Voting in Truth-Tracking Situations[1]

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Abstract We extend Condorcet's Jury Theorem (Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix. De l'imprimerie royale, 1785) to weighted voting games with voters of two kinds: a fixed (possibly empty) set of 'major' voters with fixed weights, and an ever-increasing number of 'minor' voters, whose total weight is also fixed, but where each individual's weight becomes negligible. For example, a common scenario in stock companies is that each shareholder is entitled with a number of votes (voting weight) proportional to their relative capital contribution: usually a small group of 'major' voters owns a significant number of votes – reflecting their large proportion of ownership of the capital stock – accompanied by a large 'pool' of small voters where each of these 'minor' voters has a negligible effect on voting outcomes.

As our first main result, we obtain the limiting probability that the jury will arrive at the correct decision as a function of the competence of the few major players. This result assumes that voters vote for the alternative that seems most likely correct, given the evidence. When the evidence is not perfect, however, it may be that this **informative voting strategy** is not always optimal, the reason being that the consequences of an erroneous decision can be quite costly (e.g., a risky business strategy could end up fatal). Here, the informative strategy is not in equilibrium. Minor voters are usually supposed to vote informatively no matter what, since they

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cannot affect the outcome. But if we think of the game as emerged from games with a growing number of (small) players, this conclusion is not necessarily granted.

This talk summarizes results from two papers

Lindner, I. (2008), A generalization of Condorcet's Jury Theorem to weighted voting games with many small voters, *Economic Theory*, Vol. 35(3), 607-611.

Lindner, I. and G. Owen (2009), Strategic Voting in Truth-Tracking Situations, Working paper.

A Generalization of Condorcet's Jury Theorem to Weighted Voting Games with Many Small Voters*

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Summary. We extend Condorcet's Jury Theorem (1785) to weighted voting games with voters of two kinds: a fixed (possibly empty) set of 'major' voters with fixed weights, and an ever-increasing number of 'minor' voters, whose total weight is also fixed, but where each individual's weight becomes negligible. As our main result, we obtain the limiting probability that the jury will arrive at the correct decision as a function of the competence of the few major players. As in Condorcet's result the quota $q = 1/2$ is found to play a prominent role.

Keywords and Phrases: Condorcet's Jury Theorem; Weighted voting games; Majority games.

JEL Classification Numbers: C71; D71

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1. Introduction

Condorcet (1785) considers collective decision-making, where the objective is 'truth-tracking'. The fundamental premise is that there is a unique unanimously preferred alternative (the 'truth'), but voters only have partial information and imperfect competence for detecting it. The probability of a single voter's choice being correct is taken to quantify the *competence* of a voter. Here, the quantity of interest is the *jury competence* of the decision-making body – the probability of arriving at the correct decision. Condorcet assumes equal individual competence, greater than 1/2, on a dichotomous choice. *Condorcet's Jury Theorem* shows that under simple majority rule, jury competence approaches one with increasing size of the group or increasing individual competence. Over the past decades, this celebrated result has been extended in numerous ways by statisticians, economists, political scientists, etc.[1]

The simple majority game as considered by Condorcet is a special case of a *weighted voting game* (WVG). Here, each board member is assigned a non-negative number as weight and a relative quota indicates the fraction of the total weight required for a win. The aim of this note is to provide a generalization of Condorcet's Jury Theorem to WVGs when the voters are of two kinds: a fixed (possibly empty) set of *major* (big) voters with fixed weights, and an ever-increasing number of *minor* (small) voters whose total weight is also fixed, but each individual weight becomes negligible. Using the idea that asymptotically many minor voters act like a modification of the quota for the vote among major voters,[2] the limiting jury competence is derived as a function of the competence of the few major players (as a group). As in Condorcet's result, the quota $q = 1/2$ is found to play a prominent role. We show that it maximizes the range of values of major weights for which jury competence converges to infallibility. This covers the case where major voters are absent, and Condorcet's original Jury Theorem results as a by-product.

[1] See e.g. Fey (2001) for references to recent work.

[2] Dubey and Shapley (1979) use a similar argument for analyzing asymptotic properties of the Banzhaf index.

2. The model

Consider a partition of the set of voters into two camps. The set of *major* voters is $L = \{1, ..., l\}$, where l is a natural number.[3] Each $k \in L$ is assigned a weight w_k, and let $w_L = \sum_{k \in L} w_k \in [0, 1]$ denote the combined voting weight of L. We shall consider a sequence of WVGs $\{\Gamma^\nu\}_{\nu \in \mathbb{N}}$ with a growing population of *minor* voters. In each of these games Γ^ν, the set of m^ν minor voters is denoted by $M^\nu = \{l + 1, ..., l + m^\nu\}$. For each ν, these voters have weights $\alpha_1^\nu, ..., \alpha_{m^\nu}^\nu$, which sum up to $\alpha = 1 - w_L > 0$. For any coalition $S \subset L \cup M^\nu$ we interpret $w(S)$ as the aggregate voting weight of S.

Formally, the WVG Γ^ν is described by the tuple

$$\Gamma^\nu = [q;\, w_1, \, \dots \, , w_l, \, \alpha_1^\nu, \, \dots \, , \alpha_{m^\nu}^\nu], \tag{1}$$

where $q \in (0, 1]$ is the *relative quota*. S is a *winning coalition* in Γ^ν iff $w(S) \geq q$. The latter (weak) inequality may be replaced by the strict inequality $>$. In this case we change the bracket notation in (1) to $\langle\, q;\, w_1, \, \dots \, , w_l, \, \alpha_1^\nu, \, \dots \, , \alpha_{m^\nu}^\nu \rangle$.

Put $Q^\nu := \sum_{k \leq m^\nu} [\alpha_k^\nu]^2$. Let $\{\Gamma^\nu\}_{\nu \in \mathbb{N}}$ evolve in such a way that

$$\lim_{\nu \to \infty} \alpha_{\max}^\nu / \sqrt{Q^\nu} = 0, \tag{2}$$

where $\alpha_{\max}^\nu := \max_{k \leq m^\nu} \alpha_k^\nu$. This ensures $\alpha_{\max}^\nu \to 0$ as $\nu \to \infty$, which implies $m^\nu \to \infty$.[4]

3. A Generalization of Condorcet's Jury Theorem

In a jury trial, assume a given *a priori* probability $\theta \in [0, 1]$ that the defendant is guilty of the offense charged. This models the existence of a truth independent of the jury, yet unknown its members. Each jury member (voter) k is assumed to possess a more or less reliable perception about the truth. This degree of knowledge is modeled by $p_k \in (0, 1)$, the *judgemental competence of voter k*. It is the probability that the voter will make the correct choice between the options 'guilty' or 'not

[3]Note that $l = 0$ takes care of the case where L is empty by the general convention that $\{1, \dots , 0\}$ is empty.

[4]However, it can be shown that Q^ν tends to zero so that condition (2) is stronger than $\alpha_{\max}^\nu \to 0$.

guilty'. Assume the minor voters' choices are *independent* of one another and that a common $p \in (0,1)$ exists, the probability of any minor voter making the correct decision. Hence we put $p_k = p$ for all $k \in M^\nu$.

Jury competence is measured by the likelihood of the verdict being correct. Let $C_I[\Gamma]$ denote the probability of conviction, provided the defendant is guilty. Analogously, let $C_{II}[\Gamma]$ denote the probability of acquittal in case of innocence. Jury competence then follows as

$$C[\Gamma] = \theta C_I[\Gamma] + (1-\theta)C_{II}[\Gamma]. \tag{3}$$

For the moment put $\theta = 1$, so that $C[\Gamma] = C_I[\Gamma]$ (the defendant is guilty). In the sequence of games $\{\Gamma^\nu\}_{\nu\in\mathbb{N}}$, we should expect that in the limit the continuous 'ocean' of randomly voting minor voters will be divided in such a way that the aggregate voting weight for conviction (the correct choice) is $p\alpha$. Consider the games

$$\Gamma_0 = [q - p\alpha; w_1, ..., w_l] \quad \text{and} \quad \Gamma_0' = \langle q - p\alpha; w_1, ..., w_l\rangle, \tag{4}$$

which are well-defined for $q \in \mathcal{J}(p) := (p\alpha, w_L + p\alpha)$. Γ_0 and Γ_0' can be considered limiting WVGs for the major players where the aggregate minor weight $p\alpha$ in favor of conviction is substracted from the quota q.

Let $\mathcal{B}_l = [w_L;\, w_1, w_2, ..., w_l]$ denote the *unanimity game* among the major voters in which each voter has a veto. Let $\mathcal{B}_l^* = \langle 0;\, w_1, w_2, ..., w_l\rangle$ represent the special case where the major voters operate under what Rae (1969) has called a 'rule of individual initiative': action (conviction) can be initiated by any single individual. We will show that in the sequence of games $\{\Gamma^\nu\}_{\nu\in\mathbb{N}}$, C_I converges to a limit depending on the quota q and w_L. Figure 1 gives an illustration for $p > 1/2$. Within the inner triangle $\mathcal{J}(p)$, the limit is the arithmetic mean of C_I for the games defined in (4). Outside the closure of $\mathcal{J}(p)$, the influence of the major voters is 'destroyed'.

For $\theta = 0$ (the defendant is innocent) we have $C[\Gamma] = C_{II}[\Gamma]$. Since voting for acquittal is now correct, minor voters vote for conviction with probability $1-p$. The limit scenario of C_{II} follows analogously to C_I by replacing p by $(1-p)$ and putting $C_{II} = 1 - C_I$. The resulting graph for C_{II} is homeomorphic to that in Figure 1. The inner triangle $\mathcal{J}(1-p)$ is however shifted to the left.

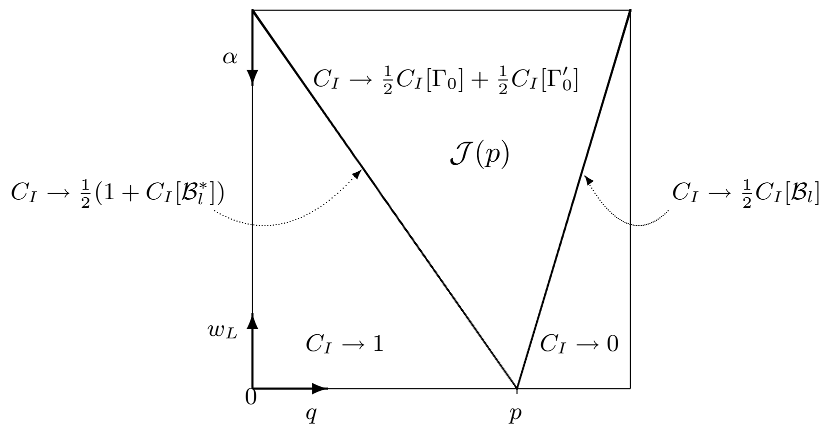

Figure 1. Limit scenario for C_I

Theorem 1. A Generalization of Condorcet Jury Theorem to WVGs
In the sequence of WVGs $\{\Gamma^\nu\}_{\nu\in\mathbb{N}}$, the limiting jury competence is a function of the competence of the few major voters. In particular, jury competence converges to

$$\lim_{\nu\to\infty} C[\Gamma^\nu] = \theta C_I + (1-\theta)C_{II}, \tag{5}$$

where C_I is given by

$$C_I = \frac{1}{2}C_I[\Gamma_0] + \frac{1}{2}C_I[\Gamma_0'], \qquad \text{if } q \in \mathcal{J}(p). \tag{6}$$

For other values of q, the right-hand side of (6) simplifies to

$$C_I = \begin{cases} 1 & \text{if } q < p\alpha, \\ 1/2\,(1 + C_I[\mathcal{B}_l^*]) & \text{if } q = p\alpha, \\ 1/2\,C_I[\mathcal{B}_l] & \text{if } q = w_L + p\alpha, \\ 0 & \text{if } q > w_L + p\alpha. \end{cases} \tag{7}$$

C_{II} is obtained by replacing p in (6) and (7) by $1-p$ and putting $C_{II} = 1 - C_I$.

The proof of Theorem 1 is available upon request. The main idea of the proof can be stated as follows. Since we assume that the minor voters' choices are independent of one another, the aggregate voting weight of any coalition of minor voters is interpreted as a sum of independent random variables. This allows us to analyze the asymptotic properties of jury competence by means of a generalized central limit theorem, the Lindeberg- Feller theorem (see e.g. Theorem 4.7 in Petrov 1995, p. 123). This method is validated as long as the weights of the minor voters are not too skewed, which is ensured by condition (2). The asymptotic behavior of jury competence begins to manifest itself at around 20 minor players. Estimations of convergence rates are available upon request. Figure 2 gives an illustration of Theorem 1.

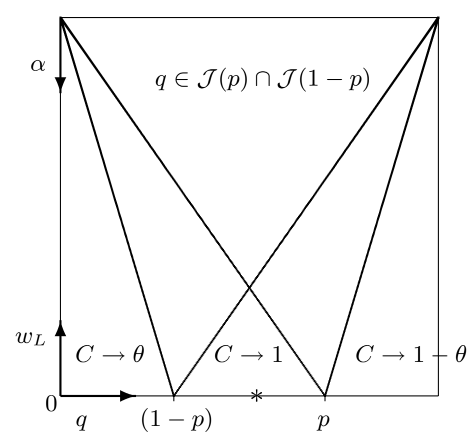

Figure 2. Generalized Jury Theorem

In the triangle-shaped area around $q = 1/2$ jury competence converges to infallibility, $C = 1$, for lower values of w_L. Note that it contains the point marked with '*' on the horizontal $w_L = 0$ (absence of major voters) and $q = 1/2$. The simple majority rule, as considered by Condorcet, is a special case of this setting in which the block of votes, $\alpha = 1$, is broken up and divided equally among an ever-increasing number of minor voters.

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Strategic Voting in Truth-Tracking Situations

Working Paper
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Abstract

In truth-tracking situations (as, for example, in a jury trial, or in deciding on a medical treatment) it seems reasonable to vote for the alternative (e.g., conviction or acquittal; surgery or radiation) that seems most likely correct, given the evidence. When the evidence is not perfect, however, it may be that this **informative voting strategy** is not always optimal, the reason being that the consequences of an erroneous decision (e.g., convicting an innocent defendant) can be quite costly. Thus the informative strategy is not in equilibrium. We analyse truth-tracking situations in weighted voting games when there is an ocean of infinitesimal players. These players are usually supposed to vote informatively no matter what, since they cannot affect the outcome. But if we think of the game as the limit of a sequence of games with an increasingly large number of (small) players, this conclusion is not necessarily granted.

1. Symmetric voting

Assume an *a priori* probability π that the defendant is guilty.[1] If guilty, then each juror receives a signal which is either G (with probability t) or NG (with probability 1-t); if not guilty, then each juror receives the signal G with probability u. The several jurors' signals are independent, contingent on defendant's guilt. We assume, of course, that $t > u$.

All jurors have the same objective, namely, to convict a guilty defendant, and to acquit an innocent defendant. They assign a cost v_1 to a type I error (convicting an innocent person) and v_2 (usually much smaller) to a type II error (acquitting a guilty person). Then, the jurors' objective function is the expected cost, namely

$$C = \pi v_2 \text{Prob}\{\text{acquittal} \mid \text{guilty}\} + (1-\pi)v_1 \text{ Prob}\{\text{conviction} \mid \text{innocent}\}$$

We treat this as a non-cooperative game, and look for equilibria. First of all, suppose all jurors vote to convict on a G signal, and acquit on NG. Let us call this, the **informative voting strategy**. The question is whether this strategy is in equilibrium.

Consider juror i's reasoning, under this assumption, and assuming she has received a signal G. Assume a relative quota q such that $k = \lceil qn \rceil$ out of n votes are needed for conviction. She knows that her vote will make a difference if and only if exactly k-1 of the other n-1 jurors vote to convict. Now, if the defendant is in fact guilty, the probability that this happens, <u>and</u> that i has received the signal G, is

$$P_a = t \binom{n-1}{k-1} t^{k-1}(1-t)^{n-k}$$

If, on the other hand, the defendant is innocent, the probability of this is

$$P_b = u \binom{n-1}{k-1} u^{k-1}(1-u)^{n-k}$$

Now, if i votes to convict, the expected loss due to i's vote is

$$(1-\pi)\, v_1 P_b$$

whereas, if she votes to acquit, the expected loss due to her vote is

[1] One possible interpretation of π is that e.g. in U.S. defendant must be indicted before coming up for trial. Indictment is work of a grand jury (as opposed to petit jury during the trial). Grand jury may make indictments easy or difficult. If grand jury makes indictments easy π may be relatively low (say, 0.3). If grand jury makes indictments difficult π would be relatively large (say, 0.8).

$$\pi \, v_2 \, P_a$$

Thus, it becomes rational for i to vote for conviction if and only if

$$\pi \, v_2 \, P_a \; \geq (1\text{-}\pi) \, v_1 P_b$$

or, equivalently, if

$$S_n \;\geq\; (1 - \pi) \; v_1 \,/\, (\;\pi v_2), \tag{1}$$

with $S_n \;:=\; (t\,/\,u)k \;[(1 - t)\,/(1 - u)]n - k$.

Similarly, we find that, given that everybody else is using the informative strategy, it is optimal for juror i to vote for acquittal, given a signal of NG, if and only if

$$S_n \;\leq\; (1 - \pi) \; v_1 \,/\, \; (\pi v_2) \quad t\,/\,u \quad (1 - u)\,/(1 - t) \tag{2}$$

We conclude.

Proposition 1.
For symmetric voting, with k out of n votes needed for conviction, and probabilities t and u respectively of obtaining a G signal in case defendant is innocent or guilty, informative voting will be in equilibrium if and only if inequalities (1) and (2) above both hold.

Note: since we assume that $t > u$, the right side of (1) is smaller than that of (2), and so at least one of these two inequalities will hold; it is possible that both of them hold.

Note: this is not the only symmetric equilibrium. There are at least two others: one in which everyone always votes to convict, regardless of the signal, and one in which everyone always votes to acquit, regardless of the signal. Since a single juror, in the minority (against all others) can never make a difference, these are clearly equilibria. We consider them *trivial* equilibria.

2. Passage to the Limit

Let us suppose now that there is an *ocean* of *infinitesimal* players. These voters are usually supposed to vote informatively no matter what, since they cannot affect the outcome. We will call this the *non-strategic case*. But if we think of the game with infinitesimal players as the limit of a sequence of games with an increasingly large number of (small) players, the question is whether this conclusion is necessarily granted. If we assume strategic behavior for small jury sizes n and let n increase at which jury size

n does a voter feel that the probability to affect the outcome (his or her voting power) is too small to make a difference and disregards strategic aspects? Under what conditions does it make a difference at all, i.e. under what conditions does the informative strategy *stay* in equilibrium for an increasing amount of players? We start by addressing the latter question.

For fixed q, consider a sequence of situations with n jurors, and $k = \lceil qn \rceil$ votes needed for conviction. In general, we have

Proposition 2. *The non-strategic case*

As $n \to \infty$ the expected loss C approaches the following limits:

$$\begin{array}{ll} (1-\pi)v_1 & \text{if } t > u > q \\ (1-\pi)v_1/2 & \text{if } u = q \\ 0 & \text{if } t > q > u \\ \pi v_2/2 & \text{if } t = q \\ \pi v_2 & \text{if } q > t > u \end{array}$$

Proof: The proof appears in the Appendix.

Note that the expected loss tends to zero for $t > q > u$. The reason is that with increasing n it becomes almost certain that a share of t voters vote "conviction" when the defendant is guilty but only a share of u voters vote to convict when the defendant is innocent. Since $t > q > u$ this ensures convergence to infallibility of the collective decision when n increases. Note also that this proposition confirms the finding of the classical Condorcet Jury Theorem which is a statement on the special case u < q = 1/2 < t.

We will next analyze the strategic case. Suppose n, the size of the jury, is fixed. What is the optimal value for q for the non-strategic case? Of course, this is not a well-posed question. Presumably, the desire is, as before, to minimize the expected loss. There is a problem, however, as it is not easy to determine how the jurors will act. We assume each will try to minimize the expected loss due to errors of either type, but, especially where there is more than one equilibrium, the "optimality" of the design will depend too much on the equilibrium chosen.

Suppose, then, that we would like to encourage the jurors to use informative voting. According to Proposition 1 the question is whether the equations (1) and (2) are met for $n \to \infty$. We get

$$S_n = \left\{ \left(t/u\right)^{k/n} \left[\left(1-t\right)/\left(1-u\right)\right]^{1-k/n} \right\}^n \tag{3}$$

We find that S_n tends to 0 if the term in curly brackets is smaller than 1. This can be achieved by setting q= k/n smaller than

$$\mu \ := \ \ln(1\text{-}t)/(1\text{-}u) \ \ / \ \ \{ \ \ln(1\text{-}t)/(1\text{-}u) \ \ - \ \ln \ t/u \ \}. \tag{4}$$

Note that it is always possible to set q equal to μ since μ is between 0 and 1. We can make an even stronger statement.

Lemma 1. *For any values of u and t with u<t it holds that*
u< μ < t .

For a proof see the appendix.

We summarize are findings as follows.

Corollary 1. *The strategic case*
For fixed q, consider a sequence of situations with n jurors, and k = ⌈qn⌉ votes needed for conviction. For increasing number of jurors we have the following statements.

- *(i)* *For q < μ acquittal eventually becomes the equilibrium strategy and expected cost approaches $C = \pi v_2$.*
- *(ii)* *For q > μ conviction eventually becomes the equilibrium strategy and expected cost approaches $C=(1\text{-}\pi)v_1$.*
- *(iii)* *For q = μ and if inequalities (1) and (2) above both hold the informative voting strategy stays in equilibrium and expected cost tend to zero.*

Next, we will answer the question whether it is always possible to choose q such that (1) and (2) hold.

Proposition 3: *Consider a sequence of situations with n jurors, and k = ⌈qn⌉ votes needed for conviction. There exist an N such the quota*

$$q^n = \frac{\ln\dfrac{1-t}{1-u} - \dfrac{1}{n}\ln\dfrac{(1-\pi)v_1}{\pi v_2}}{\ln\dfrac{1-t}{1-u} - \ln\dfrac{t}{u}} \tag{5}$$

ensures (1) and (2) to hold for all $n \geq N$.
Moreover, $\lim_{n\to\infty} q^n = \mu$ such that q^n lies between u and t for sufficiently large N.
For t = (1-u) implying the signals are equally informative, we have $\lim_{n\to\infty} q^n = 1/2$.

Proof of Proposition 3: For a fixed jury size n solving for q^n such that (1) holds with equality provides (5). Note that we assume $t > u$. Hence the right side of (2) is larger than that of (1) such that q^n will also ensure (2). With increasing n the term $\frac{1}{n}\ln\frac{(1-\pi)\nu_1}{\pi\nu_2}$ tends to zero. According to Lemma 1 this ensures that q^n is between t and u for sufficiently large n.

We summarize our findings by concluding that having any influence on the quota q it would be optimal to set q equal to (5) for sufficiently large n. If the voters behave non-strategically this ensure infallibility in the limit. In the strategic case this choice encourages informative voting which again tends to infallibility.

Note that this finding holds regardless of the quality of the signal t and u. We merely put the careful assumption that t > u. In fact, the conclusion even holds if the signals are misleading, i.e. for t and u smaller than 1/2 . (The latter could apply when evidence is falsified.)

Remark: Note that Proposition 2 confirms Condorcet's Jury Theorem in that if the signals t and 1-u are larger than ½, then the majority quota q=1/2 leads to infallibility of the jury. Similarly, when the quality of the signals is the same, i.e. t=1-u, the term μ simplifies to ½.

4. Outlook: Large Weighted Voting Games

Lindner (2008) extends Condorcet's Jury Theorem (the nonstrategic case) to weighted voting games with voters of two kinds: a fixed (possibly empty) set of 'major' voters with fixed weights, and an ever-increasing number of 'minor' voters, whose total weight is also fixed, but where each individual's weight becomes negligible. As a main result, she obtains the limiting probability that the jury will arrive at the correct decision as a function of the competence of the few major players. As in Condorcet's result the quota q = 1/2 is found to play a prominent role. The question is now how to extend this result to the strategic case, and, possibly, signals of different reliability.

Definition 1: q-chain

Let

$$N^{(0)} \subset N^{(1)} \subset N^{(2)} \subset ....$$

be an infinite increasing chain of finite non-empty sets, and let

$$N = \bigcup_{n=0}^{\infty} N^{(n)}.$$

Let w be a weight function that assigns to each $i \in N$ a positive real number w_i as weight; and let q be a real $\in (0,1)$.
For each n let $W^{(n)}$ be the weighted voting game whose assembly is $N^{(n)}$ - each voter $i \in N^{(n)}$ being endowed with the pre-assigned weight w_i - and whose relative quota is q.
We shall then say that $\left\{W^{(n)}\right\}_{n=0}^{\infty}$ is a q-chain of weighted voting games.

Given a q-chain of WVGs we associate with it the family of independent random variables $\left\{X_j \mid j \in N\right\}$, indexed by N, such that for every $j \in N$,

$$\mathrm{P}\left\{X_j = w_j\right\} = t,$$
$$\mathrm{P}\left\{X_j = 0\right\} = 1 - t,$$

if the defendant is guilty. If non-guilty t is replaced by u.

A jury member i knows that her vote will make a difference if and only the other n-1 jurors vote such that the combined weight sum of those voting in favor of conviction lies in the interval $\left[q\left(\sum_{j \in N^{(n)}} w_j \right) - w_i, q \sum_{j \in N^{(n)}} w_j \right)$. Now, if the defendant is in fact guilty, the probability that this happens, <u>and</u> that i has received the signal G, is

$$\begin{aligned} P_a &= t\, P\left[q\left(\sum_{j \in N^{(n)}} w_j \right) - w_i \le \sum_{\substack{j \in N^{(n)}, \\ j \ne i}} X_j < q \sum_{j \in N^{(n)}} w_j \right) \\ &= t\, P_{\neg i}(t) \end{aligned} \tag{6}$$

If, on the other hand, the defendant is innocent, the probability of this is

$$P_b = u\, P_{\neg i}(u)$$

Now, if i votes to convict on signal G, the expected loss due to i's vote is

$$(1\text{-}\pi)\, v_1 P_b$$

whereas, if she votes to acquit, the expected loss due to her vote is

$$\pi\, v_2\, P_a$$

Thus, it becomes rational for *i* to vote for conviction if and only if

$$\pi\, v_2\, P_a \;\geq (1-\pi)\, v_1 P_b$$

or, equivalently, if

$$P_{\neg i}(t)\;\;/\;\;P_{\neg i}(u)\;\geq \text{u/t}\;\;(1-\pi)\, v_1/(\pi\, v_2).$$

Analogously, if the defendant is in fact guilty, the probability that i is critical, and that *i* has received the signal NG, is

$$P_c = (1-t)\, P_{\neg i}(t)$$

If, on the other hand, the defendant is innocent, the probability of this is

$$P_d = (1-u)\, P_{\neg i}(u)$$

We find that, given that everybody else is using the informative strategy, it is optimal for juror *i* to vote for acquittal, given a signal of NG, if and only if

$$P_{\neg i}(t)\;\;/\;\;P_{\neg i}(u)\;\leq \text{(1-u)/(1-t)}\;\;(1-\pi)\, v_1/(\pi\, v_2).$$

In summary, the informative strategy is in equilibrium if

$$u/t\;\;(1-\pi)\; v_1/(\;\pi v_2)\;\leq \mathrm{P}_{\neg i}(\mathrm{t})/\mathrm{P}_{\neg i}(\mathrm{u})\;\leq \text{(1-u)/(1-t)}\;(1-\pi)\; v_1/(\;\pi v_2)\;. \tag{7}$$

The question is now as to the limit behavior of $P_{\neg i}(t)\,/\,P_{\neg i}(u)$ for an increasing jury size n. Note that $P_{\neg i}(t)$ and $P_{\neg i}(u)$ is a distribution of a sum of independent random variables. It is therefore tempting to work with general versions of the central limit theorem which state that for sufficiently large n $P_{\neg i}(t)$ and $P_{\neg i}(u)$ will be approximately normal distributed.

For any $i \in N$ let us put

$$S_{\neg i}^{(n)}(t) := \left(\sum_{k\in N^{(n)}} X_j\right) - X_i, \qquad \mu_{\neg i}^{(n)}(t) := E\left[S_{\neg i}^{(n)}\right], \qquad \sigma_{\neg i}^{(n)}(t) := \left(Var\left[S_{\neg i}^{(n)}\right]\right)^{1/2}. \tag{8}$$

And let $S_{\neg i}^{(n)}$ be the 'standardized' form of $S_{\neg i}^{(n)}$, i.e.

$$\overline{S}_{\neg i}^{(n)}(t) := \frac{S_{\neg i}^{(n)}(t) - \mu_{\neg i}^{(n)}(t)}{\sigma_{\neg i}^{(n)}(t)}. \tag{9}$$

Using the definition of the X_i it is easy to obtain the following explicit expression for $\mu_{\neg i}^{(n)}$ and $\sigma_{\neg i}^{(n)}$

$$\begin{aligned} \mu_{\neg i}^{(n)}(t) &= t\left(\sum\nolimits_{j \in N^{(n)}} w_j\right) - t w_i \\ \sigma_{\neg i}^{(n)}(t) &= \sqrt{t(1-t)}\sqrt{\left(\sum\nolimits_{j \in N^{(n)}} w_j{}^2\right) - w_i^2}. \end{aligned} \tag{10}$$

From (6) follows

$$P_{\neg i}(t) \approx \Phi\left(\frac{q \sum\limits_{j \in N^{(n)}} w_j - \mu_{\neg i}^{(n)}(t)}{\sigma_{\neg i}^{(n)}(t)}\right) - \Phi\left(\frac{q\left(\sum\limits_{j \in N^{(n)}} w_j\right) - w_i - \mu_{\neg i}^{(n)}(t)}{\sigma_{\neg i}^{(n)}(t)}\right) \tag{11}$$

One major difficulty is, however, that for increasing n not only the approximation error tends to zero but also both $P_{\neg i}(t)$ and $P_{\neg i}(u)$ as well as their normal approximations given by (11). With analyzing the limit behavior $P_{\neg i}(t) / P_{\neg i}(u)$ by means of the approximations we therefore face the problem of a ratio of two zero sequences and hence whether in the limit process the error terms distort the problem.

Appendix.

Proof of Proposition 2.

Let

$$C_1 = \sum_{0 \le j \le k-1} \binom{n}{j} t^j (1-t)^{n-j}$$

and

$$C_2 = \sum_{k \le j \le n} \binom{n}{j} u^j (1-u)^{n-j}$$

Then the expected costs are

$$\mathrm{C} = \pi v_2 C_1 + (1-\pi) v_1 C_2 .$$

Now

$$C_1 = \text{Prob}\{ X < k \}$$

where X is a binomial random variable with parameters n and t, and thus mean nt and variance $nt(1\text{-}t)$. For large n, this will be an approximately normal variable with this mean Φand variance. Thus we have the approximation

$$\mathrm{C}_1 \approx \Phi\left(\frac{k - nt}{\sqrt{nt\text{-}nt^2}} \right)$$

where Φ is the normal distribution function.

Since $k = \lceil nq \rceil$, we have the further approximation

$$\frac{k - nt}{\sqrt{nt\text{-}nt^2}} \approx \frac{nq - nt}{\sqrt{nt\text{-}nt^2}} = \frac{(q\text{-}t)\sqrt{n}}{\sqrt{t\text{-}t^2}}$$

Thus

$$\mathrm{C}_1 \approx \Phi\left(\frac{(q\text{-}t)\sqrt{n}}{\sqrt{t\text{-}t^2}} \right)$$

and, as $n \to \infty$, this expression converges to 0 if $q < t$, to 1 if $q > t$, and to ½ if $q = t$.

In a similar manner, C_2 converges to 0 if $q > u$, to 1 if $q < u$, and to ½ if $q = u$. This proves the theorem.

Proof of Lemma 1.

Part 1. Let

$$g(t, u) = t \ln(t/u) + (1-t) \ln[(1-t)/(1-u)].$$

We claim that, for t, u in (0, 1), this function is non-negative. In fact, it is 0 if t=u, and positive otherwise.

Clearly g = 0 if t = u. Differentiating, we have

$\partial g/\partial u = -t/u + (1-t)/(1-u)$

Setting this derivative equal to zero, we obtain

$t/u = (1-t)/(1-u)$

$t(1-u) = u(1-t)$

$u = t.$

Thus, for fixed t, there is only one stationary point (in u), and it is at u = t. If we differentiate again, we find

$\partial^2 g/\partial u^2 = t/u^2 + (1-t)/(1-u)^2,$

which is clearly positive, so that we have a local minimum. But there are no other stationary points, and thus this is the global minimum. Since g(t, t) = 0, we conclude that the function is never negative, and, in fact, is positive for all $u \neq t$.

Part 2. Consider now the function

$F(t, u) = v/w,$

where

$v = \ln(1-t) - \ln(1-u)$
$w = \ln(1-t) - \ln(1-u) - \ln t + \ln u.$

Clearly, v = w = 0 whenever u = t.
Thus F(t, t) is undetermined. We can however fix t and apply L'Hôpital's rule (in u)

$\text{Lim}_{u \to t}\, v/w = \text{Lim}\, v'/w' = \text{Lim}\, [1/(1-u)]/[1/(1-u) + 1/u] = \text{Lim}\, u = t.$

We conclude that setting F(t, t) = t gives us a continuous function.

Next we differentiate F with respect to t. We find

$$w\, \partial v/\partial t \;=\; -\,[\ln(1\text{-}t) - \ln(1\text{-}u) + \ln u - \ln t] \,/\, (1\text{-}t)$$

$$v\, \partial w/\partial t \;=\; -\,[\ln(1\text{-}t) - \ln(1\text{-}u)] \,/\, (1\text{-}t) \;-\, [\ln(1\text{-}t) - \ln(1\text{-}u)] \,/\, t$$

Some algebra now gives us

$$w\, \partial v/\partial t \;-\, v\, \partial w/\partial t \;=\; \frac{t(\ln t - \ln u) + (1\text{-}t)\,[\ln(1\text{-}t) - \ln(1\text{-}u)]}{t(1\text{-}t)} \;=\; \frac{g(t, u)}{t(1\text{-}t)}$$

where g is as in part 1 above. Thus

$$\partial F/\partial t \;=\; g(t, u) \,/\, [t(1\text{-}t)w^2]$$

All the terms appearing in this last expression are positive so long as $t, u \in (0, 1)$, $t \neq u$. Thus (for fixed u) F is an increasing function in t. It follows that, if $t > u$,

$$F(t, u) > F(u, u) = u$$

Finally, note that F is symmetric in t and u. Thus (for fixed t) F is also an increasing function in u. Hence, if $u < t$,

$$F(t, u) < F(t, t) \;=\; t.$$

We conclude that, for $u < t$,

$$u < F(t, u) < t.$$

References

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