Critical Strategies Under Approval Voting: Who Gets Ruled In And Ruled Out

Steven J. Brams
Department of Politics
New York University
New York, NY 10003
USA
steven.brams@nyu.edu

M. Remzi Sanver
Department of Economics
Istanbul Bilgi University
80310, Kustepe, Istanbul
TURKEY
sanver@bilgi.edu.tr

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Abstract

We introduce the notion of a "critical strategy profile" under approval voting (AV), which facilitates the identification of all possible outcomes that can occur under AV. Included among AV outcomes are those given by scoring rules, single transferable vote, the majoritarian compromise, Condorcet systems, and others as well. Under each of these systems, a Condorcet winner may be upset through manipulation by individual voters or coalitions of voters, whereas AV ensures the election of a Condorcet winner as a strong Nash equilibrium wherein voters use sincere strategies. To be sure, AV may also elect Condorcet losers and other lesser candidates, sometimes in equilibrium. This multiplicity of (equilibrium) outcomes is the product of a social-choice framework that is more general than the standard preference-based one. From a normative perspective, we argue that voter judgments about candidate acceptability *should* take precedence over the usual social-choice criteria, such as electing a Condorcet or Borda winner.

Keywords: approval voting; elections; Condorcet winner/loser; voting games; Nash equilibrium.

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1. Introduction

Our thesis in this paper is that *several* outcomes of single-winner elections may be socially acceptable, depending on voters' individual views on the acceptability of the candidates. To support this thesis, we extend the social-choice framework to include information not only on how voters rank candidates but also on where they draw the line between those they consider acceptable and those they consider unacceptable.

This new information is precisely that which is elicited by approval voting (AV), whereby voters can approve of as many candidates as they like or consider acceptable. This gives them the opportunity to be sovereign by expressing their approval for any set of candidates, which no other voting system permits. In so doing, AV better enables voters both to elect and to prevent the election of candidates, as we show in the paper.

The set of AV outcomes may be quite large. For example, Condorcet winners are always AV outcomes, as are scoring-rule winners, which include plurality-voting and Borda-count winners. Social-choice rules that are neither Condorcet nor rely on scoring, including the Hare system of single transferable vote (STV) and the majoritarian compromise (MC), also give AV outcomes. Still other outcomes that none of these systems yields may be AV outcomes.

Although AV may lead to a plethora of outcomes, they are not haphazard choices that can easily be upset when voters are manipulative. The candidate elected by AV will often be a candidate at a Nash equilibrium, from which voters with the same preferences will have no incentive to depart. Moreover, AV elects a unique Condorcet winner (if one exists) as a *strong* Nash equilibrium, which yields outcomes that are invulnerable to departures by any set of voters. In allowing for other Nash-equilibrium outcomes, including even Condorcet losers, AV does not rule out other outcomes that may, on occasion, be more acceptable to voters.

Saari and Van Newenhizen (1988a) and Saari (1994, 2001), among others, have argued that it is a vice that AV can lead to a multiplicity of outcomes.[1] We argue, on the contrary, that voters' judgments about candidate acceptability *should* take precedence over standard social choice criteria such as electing a Condorcet or Borda winner, which—as we will show—may clash with these judgments.[2]

Because the notion of acceptability is absent from preference-based systems, these systems do not distinguish voters who judge only a first choice acceptable from voters who judge all except a last choice acceptable. We believe this is critical information that the social choice should reflect. Hence, we expand the standard social-choice framework to incorporate it.

In section 2, we define preferences and describe admissible and sincere strategies under AV. In section 3 we characterize AV outcomes and describe the "critical strategy profile"—a key concept in this paper—that produces them. We compare AV outcomes with those given by other voting systems. Among other things, we show that no "fixed rule," in which voters vote for a predetermined number of candidates, always elects a unique Condorcet winner, suggesting the need for a more flexible system.

[1] The critique of AV by Saari and Van Newenhizen (1988a) provoked an exchange between Brams, Fishburn, and Merrill (1988a, 1988b) and Saari and Van Newenhizen (1988b) over whether the plethora of AV outcomes more reflected AV's "indeterminacy" (Saari and Van Newenhizen) or its "responsiveness" (Brams, Merrill, and Fishburn); other critiques of AV are referenced in Brams and Fishburn (2005).
[2] In fact, it is an old story that standard social choice criteria may clash among each other. For example, even when there is a Condorcet winner, who can defeat every other candidate in pairwise contests, there may be a different Borda-count winner, who on the average is ranked higher than a Condorcet winner. See Nurmi (1999, 2002) and Brams and Fishburn (2002) for other examples that have produced many of the so-called paradoxes of social-choice theory.

The stability of outcomes under AV is analyzed in section 4. Besides the Nash-equilibrium results alluded to above, we show that Condorcet voting systems, which guarantee the election of Condorcet winners when voters sincerely rank candidates, may not elect Condorcet winners in equilibrium.

In section 5 we conclude on a normative note that voters *should* be able to express their approval of any set of candidates. Likewise, a voting system *should* allow for the possibility of multiple acceptable outcomes, especially in close elections. That AV more than other voting systems is responsive in this way we regard as a virtue. That it singles out as strong Nash-equilibrium outcomes unique Condorcet winners may or may not be desirable. We discuss these and other questions related to the nature of acceptable outcomes, where we suggest that "acceptability" replace the usual social-choice criteria for assessing the satisfactoriness of election outcomes.

2. Preferences and Strategies under AV

Consider a set of voters choosing among a set of candidates. We denote individual candidates by small letters *a*, *b*, *c*, …. A voter's *strict preference relation* over candidates will be denoted by *P*, so *aPb* means that a voter strictly prefers *a* to *b*, which we will denote by the following left-to-right ranking (separated by a space): *a b*. We assume in the subsequent analysis that all voters have strict preferences, so they are not indifferent among two or more candidates.[3]

We assume that every voter has a *connected* preference: For any *a* and *b*, either *a b* or *b a* holds. Moreover, *P* is *transitive*, so *a c* whenever *a b* and *b c*. The list of preferences of *all* voters is called a *preference profile* **P**.

An *AV strategy S* is a subset of candidates. Choosing a strategy under AV means voting for all candidates in the subset and no candidates outside it. The list of strategies of all voters is called a *strategy profile* **S**.

The number of votes that candidate *i* receives at **S** is the number of voters who include *i* in the strategy *S* that they select. For any **S**, there will be a set of candidates ("winners") who receive the greatest number of votes.

We assume that voters use admissible and sincere strategies. An AV strategy *S* is *admissible* if it is not dominated in a game-theoretic sense. Admissible strategies under AV involve always voting for a most-preferred candidate and never voting for a least-preferred candidate (Brams and Fishburn, 1978, 1983).

An AV strategy is *sincere* if, given the lowest-ranked candidate that a voter approves of, he or she also approves of all candidates ranked higher. Thus, if *S* is sincere, there are no "holes" in a voter's approval set: Everybody ranked above the lowest-ranked candidate that a voter approves of is also approved; and everybody ranked below is not approved.[4]

A strategy profile **S** is said to be sincere if and only if the strategy *S* that every voter chooses is sincere, based on each voter's preference *P*. Sincere strategies are always admissible if we exclude "vote for everybody," which we henceforth do. To be sure, the sincerity assumption is a simplification, but it does not significantly alter our main findings.

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[3] This restriction simplifies the analysis; its relaxation to allow for voter indifference among candidates has no significant effect on our findings.

[4] Admissible strategies may be insincere if there are four or more candidates. For example, if there are exactly four candidates, it may be admissible for a voter to approve of his or her first and third choices without also approving of a second choice (see Brams and Fishburn, 1983, pp. 25-26, for an example). However, the circumstances under which this happens are sufficiently rare and nonintuitive that we henceforth suppose that voters choose only sincere strategies under AV.

As we will illustrate shortly, voters may have multiple sincere strategies, which is precisely what creates the indeterminacy that some analysts find problematic but which we find helpful. In expanding the opportunities for voters to express themselves, AV abets the discovery of consensus choices.

As an illustration of these concepts, assume that there are 7 voters, who can be grouped into three different types and who vote for the set of four candidates $\{a, b, c, d\}$:

Example 1. (i) 3 voters: *a b c d*; (ii) 2 voters: *b c a d;* (iii) 2 voters: *d b c a.*

The three types define the preference profile **P** of all 7 voters. For simplicity, we assume in this example and later that all voters of each type choose the same strategy *S*.

Voters of type (1) have three sincere strategies: $\{a\}$, $\{a, b\}$, and $\{a, b, c\}$, which for convenience we write as *a*, *ab*, and *abc*. A typical sincere strategy profile of the 7 voters is **S** = (*a*, *a*, *a*, *bc*, *bc*, *dbc*, *dbc*), whereby the 3 voters of type (1) approve of only their top candidate, the 2 voters of type (2) approve of their top two candidates, and the 2 voters of type (3) approve of all candidates except their lowest-ranked. The number of votes of each candidate at **S** is 4 votes for *b* and *c*, 3 votes for *a*, and 2 votes for *d*. Hence, AV selects candidates $\{b, c\}$ as the (tied) winners at **S**.

3. Election Outcomes under AV

Given a preference profile **P**, we consider the set of all candidates that *can* be chosen by AV when voters use sincere strategies. We call this the set *AV outcomes* at **P**. Clearly, a candidate ranked last by all voters cannot be in this set, because it is inadmissible for any voter to vote for this candidate.

Define an *AV critical strategy profile* for candidate i at preference profile **P** as follows: Every voter who ranks i as his or her worst candidate votes only for the candidate that he or she ranks top. The remaining voters vote for i *and* all candidates they prefer to i.

Let $C_i(\mathbf{P})$ be the AV critical strategy profile of candidate i at **P**. In Example 1, the critical strategy profile for candidate a is $C_a(\mathbf{P}) = (a, a, a, bca, bca, d, d)$, giving a 5 votes compared to 2 votes each for b, c, and d. It can easily be seen that $C_i(\mathbf{P})$ is admissible and sincere.

We next give three lemmata that provide a theoretical foundation for several of our subsequent propositions. They (i) show that under AV candidate i cannot do better than at $C_i(\mathbf{P})$; (ii) characterize AV outcomes; and (iii) characterize outcomes that can never by chosen under AV.

Lemma 1. *Assume all voters choose sincere strategies. The AV critical strategy profile for candidate i,* $C_i(\mathbf{P})$*, maximizes the difference between the number of votes that i receives and the number of votes that every other candidate j receives.*

Proof. Clearly, no other sincere strategy profile yields candidate i more votes than its AV critical strategy profile $C_i(\mathbf{P})$. Now consider the number of votes received by any other candidate j at $C_i(\mathbf{P})$. Candidate j will receive no fewer and sometimes more votes if there are the following departures from $C_i(\mathbf{P})$:

(i) a voter who ranked candidate i last, and therefore did not vote for him or her, votes for one or more candidates ranked below his or her top-ranked choice (possibly including candidate j); or

(ii) a voter who did not rank candidate *i* last votes for one or more candidates ranked below *i* (possibly including candidate *j*) unless *i* is ranked next-to-last.

In either case, candidate *j* never gets fewer, and may get more, votes when there are these departures from candidate *i*'s critical strategy profile $C_i(\mathbf{P})$. Besides (i) and (ii), the only other possible departure would be one in which a voter does not vote for candidate *i,* who is not ranked last, but only candidates above *i*. But this gives candidate *i* fewer votes than $C_i(\mathbf{P})$ and *j* possibly as many. Q.E.D.

Using Lemma 1, we give a simple way to determine whether any candidate *i* is an AV outcome:

Lemma 2. *Candidate i is an AV outcome if and only if i is chosen at his or her critical strategy profile* $C_i(\boldsymbol{P})$*.*

Proof. The "if" part is a direct consequence of the fact that $C_i(\mathbf{P})$ is sincere. To show the "only if" part, suppose candidate *i* is not chosen by AV at $C_i(\mathbf{P})$. By Lemma 1, $C_i(\mathbf{P})$ maximizes the difference between the number of votes that *i* receives and the number of votes that any other candidate *j* receives, so there is no other sincere, admissible strategy profile at which *i* can be chosen by AV. Q.E.D.

Using Lemma 2, we give a characterization of candidates that *cannot* be AV outcomes.

Lemma 3. *Given any preference profile* $\boldsymbol{P}$ *and any candidate i, i cannot be an AV outcome if and only if there exists some other candidate j such that the number of voters* *who consider j as their best choice and i as their worst choice exceeds the number of voters who prefer i to j.*

Proof. Given any preference profile **P** and any two candidates i and j, voters can be partitioned into three (disjoint) classes: (i) those who prefer i to j; (ii) those who consider j as the best choice and i as the worst choice; and (iii) those who prefer j to i but do not fall into class (ii). At critical strategy profile $C_i(\mathbf{P})$, the voters in class (i) will vote for i but not j; those in class (ii) will vote for j but not i; and those in class (iii) will vote for both i and j. Setting aside class (iii), which gives each candidate the same number of votes, candidate i cannot be selected at $C_i(\mathbf{P})$ if and only if the number of voters in class (ii) exceeds the number of voters in class (i). Hence, by Lemma 2 candidate i cannot be an AV outcome. Q.E.D.

In effect, Lemma 3 extends Lemma 2 by saying precisely when candidate i will be defeated by candidate j and cannot, therefore, be an AV outcome.

AV can generate a plethora of outcomes. Consider again Example 1, in which we showed earlier that AV selects candidate a at $C_a(\mathbf{P})$. Similarly, AV selects candidates b and $\{b, c\}$, all with 7 votes, at critical strategy profiles $C_b(\mathbf{P}) = \{ab, ab, ab, b, b, db, db\}$ and $C_c(\mathbf{P}) = \{abc, abc, abc, bc, bc, dbc, dbc\}$. However, $C_d(\mathbf{P}) = \{a, a, a, b, b, d, d\}$, so candidate a (3 votes) rather than candidate d (2 votes) is chosen at candidate d's critical strategy profile.[5] In sum, the set of AV outcomes that are possible in Example 1 is $\{a, b, \{b, c\}\}$.

[5] That d cannot be chosen also follow from Lemma 3: More voters (3) consider a as their best choice and d as their worst choice than prefer d to a (2).

A candidate is a *Pareto candidate* if there is no other candidate that all voters rank higher. Example 1 illustrates three things about the tie-in of Pareto candidates and AV outcomes: (i) *a* and *b* are Pareto candidates and AV outcomes; (ii) *c* is not a Pareto candidate but is a component of an AV outcome (it ties with *b* at $C_c(\mathbf{P})$); and (iii) *d* is a Pareto candidate but not an AV outcome. These observations are generalized by the following proposition:

Proposition 1. *The following are true about the relationship of Pareto candidates and AV outcomes:*

(i) At every preference profile $\mathbf{P}$*, there exists a Pareto candidate that is an AV outcome or a component of an AV outcome;*

(ii) Not every Pareto candidate is necessarily an AV outcome; and

(iii) A non-Pareto candidate may be a component of an AV outcome but never a unique AV outcome.

Proof. To show (i), take any preference profile $\mathbf{P}$. Assume that every voter votes only for his or her top choice. Then the one or more candidates chosen by AV, because they are top-ranked by some voters, must be Pareto candidates. To show (ii), it suffices to check the critical strategy profile $C_d(\mathbf{P})$ of Example 1, wherein candidate *d* is not an AV outcome but is a Pareto candidate because *d* is top-ranked by the 2 type (3) voters.

In Example 1, we showed that *c* is not a Pareto candidate but is a component of an AV outcome. To show that a non-Pareto candidate can never be a unique AV outcome and prove (iii), consider any $\mathbf{P}$ at which there exists a non-Pareto candidate *i* that is a component of an AV outcome. Take any sincere strategy profile $\mathbf{S}$ where this outcome is selected. Because *i* is not a Pareto candidate, there exists some other candidate *j* that every voter prefers to *i*. Hence, every voter who voted for *i* at **S** must have voted for *j* as well, which implies that *i* and *j* tie for the most votes. Indeed, *all* candidates *j* that Pareto dominate *i* will be components of an AV outcome at **S**. Because at least one of the candidates *j* that Pareto-dominate *i* must be ranked higher by one or more voters than all other candidates *j*, AV picks a Pareto candidate that ties candidate *i*. Q.E.D.

In Example 1, candidate *b* is the *Condorcet winner*, who can defeat all other candidates in pairwise contests, and candidate *d* is the *Condorcet loser*, who is defeated by all other candidates in pairwise contests. Not surprisingly, *b* is an AV outcome but *d* is not. However, consider the following 7-voter, 3-candidate example:

Example 2. (i) 3 voters: *a b c*; (ii) 2 voters: *b c a*; (iii) 2 voters: *c b a*.

Notice that the 2 type (ii) and the 2 type (iii) voters prefer candidates *b* and *c* to candidate *a*, so *a* is the Condorcet loser. But because the critical strategy profile of candidate *a* is $C_a(\mathbf{P}) = (a, a, a, b, b, c, c)$, *a* is an AV outcome—as are also candidates *b* and *c*, rendering all three candidates in this example AV outcomes.

We summarize the Condorcet properties of AV outcomes with our next proposition:

Proposition 2. *Condorcet winners are always AV outcomes, whereas Condorcet losers may or may not be AV outcomes.*

Proof. If candidate *i* is a Condorcet winner, a majority of voters prefer *i* to every other candidate *j*. This implies that fewer voters rank *j* as their best choice and *i* as their worst choice, which by Lemma 3 implies that candidate *i* is an AV outcome. That a

Condorcet loser may not be an AV outcome is shown by candidate *d* in Example 1, whereas candidate *a* in Example 2 shows that a Condorcet loser may be an AV outcome. Q.E.D.

Define a *fixed rule* as a voting system in which voters vote for a predetermined number of candidates. “Limited voting” uses a fixed rule; this system is frequently used in multiwinner elections, such as for a city council, in which voters can vote for, and only for, the number of candidates to be elected.

Proposition 3. *No fixed rule may elect a unique Condorcet winner.*

Proof. Consider the following 17-voter, 3-candidate example (Moulin, 1988):

Example 3. (i) 6 voters: *a b c*; (ii) 4 voters: *b a c*; (iii) 4 voters: *b c a;* (iv) 3 voters*: c a b.*

Vote-for-1 and vote-for-2 both elect candidate *b*. Thus, neither of the fixed rules elects the unique Condorcet winner, candidate *a*. Q.E.D.

By contrast, several sincere strategies, including $C_a(\mathbf{P})$ = (*a*, *a*, *bda*, *bda*, *ca*)—in which different voter types vote for different numbers of candidates—elect *a*. Clearly, the flexibility of AV may be needed to elect a unique Condorcet winner.

We next turn to scoring rules and analyze the relationship between the winner they select and AV outcomes. The best-known scoring rule is the Borda count (BC): Given that there are *n* candidates, BC awards $n - 1$ points to each voter’s first choice, $n - 2$ points to each voter’s second choice, …, and 0 points to his or her worst choice.

In Example 1, the BC winner is candidate *b*, who receives from the three types of voters a Borda score of 3(2) + 2(3) + 2(2) = 16 points. In Example 2, the BC winner is also candidate b, who receives from the three types of voters a Borda score of 3(1) + 2(2) + 2(1) = 9 points. In these examples, the BC winners coincide with the Condorcet winners, making them AV outcomes (Proposition 2), but this need not be the case, as we will illustrate shortly.

There are other *scoring rules* besides BC, so we begin with a definition. Given m candidates, fix a non-increasing vector $(s_1,..., s_m)$ of real numbers ("scores") such that $s_i \geq s_{i+1}$ for all $i \in \{1,..., m - 1\}$ and $s_1 > s_m$. Each voter's k^{th} best candidate receives score s_k. A candidate's score is the sum of the scores that he or she receives from all voters.

For a preference profile **P**, a scoring rule selects the candidate or candidates that receive the highest score. A scoring rule is said to be *strict* if it is defined by a decreasing vector of scores, $s_i > s_{i+1}$, for all $i \in \{1,..., m - 1\}$.

We next show that all scoring-rule winners, whether they are Condorcet winners or not, are AV outcomes, but candidates that are selected by no scoring rule may also be AV outcomes:

Proposition 4. *At all preference profiles* ***P****, a candidate chosen by any scoring rule is an AV outcome. There exist preference profiles* ***P*** *at which a candidate is not chosen by any scoring rule but is, nevertheless, an AV outcome.*

Proof. We begin by proving the first statement. Take any preference profile **P** and any candidate i chosen by a scoring rule at **P**. Let $(s_1,..., s_m)$ be the scoring-rule vector that results in the election of candidate i at **P**. By a normalization of the scores, we can without loss of generality assume that $s_1 = 1$ and $s_m = 0$.

Note that AV can be seen as a variant of a nonstrict scoring rule, whereby every voter gives a score of 1 to the candidates in his or her strategy set S (approved candidates) and a score of 0 to those not in this set. AV chooses the candidate or candidates with the highest score.[6]

Let $r_k(x)$ denote the number of voters who consider candidate x to be the k^{th} best candidate at **P**. Because candidate i is picked by the scoring rule $(s_1,..., s_m)$, it must be true that

$$s_1[r_1(i)] + s_2[r_2(i)] + ... + s_m[r_m(i)] \geq s_1[r_1(j)] + s_2[r_2(j)] + ... + s_m[r_m(j)] \qquad (1)$$

for every other candidate j.

To show that the scoring-rule winner, candidate i, is an AV outcome, consider i's critical strategy profile $C_i(\mathbf{P})$. There are two cases:

Case (i): Voters rank candidate i last. Under a scoring rule, these voters give a score of 0 to candidate i, a score of 1 to their top choices, and scores between 0 and 1 to the remaining candidates. Under AV, these voters give a score of 0 to candidate i, a score of 1 to their top choices, and scores of 0 to the remaining candidates at $C_i(\mathbf{P})$.

Thus, candidate i does the same under the scoring rule as under AV (left side of inequality (1)), whereas all other candidates j do at least as well under the scoring rule as under AV (right side of inequality (1)). This makes the sum on the right side for the scoring rule at least as large as, and generally larger than, the sum of votes under AV, whereas the left side remains the same as under AV. Consequently, if inequality (1) is satisfied under the scoring rule, it is satisfied under AV at $C_i(\mathbf{P})$.

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[6] Of course, AV is not a scoring rule in the classical sense whereby voters give scores to candidates according to the same predetermined vector. The restrictions on the vector that sincere, admissible strategies impose is that (i) the first component (score of the top candidate) be 1, (ii) the m^{th} component (score of the bottom candidate) be 0, (iii) all components representing candidates at or above the lowest candidate a voter approves of are 1, and (iv) all components below the component representing this candidate are 0.

Case (ii): Voters do not rank candidate i last. Under a scoring rule, these voters give candidate *i* a score of s_k if they rank him or her k^{th} best. Under AV, these voters give a score of 1 to candidate *i* at $C_i(\mathbf{P})$. Thus, every s_k on the left side of equation (1) is 1 for candidate *i* under AV, which makes the sum on the left side at least as large as, and generally larger than, the sum under a scoring rule. By comparison, the sum on the right side for all other candidates *j* under AV is less than or equal to the sum on the left side, with equality if and only if candidate *j* is preferred to candidate *i* by all voters. Consequently, if inequality (1) is satisfied under the scoring rule, it is satisfied under AV at $C_i(\mathbf{P})$.

Thus, in both cases (i) and (ii), the satisfaction of inequality (1) under a scoring rule implies its satisfaction under AV at candidate *i*'s critical strategy profile, $C_i(\mathbf{P})$. Hence, a candidate chosen under any scoring rule is also an AV outcome.

To prove the second statement, consider the following 7-voter, 3-candidate example (Fishburn and Brams, 1983, p. 211):

Example 4. (i) 3 voters: *a b c*; (ii) 2 voters: *b c a*; (ii) 1 voter: *b a c*; (iv) 1 voter: *c a b.*

Because candidate *b* receives at least as many first choices as *a* and *c*, and more first and second choices than either, every scoring rule will select *b* as the winner. But *a* is the Condorcet winner and, hence, an AV outcomes by Proposition 2.[7] Q.E.D.

We next show the outcomes of two social choice rules that are not scoring rules, the Hare system of single transferable vote (STV) and the majoritarian compromise (MC), are always AV outcomes, whereas the converse is not true—AV outcomes need not be STV or MC outcomes.[8] Before proving this result, we illustrate STV and MC with a 9-voter, 3-candidate example:[9]

Example 5. (i) 4 voters: *a c b* 2; (ii) 2 voters: *b c a*; (iii) 3 voters: *c b a.*

Under STV, candidates with the fewest first-choice—and successively lower-choice—votes are eliminated; their votes are transferred to second-choice and lower-choice candidates in their preference rankings until one candidate receives a majority of votes. To illustrate in Example 5, because candidate *b* receives the fewest first-choice votes (2)—compared with 3 first-choice votes for candidate *c* and 4 first-choice votes for candidate *a*—*b* is eliminated and his or her 2 votes go to the second choice of the 2 type (2) voters, candidate *c*. In the runoff between candidates *a* and *c*, candidate *c*, now with votes from the type (2) voters, defeats candidate *a* by 5 votes to 4, so *c* is the STV winner.

Under MC, first-choice, then second-choice, and then lower-choice votes are counted until at least one candidate receives a majority of votes; if more than one candidate receives a majority, the candidate with the most votes is elected. Because no candidate in Example 5 receives a majority of votes when only first choices are counted, second choices are next counted and added to the first choices. Candidate *c* now receives

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[7] Example 4 provides an illustration in which BC, in particular, fails to elect the Condorcet winner.

[8] Ideally, of course, it would be desirable to prove this result for all voting systems, but we know of no general definition of a voting system that encompasses all those that have been used or proposed, in contrast to scoring systems and, as we will show later, Condorcet systems (Brams and Fishburn, 2002).

[9] These two voting systems, among others, are discussed in Brams and Fishburn (2002). MC, which is less well known than STV, was proposed independently as a voting procedure (Hurwicz and Sertel, 1997; Sertel and Yilmaz, 1999; Sertel and Sanver, 1999; Slinko, 2002) and as a bargaining procedure under the rubric of "fallback bargaining" (Brams and Kilgour, 2001). As a voting procedure, the threshold for winning is assumed to be simple majority, whereas as a bargaining procedure the threshold is assumed to be unanimity, but qualified majorities are also possible under either interpretation.

the support of all 9 voters, whereas *a* and *b* receive 4 and 5 votes, respectively, so *c* is the MC winner.

Proposition 5. *At all preference profiles* **P***, a candidate chosen by STV or MC is an AV outcome. There exist preference profiles* **P** *at which a candidate chosen by AV is neither an STV nor an MC outcome.*

Proof. We start by showing that every STV outcome is an AV outcome. By Lemma 3, there exists a candidate *j* such that the number of voters who rank *j* as their best candidate and *i* as their worst candidate exceeds the number of voters who prefer *i* to *j*. A fortiori, the number of voters who consider *j* as their best candidate exceeds those who consider *i* as their best candidate.

This result holds for any subset of candidates that includes both *i* and *j*. Hence, STV will never eliminate *j* in the presence of *i*, showing that *i* cannot be an STV winner.

Neither can *i* be an MC winner, because *j* will receive more first-place votes than *i*. If this number is not a majority, the descent to second and still lower choices continues until at least one candidate receives a majority. Between *i* and *j*, the first candidate to receive a majority will be *j*, because *j* receives more votes from voters who rank him or her first than there are voters who prefer *i* to *j*. Thus, *j* will always stay ahead of *i* as the descent to lower and lower choices continues until *j* receives a majority.

To show that AV outcomes need not be STV or MC outcomes, consider Example 4, in which the Condorcet winner, candidate *c*, is chosen under both STV and MC. Besides *c*, AV may also choose candidate *a* or candidate *b*: *a* is an AV outcome at critical strategy profile $C_a(\mathbf{P}) = (a, a, a, a, b, b, c, c, c)$; and *b* is an AV outcome at critical strategy profile $C_b(\mathbf{P}) = (a, a, a, a, b, b, cb, cb, cb)$. Q.E.D.

So far we have shown that AV yields at least as many, and generally more, (Pareto) outcomes than any scoring rule and two nonscoring voting systems. To be sure, one might question whether the three possible AV outcomes in Example 4 have an equal claim to being *the* social choice. Isn't candidate *c*, the Condorcet winner, BC winner, STV winner, and MC winner—and ranked last by no voters—the best overall choice? By comparison, candidate *b* is only a middling choice; and candidate *a*, who is the plurality-vote (PV) winner, is the Condorcet loser.[10]

Just as AV allows for a multiplicity of outcomes, it also enables voters to prevent them.

Proposition 6. *At every preference profile* ***P*** *at which there is not a unique AV outcome, AV can prevent the election of every candidate*, *whereas scoring rules, STV, and MC cannot prevent the election of all of them.*

Proof. In the absence of a unique AV outcome, there is no candidate that can be assured of winning, which implies that every candidate can be prevented from winning. To show that scoring rules, STV, and MC cannot prevent the election of all candidates when AV can, consider the following 3-voter, 3-candidate example:

Example 6. (i) 1 voter: *a b c*; (ii) 1 voter: *b a c* (iii) 1 voter: *c b a.*

The Condorcet winner, *b*, wins under every scoring system, including BC, and also under MC. Under STV, either *a* or *b* may win, depending on which of the three

[10] Note that PV is a degenerate scoring rule, under which a voter's top candidate receives 1 point and all other candidates receive 0 points. By Proposition 4, sincere outcomes under PV are always AV outcomes but not vice-versa. As a case in point, candidate *a* is the sincere PV outcome in Example 5, whereas candidates *b* and *c* are also sincere AV outcomes.

candidates is eliminated first. Thus, only *c* is prevented from winning under these other systems, whereas every candidate can be prevented from winning under AV. Q.E.D.

We have seen that AV allows for outcomes that BC, MC, and STV do not (e.g., *c* in Example 6 when there is a three-way tie). At the same time, it may preclude outcomes (e.g., *b* in Example 6) that other systems cannot prohibit. In effect, voters can fine-tune their preferences under AV, making outcomes responsive to information that transcends these preferences.

We next consider not only what outcomes can and cannot occur under AV but also what outcomes are likely to persist because of their stability. While we know that non-Pareto candidates cannot win a clear-cut victory under AV (Proposition 1), might it be possible for Condorcet losers to be AV outcomes *and* stable? To answer this question, we will distinguish two types of stability.

4. Stability of AV Outcomes

As earlier, we assume that voters choose sincere, admissible strategies under AV. Now, however, we suppose that they may not draw the line between acceptable and unacceptable candidates as they would if they were truthful. Instead, they may vote strategically in order to try to obtain a preferred outcome.

To determine what is "preferred," we extend preference to sets. We assume that a voter whose preference is *a b* will prefer *a* to $\{a, b\}$, and $\{a, b\}$ to *b*. In assessing the stability of outcomes, we assume that preferences over sets which satisfy this condition are admissible.

We define two kinds of stability, the first of which is the following: Given a preference profile **P**, a nontied AV outcome is *stable* if there exists a strategy profile **S** such that no voters of a single type have an incentive to switch their strategy to another sincere strategy in order to induce a preferred outcome.[11] In analyzing the stability of AV outcomes, we need confine our attention only to those outcomes stable at $C_i(\mathbf{P})$ because of the following proposition:

Proposition 7. *A nontied AV outcome i is stable if and only if it is stable at its critical strategy profile,* $C_i(\mathbf{P})$.

Proof. The "if" part follows from the existence of a strategy profile, $C_i(\mathbf{P})$, at which outcome *i* is stable. To show the "only if" part—that if an AV outcome is not stable at its critical strategy profile, then it cannot be stable at any other strategy profile—assume candidate *i* is unstable at $C_i(\mathbf{P})$. At any other strategy profile $\mathbf{S'}$, candidate *i* receives no more approval votes and generally fewer than at $C_i(\mathbf{P})$ by Lemma 1. Hence, those voters who switch to different sincere strategies to induce the election of a preferred candidate at $\mathbf{S}$ can also do so at $\mathbf{S'}$. Q.E.D.

The strategies of voters associated with a stable AV outcome at $C_i(\mathbf{P})$ define a Nash equilibrium of a voting game in which the voters have complete information about each others' preferences and make simultaneous choices.[12]

Neither candidate *a* nor candidate *b* is a stable AV outcome in Example 5. At critical strategy profile $C_a(\mathbf{P}) = (a, a, a, a, b, b, c, c, c)$ that renders candidate *a* an AV outcome, if the 2 type (ii) voters switch to strategy *bc*, candidate *c*, whom the type (ii)

[11] Treating voters of one type, all of whose members have the same preference, as single (weighted) voters provides the most stringent test of stability. This is because any outcome that can be destabilized by the switch of individual voters (of one type) can be destabilized by the switch of all voters of that type, but the converse is not true: Outcomes may be stable when some but not all voters of one type switch. Our definition of stability precludes outcomes of the latter kind from being stable.

[12] For an analysis of Nash equilibria in voting games under different rules and information conditions from those given here, see Myerson (2002) and references cited therein.

voters prefer to candidate *a*, wins. At critical strategy profile $C_b(\mathbf{P}) = (a, a, a, a, b, b, cb, cb, cb)$ that renders candidate *b* an AV outcome, the 4 type (i) voters have an incentive to switch to strategy *ac* to induce the selection of candidate *c*, whom they prefer to candidate *b*.

Although AV outcomes *a* and *b* in Example 5 are not stable at their critical strategy profiles, AV outcome *c* most definitely is stable at its critical strategy profile, $C_c(\mathbf{P}) = (ac, ac, ac, ac, bc, bc, c, c, c)$: No switch on the part of the 4 type (i) voters to *a*, of the 2 type (ii) voters to *b*, or of the 3 type (ii) voters to *cb* can lead to a preferred outcome for any of these types—or, indeed, change the outcome at all (because candidate *c* is the unanimous choice of all voters at *c*'s critical strategy profile).

Not only can no single switch by any of the three types induce a preferred outcome for the switchers at $C_c(\mathbf{P})$, but no *coordinated* switches by two or more types can induce a preferred outcome. Thus, for example, if the *ac*-voters switched from *ac* to *a*, and the *bc* voters switched from *bc* to *b*, they together could induce AV outcome *a*, which the 4 type (i) voters would clearly prefer to outcome *c*. But *a* is the worst choice of the 2 type (ii) voters, so they would have no incentive to coordinate with the type (i) voters to induce this outcome.

That AV outcome *c* is, at the critical strategy profile of candidate *c*, invulnerable to coordinated switches leads to our second type of stability: Given a preference profile **P**, an outcome is *strongly stable* if there exists a strategy profile **S** such that no types of voters, coordinating their actions, can form a coalition *K*, all of whose members would have an incentive to switch their AV strategies to other sincere strategies in order to induce a preferred outcome.

We assume that the coordinating players in K are allowed to communicate to try to find a set of strategies to induce a preferred outcome *for all of them*. These strategies define a *strong* Nash equilibrium of a voting game in which voters have complete information about each others' preferences and make simultaneous choices.

Proposition 8. *A nontied AV outcome i is strongly stable if and only if it is strongly stable at its critical strategy profile,* $C_i(\mathbf{P})$.

Proof. Analogous to that of Proposition 7.

An AV stable outcome need not be strongly stable. To illustrate this weaker form of stability, consider AV outcome *a* in Example 1 and its critical strategy profile, $C_a(\mathbf{P})$ = (*a*, *a*, *a*, *bca*, *bca*, *d*, *d*). The 2 type (ii) voters cannot upset this outcome by switching from *bca* to *bc* or *b*, nor can the 2 type (iii) voters upset it by switching from *d* to *db* or *dbc*. However, if these two types of voters cooperate and form a coalition K, with the 2 type (ii) voters choosing strategy *b* and the 2 type (ii) voters choosing strategy *db*, they can induce the selection of Condorcet winner *b*, whom both types prefer to candidate *a*. At critical strategy profile $C_a(\mathbf{P})$, therefore, AV outcome *a* is stable but not strongly stable, whereas AV outcome *b* is strongly stable at its critical strategy profile, $C_b(\mathbf{P})$ = (*ab*, *ab*, *ab*, *b*, *b*, *db*, *db*).

If an AV outcome is neither strongly stable nor stable, it is *unstable*.

Proposition 9. *There may be no strongly stable or stable nontied AV outcomes—that is, every nontied AV outcome may be unstable.*

Proof. Consider the following 3-voter, 3-candidate example:[13]

Example 7. (i) 1 voter: *a b c*; (ii) 1 voter: *b c a*; (iii) 1 voter: *c a b*.

The critical strategy profile that elects candidate *a* is $C_a(\mathbf{P}) = (a, b, ca)$. If voter (ii) switches to *bc*, he or she can induce preferred outcome $\{a, c\}$. In a similar manner, it is possible to show that neither candidate *b* nor candidate *c* is a stable AV outcome. Q.E.D.

We next characterize strongly stable outcomes.

Proposition 10. *A nontied AV outcome is strongly stable if and only if it is a unique Condorcet winner.*

Proof. To prove the "if" part, suppose candidate *i* is a unique Condorcet winner at **P**. We will show that *i* is a nontied AV outcome that is strongly stable at its critical strategy profile, $C_i(\mathbf{P})$. Clearly, *i* is a nontied AV outcome at $C_i(\mathbf{P})$ by Proposition 2. To show its strong stability, suppose there exists a coalition of voters *K*, comprising one or more types, that prefers some other candidate *j* to candidate *i* and coordinates to induce the selection of *j*. Because candidate *i* is a unique Condorcet winner, however, the cardinality of *K* is strictly less than the cardinality of coalition *L*, whose members prefer *i* to *j*. The members of *L* vote for *i* but not for *j* at $C_i(\mathbf{P})$. Hence, whatever sincere, admissible strategy switch the members of *K* consider at candidate *i*'s critical strategy profile to induce the election of candidate *j*, *j* will receive fewer votes than *i*, proving that *i* is a strongly stable AV outcome.

[13] Example 7 is the standard example of the Condorcet paradox, or cyclical majorities, in which there is no Condorcet winner.

To prove the "only if" part, suppose that candidate *i* is not a unique Condorcet winner. Consequently, there exists a candidate *j* and a majority coalition of voters *K*, comprising one or more types, that prefers *j* to *i*. We will now show that *i* is not a strongly stable AV outcome at its critical strategy profile, $C_i(\mathbf{P})$, which by Proposition 8 shows that *i* is not a strongly stable AV outcome. Suppose AV does not elect *i* at $C_i(\mathbf{P})$. Then *i* is not an AV outcome and hence not a strongly stable one. Now suppose that AV elects *i* at $C_i(\mathbf{P})$. Because the members of *K* can change their strategies to elect *j*, whom they prefer to *i*, *i* is not a strongly stable AV outcome. Q.E.D

We next show that Condorcet losers as well as winners may be stable AV outcomes.

Proposition 11. *A unique Condorcet loser may be a stable AV outcome, even when there is a different outcome that is a unique Condorcet winner (and therefore strongly stable).*

Proof. Consider the following 7-voter, 5-candidate example:

Example 8. (i) 3 voters: *a b c d e*; (ii) 1 voter: *b c d e a*; (iii) 1 voter: *c d e b a*;
(iv) 1 voter: *d e b c a*; (v) 1 voter: *e b c d a.*

Candidate *a* is the Condorcet loser, ranked last by 4 of the 7 voters. But at its critical strategy profile, $C_a(\mathbf{P}) = (a, a, a, b, c, d, e)$, candidate *a* is a stable AV outcome, because none of the four individual voters, by changing his or her strategy, can upset *a*, who will continue to receive 3 votes.

Consider the critical strategy profile of candidate *b*, $C_b(\mathbf{P}) = (ab, ab, ab, b, cdeb, deb, eb)$, who receives 7 votes, compared with 3 votes each for *a* and *e*, 2 votes for *d*, and

1 vote for *c*. Again, no single type of voter can upset this outcome, nor can any coalition, because candidate *b* is the unique Condorcet winner, making him or her strongly stable. Q.E.D.

Whether a Condorcet loser, like candidate *a* in Example 8, "deserves" to be an AV winner—and a stable one at that—depends on whether voters have sufficient incentive to unite in support of a candidate like Condorcet winner *b*, who is the first choice of only one voter. If they do not rally around *b*, and the type (i) voters vote only for *a*, then *a* is arguably the more acceptable choice.

A *Condorcet voting system* is one that always elects a Condorcet winner, if one exists, when voters are sincere—that is, when they rank candidates according to their preferences. (Note that this use of "sincerity" is different from that of AV, which, as we noted earlier, allows for multiple sincere strategies.) A Condorcet winner, however, may not be elected as a Nash equilibrium under a Condorcet voting system, much less a strong one (as under AV).

Proposition 12. *No Condorcet voting system ensures the election of a unique Condorcet winner as a Nash equilibrium.*

Proof. Consider the following example, in which there is no Condorcet winner:

Example 9. (i) 2 voters: *a d b c*; (ii) 2 voters: *b d c a*; (iii) 1 voter: *c a b d*.

In the absence of a Condorcet winner, we assume that different candidates may be chosen by a Condorcet voting system.

We first show that by changing the preference ranking of each of the three voter types, one at a time, in Example 9, we can render different candidates Condorcet winners.

However, if a Condorcet voting system chooses a candidate or candidates preferred by this type in Example 9, then the Condorcet winner is not a Nash-equilibrium outcome. To prevent this from happening, we must preclude the possibility of choosing the preferred candidate(s). It turns out that the Condorcet winners we show can occur, and whose choice as an equilibrium outcome can be upset by some voter type changing its ranking to that in Example 9, preclude the possibility of a Condorcet voting system's choosing *a*, *b*, *c*, or *d* in this example.[14]

To begin, assume the preference ranking of the 2 type (ii) voters in Example 9 is *b d a c*, but the other two types have the same preferences as shown in Example 9. Then candidate *a* is the Condorcet winner. If a Condorcet voting system would choose either candidate *b* or *d* in Example 9, then it would be in the interest of the type (ii) voters to switch to *b d c a*, as given in Example 9, to obtain a preferred outcome.

Assume the preference ranking of the type (iii) voter is *c d a b*. Then candidate *d* is the Condorcet winner. If a Condorcet voting system would choose candidate *c* in Example 9, then it would be in the interest of the type (iii) voter to switch to *c a b d*, as given in Example 9, to obtain a preferred outcome.

Finally, assume the preference ranking of the 2 type (i) voters is *a c d b*. Then candidate *c* is the Condorcet winner. If a Condorcet voting system would choose candidate *c* in Example 9, then it would be in the interest of the type (i) voters to switch to *a d b c*, as given in Example 9, to obtain a preferred outcome.

[14] These choices do not preclude the possibility of the Condorcet voting system's choosing most subsets of candidates in Example 9, such as {*b*, *c*}, which may or may not be preferred to a Condorcet winner. Thus, Proposition 9 is applicable to social choice functions, which are "resolute" social choice rules that choose single candidates, and not to social choice correspondences, which may choose nonsingleton subsets of candidates.

In summary, we have shown that three of the four candidates in Example 9 can be rendered a Condorcet winner by changing the preference ranking of one voter type. If this is the true ranking of these voter(s), it is always in their interest to misrepresent their preferences to those shown in Example 9, given that a Condorcet voting system chooses the candidates we postulated in each of the above cases. But to prevent this from happening in all the cases, we must preclude the possibility of the Condorcet voting system's choosing all four candidates in Example 9—and thereby undermining the Nash-equilibrium status of the unique Condorcet winners in the modifications of Example 9. Q.E.D.

Proposition 12 casts doubt on the efficacy of Condorcet voting systems, such as those of Black or Copeland (Brams and Fishburn, 2002), to do what they purport to do *in equilibrium*. By contrast, AV always ensures that Condorcet winners can be elected as strong Nash equilibria.[15]

5. Conclusions

AV outcomes subsume all outcomes that other voting systems we examined choose, including scoring rules like the Borda count, single transferable vote (STV), and the majority compromise (MC). In addition, they include outcomes that none of these other systems selects, rendering AV outcomes more inclusive.

AV outcomes, however, are not an indiscriminate set. Thus, they always include Pareto candidates and never include non-Pareto candidates as unique winners. But which

[15] Under a somewhat weaker definition of strong stability, the equivalence of strong Nash-equilibrium outcomes and Condorcet winners is shown for a large class of voting rules in Sertel and Sanver (2004).

outcome is chosen critically depends on where voters draw the line between acceptable and non-acceptable candidates.

Preference-based systems are not responsive to this information and so limit the field of candidates that can win. Despite the bigger menu that AV allows, however, voters are better able to prevent the election of candidates under AV than under other voting systems.

Only under AV are strongly stable outcomes always Condorcet winners when voters choose sincere strategies. By contrast, Condorcet voting systems that purport always to elect Condorcet winners may fail to do so when voters are strategic.

AV allows for other stable outcomes, though not strongly stable ones, such as Borda-count winners and even Condorcet losers. Indeed, we see nothing wrong in such candidates winning if they are the most approved by voters—especially if "majority tyranny" is a concern (Baharad and Nitzan, 2002)—though a number of studies suggest that AV is likely to elect Condorcet winners when they exist.

Beyond our normative views on the desirability as well as the practicality of AV, it is worth noting that basing social choice on acceptability rather than on traditional social-choice criteria is a radical departure from the research program initiated by Borda and Condorcet in late 18th-century France (McLean and Urken, 1995). While we do not eschew these criteria, they should not be the be-all and end-all for judging whether outcomes are acceptable or not.

Rather, we believe, the pragmatic judgments of sovereign voters about who is acceptable and who is not should be decisive. This is information that enriches the standard social-choice framework and should, therefore, be incorporated in it.

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