8,98%	8,91%	-0,07%	-0,007
France	7,91%	7,87%	-0,04%	-0,005
United Kingdom	7,76%	7,72%	-0,04%	-0,005
Italy	7,65%	7,62%	-0,04%	-0,005
Spain	6,68%	6,65%	-0,03%	-0,004
Poland	6,21%	6,18%	-0,02%	-0,004
Romania	4,74%	4,73%	-0,01%	-0,003
Netherlands	4,17%	4,16%	-0,01%	-0,002
Greece	3,50%	3,50%	0,00%	0,000
Portugal	3,41%	3,41%	0,00%	0,000
Belgium	3,41%	3,41%	0,00%	0,000
Czech Republic	3,37%	3,37%	0,00%	0,000
Hungary	3,33%	3,34%	0,00%	0,001
Sweden	3,19%	3,19%	0,00%	0,001
Austria	3,05%	3,06%	0,01%	0,002
Bulgaria	2,95%	2,96%	0,01%	0,002
Denmark	2,53%	2,54%	0,01%	0,005
Slovakia	2,52%	2,53%	0,01%	0,005
Finland	2,50%	2,51%	0,01%	0,005
Ireland	2,29%	2,30%	0,02%	0,007
Lithuania	2,06%	2,08%	0,02%	0,009
Latvia	1,74%	1,76%	0,02%	0,013
Slovenia	1,65%	1,68%	0,02%	0,015
Estonia	1,41%	1,43%	0,03%	0,019
Cyprus	1,14%	1,17%	0,03%	0,026
Luxembourg	0,96%	0,99%	0,03%	0,034
Malta	0,91%	0,94%	0,03%	0,036

Table 2: Comparison of voting weights $w_i(c^*)$ and voting powers $\beta_i[\mathcal{W}_{c^*}](q)$ for the optimal quota $q^*(c^*) = 0,6102$, where $c^* = 0{,}919612$.

$\beta_i[\mathcal{W}_{c^*}](q^*)/w_i(c^*)$ and 1 is stated for all $i = 1, \ldots, 27$. The squared cumulative error $\mathrm{SCE}(q^*(c^*))$ of the analyzed voting system is equal to 0,0043 and is therefore insignificant, yet it does not vanish completely. Hence, we can conclude that using the appropriate quota and slightly adjusted voting weights will lead to a fair voting system, in which the two contrary principals "One person, one vote" and "One state, one vote" are both taken into account via the presented convex combination.

Voting system	Mean majority deficit	Ratio
Treaty of Nice (TN)	3,9745	10,1293
Minimal MMD with TN weights	0,3924	
Jagiellonian Compromise (JC)	1,0294	2,7723
Minimal MMD with JC weights	0,3713	
Mixed fairness model (MFM)	0,9754	3,0271
Minimal MMD with MFM weights	0,3222	

Table 3: Mean majority deficit: Comparison of the mean majority deficit of the voting system according to (a) the Treaty of Nice, (b) the Jagiellonian Compromise and (c) the observed mixed fairness model to their possible minimal values.

Finally, we want to analyze and compare the mean majority deficits of the current situation afforded by the Treaty of Nice, the Jagiellonian Compromise, and the derived mixed fairness model from this section. The minimal value for the expected majority deficit for the voting weights from the Treaty of Nice is equal to 0,3924. A comparison of this value to the current mean majority deficit, shows that it is 10 times higher than the presented possible minimum. This is due to the relatively high quota of 73,9% that is applied in the Council of Ministers of the European Union. The same comparison for the Jagiellonian Compromise establishes a less drastic disparity, since the mean majority deficit is only about 2,77 times higher than the possible minimum for the given weights. The same is true for the adapted mixed fairness model. Nevertheless, the ratio of about 3,03 is slightly higher than for the Jagiellonian Compromise. One can interpret this fact as a trade-off for the adjustment of the voting weights according to the desired mixed fairness model. In general, we can conclude that the Jagiellonian Compromise and the mixed fairness model are in better compliance with the fairness principle of the minimal mean majority deficit than the current situation.

5 Optimal Quota

Now, we want to prove that the optimal quota for a mixed fairness model with the mixture parameter c is given by (8), that is

$$q^*(c) = \frac{1}{2}\left(1 + \sqrt{\sum w_i(c)^2}\right).$$

However, we first have to remark that we must to restrict the domain of c to $(0.5, 1]$. The necessity of this constraint derives from the following reasoning. In the mixed fairness model we assign the weight c to the quota dependent principle[9] and the weight $1-c$ to the quota independent idea[10]. As presented in the previous section, we desire to create a voting system which yields a similar quota dependency as for the Jagiellonian Compromise, and therefore it is clear that we only can achieve such a result if the quota dependent idea receives a higher weight than the quota independent principle.

Now, let $c \in (0.5, 1]$ be a fixed value and hence the voting weights $w_j(c)$ for all $j \in \{1, \dots, m\}$ will not vary in the following calculations. Our goal is to find a quota $q^*(c)$ such that we obtain $\frac{\beta_j(q^*(c))}{w_j(c)} \approx 1$ for all $j \in \{1, \dots, m\}$, and therefore all of these ratios must to be equal to each other. We will first perform a pairwise comparison for any two players j and k with $w_j(c) > w_k(c)$[11] and hence $\sigma_j^2 \leq \sigma_k^2$. Using the notation $x^*_{j,k} := q^*_{j,k} - \frac{1}{2}$, where $q^*_{j,k}$ is the optimal quota such that player j and k have the same ratios of voting weight to voting power, we obtain

$$\begin{aligned}
&\frac{\beta_j(q^*_{j,k})}{w_j(c)} = \frac{\beta_k(q^*_{j,k})}{w_k(c)} \\
\Leftrightarrow\quad &\frac{w_k(c)}{w_j(c)} = \frac{\beta_k(q^*_{j,k})}{\beta_j(q^*_{j,k})} = \frac{\psi_k(q^*_{j,k})}{\psi_j(q^*_{j,k})} \overset{(2)}{\approx} \frac{\frac{w_k(c)}{\sigma_k}}{\frac{w_j(c)}{\sigma_j}} \cdot \frac{\varphi\left(\frac{x^*_{j,k}}{\sigma_k}\right)}{\varphi\left(\frac{x^*_{j,k}}{\sigma_j}\right)} \\
\Leftrightarrow\quad &\frac{\sigma_j}{\sigma_k} \exp\left(\frac{{x^*_{j,k}}^2}{2}\left(\frac{1}{\sigma_j^2} - \frac{1}{\sigma_k^2}\right)\right) = 1 \qquad (9)
\end{aligned}$$

[9] That is the Jagiellonian Compromise which satisfies the "One person, one vote" condition.

[10] That is the unitary voting system with equal voting weights for all players. It produces the same voting power for all voters and for all cogitable quotas.

[11] If we have that $w_j(c) = w_k(c)$ for $j \neq k$, then trivially the ratios of voting power and voting weight are equal to each other, regardless which quota is applied.

Solving equation (9) for $x^*_{j,k}$ we get

$$x^*_{j,k} = \sqrt{\frac{\ln\left(\frac{\sigma_k^2}{\sigma_j^2}\right)}{\frac{1}{\sigma_j^2} - \frac{1}{\sigma_k^2}}} = \sqrt{\frac{4\sigma_j^2\sigma_k^2}{w_j(c)^2 - w_k(c)^2} \cdot \ln\left(1 + \frac{w_j(c)^2 - w_k(c)^2}{4\sigma_j^2}\right)} \quad (10)$$

Recall that a requirement for the validation of this approximation was that $\max_{i=1,\dots,n} w_i(c) << \sqrt{\sum_{j=1}^n w_j(c)^2}$, and therefore we have that $\frac{\frac{1}{2}w_i(c)}{\sigma_i} <<$ 1. Thus, $\frac{w_j(c)^2 - w_k(c)^2}{4\sigma_j^2} > 0$ will be very close to 0 such that we can apply the approximation $\ln(1+\varepsilon) \approx \varepsilon$, which follows from Taylor's theorem. Hence, we can further simplify

$$\begin{aligned} x^*_{j,k} &\approx \sqrt{\frac{4\sigma_j^2\sigma_k^2}{w_j(c)^2 - w_k(c)^2} \cdot \frac{w_j(c)^2 - w_k(c)^2}{4\sigma_j^2}} = \sqrt{\sigma_k^2} \\ &= \sqrt{\frac{1}{4}\sum_{i=1}^m w_i(c)^2 - \frac{1}{4}w_k(c)^2} \approx \frac{1}{2}\sqrt{\sum_{i=1}^m w_i(c)^2}. \end{aligned} \quad (11)$$

The last approximation in (11) follows again from the fact that $\max_{i=1,\dots,n} w_i^2(c) << \sum_{j=1}^n w_j(c)^2$. Therefore, we obtain the result that for any pairs j and k, the ratios of relative voting power and relative voting weight are approximately equal if we choose the quota $q^*_{j,k} = q^* = \frac{1}{2}\left(1 + \sqrt{\sum w_i(c)^2}\right)$, which is not dependent on the choice of j and k. As we consider relative values for voting power and voting weight, we can deduce that with the right choice of the quota, all ratios will be approximately equal to 1.

6 Conclusion and Future Directions

We have analyzed different fairness principles for the current voting system in the Council of Ministers in the European Union. We have seen that the allocation of the voting weights according to the Treaty of Nice is not fair, but fairness can be achieved by looking at a combination of two opposed ideas. With a special quota one can achieve concordance between voting weights and voting powers. This leads to a simple and transparent voting system, since the voting weights (which were set to the desired values of the relative voting power) give complete information about the allocation of voting power. Furthermore the mixed fairness model is easily extendible if new member states join the existing voting system.

Note that the combination of mixed fairness models together with a special quota is a generalization of the existing theory. For example, the Jagiellonian Compromise is a special case of such a mixed fairness model, equipped with the mixture parameter $c = 1$. Furthermore, our formula (8) converges towards $\frac{1}{2}$ if the number m of voters gets infinitely large. This has two positive effects. Firstly, we have that with a growing number of players, the mean majority deficit will decline towards its possible minimum. And secondly, this observation is consistent with Penrose Limit Theorem [12].

Finally, we must note that the presented mixed fairness model is not applicable to all existing voting systems. This is for instance the case for the House of Representatives of the United States of America. The reason for this is that a linear regression model is not appropriate in this case. Thus, we should develop a more general mixed fairness model which also takes other fairness ideas like the minimization of the mean majority deficit into account.

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