The Probability of Conflicts in a U.S. Presidential Type Election

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Abstract

In a two candidate election, it might be that a candidate wins in a majority of districts while he gets less vote than his opponent in the whole country. In Social Choice Theory, this situation is known as the compound majority paradox, or the referendum paradox. Although occurrences of such paradoxical results have been observed worldwide in political elections (e.g. United States, United Kingdom, France), no study evaluates theoretically the likelihood of such situations. In this paper, we propose four probability models in order to tackle this issue, for the case where each district has the same population. For a divided electorate, our results prove that the likelihood of this paradox rapidly tends to 20% when the number of districts increases. This probability decreases with the number of states when a candidate receives significatively more vote than his opponent over the whole country.
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1 Introduction

The 2000 US presidential elections remind us that voting paradoxes are not only theoretical issues for economists and political scientists; They sometimes

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Table 1: 2000 US Presidential Elections

Candidates	Popular Vote	Electoral College
Gore	48.4 %	267
Bush	47.9 %	271
Nader	2.7 %	0
Buchanan	0.4 %	0
Browne	0.4 %	0

happen! With 48.4% of the popular vote, A. Gore could only win 21 States among 51, for a total of 267 electors in the Electoral College, while G.W. Bush get 271 electors with less support from the popular vote (See Table 1).

This paradox is known in Social Choice literature as the referendum paradox (see Nurmi [15]). It may occur each time the decision is not taken directly by the voters by referenda, but through representatives locally elected. Then, the decision taken by the representatives may not reflect the will of the voters. Note that this paradox is related to other voting paradoxes, such as the Ostrogorski paradox (see Ostrogorski [16], Nurmi [15], Laffond and Lainé [12], Saari and Sieberg [18]). If the decision is taken through a more complex hierarchy of committees, the same problem can occur (see Galam [7]).

In fact, the occurrence of the referendum paradox has been observed in many democracies. The US presidential elections displayed the paradox in 1824, 1876, 1888 and 2000 (see Leip [13], Saari [17]). Since World War Two, it happened twice in the United Kingdom (see Table 2). This paradox may become frequent in French local elections: since 1992, many aspects of the local public policies are no longer decided directly by the cities themselves, but through communities of cities. For example, we have been able to identify such a paradox in the "Grand Caen" area, which gathers 18 cities in the surrounding of Caen. Though the left parties get a large majority of votes over the whole area and control 13 cities since the 2001 elections, they have been defeated in the main towns. In turns, they only control 26 seats over 70 in the council of the community (see Table 3). As there are nowdays about 90 communities of cities in France, the case of Caen is probably not a unique one.

Table 2: Parliamentary Elections in the United Kingdom

Year	Labour		Conservative		Others	
	Votes (%)	Seats	Votes (%)	Seats	Votes (%)	Seats
1951	48.7	295	48.0	321	3.7	9
1974	37.2	304	38.2	296	24.6	38

Thus two-level voting may clearly lead to undesired results. Moreover, one of the main assumption of Public Economy, the fact that majority voting picks out the prefered policy of the median voter when the preferences are single peaked (see Black [1], Downs [5]) is no longer true; The social compromise may be beaten by a more extremist policy in a federal system. Thus, as many local, federal, national and international[1] bodies are committees of representatives designated by the adequate districts, it is of importance to get a better theoretical knowledge about the occurrence of the referendum paradox. This is what we intend to do in this paper, by adapting models used in statistics, physics, and social choice to treat this issue. Of course, our study is quite preliminary and uses some simplifying assumptions. We propose to study this paradox:

- When there are two parties. Although possible, the study of the case of more than two parties is left for further research.
- When all the states have the same population size.
- For a large number of voters in each state and for small population sizes whenever it is possible.
- From $N = 3$ states (or consistuencies, districts, cities,...) to $N = 100$. We get exact values for some simple cases and computer estimations otherwise.
- With 4 different models. The first two are based upon adapted versions of the Impartial Culture and Impartial Anonymous Culture (respectively abbreviated IC and IAC hereafter) assumptions that are used in

[1] As an example, this problem may arise in the European Union. The Treaty of Nice assigns to each country within the European Union a number of vote in the Council of Minister. It is not sure that a policy supported by a majority of citizens will exceed the threshold of 74% of the votes in the Council. For more details about the Nice Treaty and the distribution of power among member States, see Felsenthal and Machover [6] and Bobay [2].

Table 3: Local Elections in Grand Caen Area

Town	Population	Votes			Seats	
		Left	Right	Others	Left	Right
Authie	999	0	413	96		2
Breteville	4 489	993	1 066	0		2
Caen	117 157	14 575	20 065	0		26
Cambes	1 525	287	409	15		2
Carpiquet	1 884	428	563	51		2
Colombelles	6 272	1362	0	0	3	
Cormelles	4 644	702	1371	0		3
Cuvervilles	1 797	586	0	174	2	
Démouville	3 128	1 034	0	89	2	
Epron	1 798	442	281	75	2	
Fleury	4 305	1 065	0	0	2	
Giberville	4 639	1 496	0	0	3	
Hérouville	24 374	3 874	3 074	0		7
Ifs	9 290	2 436	1 100	0	3	
Louvigny	1 785	651	0	322	2	
Mondeville	10 678	4 126	0	0	3	
Saint-Contest	2 030	530	478	73	2	
Saint-Germain	2 554	748	0	31	2	
Total	203 348	35 304	28 820	926	26	44
Percent		54.3	44.3	1.4	37.1	62.9

The elections took place on the 11th of March 2001 for the first round, and on the 18th of March for the run-offs. The voting rule is different according to the number of inhabitants in the city.

If the city has more than 3500 inhabitants, only complete lists of candidates can run into the competition. The winner is the list which gets 50% of the vote on the first round or the greatest number of votes in the second round. In Hérouville, the two left lists did not merge for the second round and let the right list win with 44.2% of the popular vote.

In small towns, individual candidates can be proposed and voters can put as many manes as seats on their ballots. A candidate needs 50% of the votes to be elected in the first round. In the run-off, the canddiates who received the highest number of votes are selected. Thus, in small towns, we took into account the figures from the round where more delegates were designated. Next, we add up the votes received respectively by the left, right and non-partisan candidates and divided them by the number of seats in competition for that round.

Social Choice Theory; They assume that each party has equal chance to win. The other two propose ways to introduce a systematic bias in favor of one candidate.

In section 2, we adapt the two classical models that are generally used in social choice theory, for the case $N = 3$. In this simple case, we are able to derive rather easily formulas for the computation of the probability of the referendum paradox and can also illustrate the differences between both models with figures. Section 3 displays the figures under the IC and IAC models for the case $N > 3$ When $N = 4$ or $N = 5$, we are still able to derive exact formuals, but as the computations become tedious, the details of the computations are presented in the annexes. For all the other cases, we rely on computer estimations in order to evaluate the likelihood of the referendum paradox. We present two new models in section 4, by taking into account the fact that one party is more likely to win. We discuss our results in section 5.

2 Adapting the Classical Models: The Case $N = 3$

In order to present and compare the different models we shall use, we first present them for the case $N = 3$ states (or groups, districts, consistuencies). Let n be the number of voters in each state. We denote by n_i the number of voters who, in state i, vote for candidate A, $i = 1, 2, ..., N$. The other voters are assumed to vote for candidate B; There is no abstention. A *voting situation* is a vector $\mathbf{n} = (n_1, n_2, ..., n_N)$ with $0 \leq n_i \leq n$. For $N = 3$, $\mathbf{n} = (n_1, n_2, n_3)$. A conflict between a decision made by a majority of states and a decision made nationwide through a referendum occurs if, for example, States 1 and 2 vote for candidate A, while a majority of voters prefer B. This situation is described by inequalities (2.1), (2.2) and (2.3):

$$n_1 > n/2 \tag{2.1}$$
$$n_2 > n/2 \tag{2.2}$$
$$n_1 + n_2 + n_3 < 3n/2 \tag{2.3}$$

There are five other cases leading to a paradox, similar to this one. Thus, we only need to estimate the probability that inequalities (2.1) to (2.3) are met with an adequate probability model describing the behavior of the voters.

2.1 Impartial Culture model

The Impartial Culture condition has been introduced in Social Choice literature by Guilbaut [10], for the study of the Condorcet paradox. It assumes that each voter picks his preference randomly among the possible preference types according to an uniform probability distribution. In our case, each voter has a probability $\frac{1}{2}$ to cast his vote in favor of candidate A, and a probability $\frac{1}{2}$ to cast his vote for candidate B. The distribution of n_i is a binomial law; When the number of voters is large in each state, the distribution of n_i tends to a normal law, of mean $n/2$ and variance $\sigma = \sqrt{n}/2$. For each n_i, $i = 1, \dots, N$, let

$$x_i = \frac{1}{\sigma}\left(n_i - \frac{n}{2}\right)$$

The Central Limit Theorem implies the following convergence for the density function as $n \to \infty$:

$$f(x_i) \mapsto \frac{1}{\sqrt{2\pi}} e^{\frac{-x_i^2}{2}}.$$

For the three states case, the joint distribution of $\mathbf{x} = (x_1, x_2, x_3)$ as $n \to \infty$ is given by:

$$f(\mathbf{x}) \mapsto \frac{1}{(\sqrt{2\pi})^3} e^{\frac{-|\mathbf{x}|^2}{2}}$$

where $|\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2$. By subtracting or dividing the number of voters by the same constant, the quantities change but the comparison between them is unchanged, therefore one can claim that $\mathbf{n}$ satisfies conditions (2.1),(2.2),(2.3) if and only if $\mathbf{x}$ satisfies $(2.1)'$, $(2.2)'$ and $(2.3)'$:

$$\begin{aligned} x_1 &> 0 && (2.1)' \\ x_2 &> 0 && (2.2)' \\ -x_1 - x_2 - x_3 &> 0 && (2.3)' \end{aligned}$$

Let $P_{IC}(\infty, 3)$ be the probability of the referendum paradox for three states of population $n \to \infty$ under the IC condition. Thus, $P_{IC}(\infty, 3)$ is equal to six times:

$$I = \frac{1}{(\sqrt{2\pi})^3} \int_{C_1} e^{\frac{-|\mathbf{x}|^2}{2}} dx_1 dx_2 dx_3$$

where $C_1 = \{x \in \mathbb{R}^3 : x \text{ satisfies } (2.1)',\ (2.2)' \text{ and } (2.3)'\}$. We must integrate I in the triangular cone delimited by the three straight lines Oa, Ob and Oc (See Figure 1). We write as $r^2 d\Omega dr$ the volume element $dx_1 dx_2 dx_3$ where $d\Omega$ is the element of solid angle and $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ and the integration on r is straighfoward. We observe that

$$I = \frac{1}{4\pi} \int d\Omega$$

Hence, computing the desired probability reduces to find the measure of the cone C_1 and to divide it by the surface of the sphere, 4π. Therefore, the measure of the cone is exactly the solid angle of this cone.

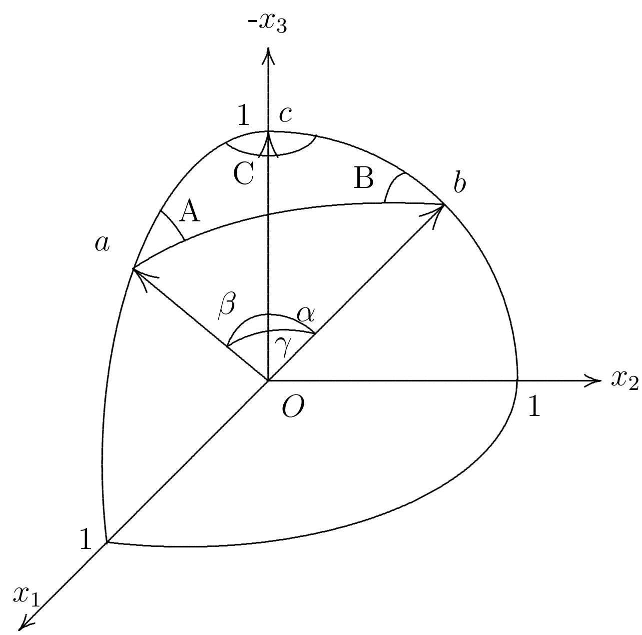

Figure 1: The cone C_1 of conflicts with the IC model

In fact, C_1 defines a spherical triangle on the surface of the unit sphere in $\mathbb{R}^3$ (see Figure 1). To perform this surface, we notice that the respective angles between $ObOc$, $OaOc$ and $OaOb$ are $\alpha = \beta = \frac{\pi}{4}$ and $\gamma = \frac{\pi}{2}$. Now, the general formulas which give the relations among A, B and C, the angles of the spherical triangle and α, β and γ are of the form:

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A \tag{2.4}$$

plus two other formulas obtained by permutation. We immediatly obtain $A = B = \arccos\frac{\sqrt{3}}{3}$ and $C = \frac{\pi}{2}$. Now, the area of a spherical triangle on a sphere of radius 1 (i.e. the solid angle we are looking for) is:

$$S = A + B + C - \pi \tag{2.5}$$

$$= 2\arccos\left(\frac{\sqrt{3}}{3}\right) - \frac{\pi}{2} \tag{2.6}$$

To obtain the probability $P_{IC}(\infty, 3)$ we divide S by 4π, the surface of the sphere, and multiply it by 6 (as there are 6 similar cases):

$$P_{IC}(\infty, 3) = \frac{3 \arccos\left(\frac{\sqrt{3}}{3}\right)}{\pi} - \frac{3}{4} \approx 0.16226.$$

2.2 Impartial Anonymous Culture

The IAC model is based upon the assumption that every voting situation is equally likely. It has been first introduced in Social Choice literature by Gehrlein and Fishburn [9]. Given that the total number of voting situations is $(n+1)^N$, the probability of the paradox under IAC when the number of voters in each state is n can be written as

$$P_{IAC}(n, N) = \frac{|X(n, N)|}{(n+1)^N}$$

where $X(n, N)$ denotes the set of situations giving rise to the paradox. For small numbers of states, this model makes possible not only the computation of the limiting probability $P_{IAC}(\infty, N)$, but also the derivation of closed form representation for $P_{IAC}(n, N)$ as a function of n. In what follows, we will assume that n is odd.

For n odd, it is easily seen that a voting situation is consistent with the inequalities (2.1) to (2.3) if and only if:

$$\frac{n+1}{2} \leq n_1 \leq n-1, \quad \frac{n+1}{2} \leq n_2 \leq \frac{3n-1}{2} - n_1, \quad 0 \leq n_3 \leq \frac{3n-1}{2} - n_1 - n_2$$

The corresponding number of situations is obtained as:

$$\sum_{n_1=\frac{n+1}{2}}^{n-1} \sum_{n_2=\frac{n+1}{2}}^{\frac{3n-1}{2}-n_1} \sum_{n_3=0}^{\frac{3n-1}{2}-n_1-n_2} 1 = \frac{(n+1)(n^2+2n-3)}{48}.$$

As the popular winner can be either A or B and there are three ways to choose the single state that gives a majority to A, we obtain the desired probability by multiplying the above number by six and by dividing by $(n+1)^3$:

$$P_{IAC}(n, 3) = \frac{n^2+2n-3}{8(n+1)^2}.$$

In the limit of an infinite number of voters, we obtain $P_{IAC}(\infty, 3) = 1/8$. Observe that this value could be directly obtained in the following way: Let

$y_i = n_i/n$ and assume that n tends to infinity; Then:

$$P_{IAC}(\infty, 3) = 6 \int_{1/2}^{1} \int_{1/2}^{3/2-y_1} \int_{0}^{3/2-y_1-y_2} dy_1 dy_2 dy_3 = 1/8 = 0.125.$$

In order to provide a graphic comparison between the IC and IAC models, define $z_i = y_i - 0.5$. For the IAC model with $n \to \infty$, the density of the points (z_1, z_2, z_3) is uniform on the cube $[-0.5, 0.5]^3$ and the probability that inequalities $(2.1)'$ to $(2.3)'$ are satisfied is equal to the volume of the shaded pyramid, divided by the volume of the cube, i.e., 1. For the IC model, the density is uniform on the surface of the unit sphere while it is uniform on all the cube $[-0.5, 0.5]^3$ with the IAC assumption.

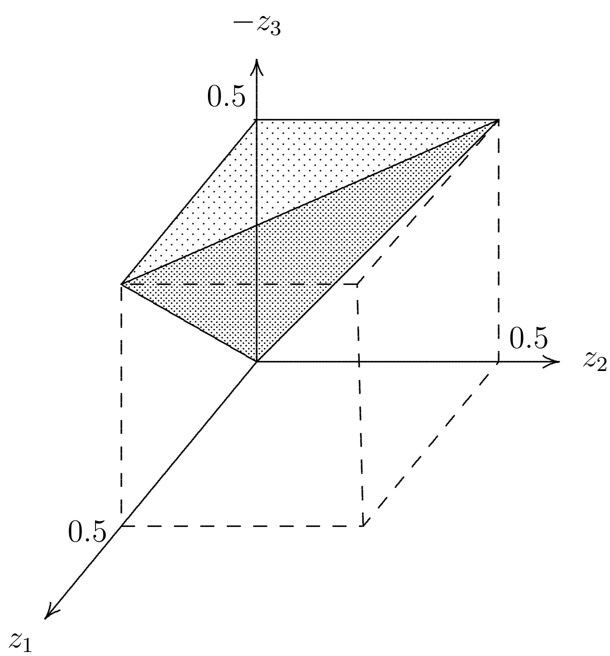

Figure 2: Conflicts with the IAC model

3 Probablities of Conflicts for an arbitrary number of states

3.1 The case $N = 4$

In the four-state case, the referendum paradox occurs when three states vote in favor of candidate A while a majority of voters prefer B nationwide. There are eight possible cases, one of them being described by inequalities (3.1) to (3.4):

$$n_1 > n/2 \quad (3.1)$$
$$n_2 > n/2 \quad (3.2)$$
$$n_3 > n/2 \quad (3.3)$$
$$n_1 + n_2 + n_3 + n_4 < 2n \quad (3.4)$$

Using arguments and techniques similar to the ones of Section 2, we can evaluate the probability of the referendum paradox under IC and IAC assumptions.

Proposition 3.1 *Let $P_{IC}(\infty, 4)$ be the probability a referendum paradox with four states and a large n under the IC model. Then,*

$$P_{IC}(\infty, 4) = \int_0^1 \frac{6 \arccos\left(\frac{1}{t^2+1}\right) t}{\sqrt{3+t^2}\sqrt{6+5\,t^2+t^4}\pi^2} dt = \frac{1}{24} \approx 0.041666$$

Proof: see Annexe I.

Proposition 3.2 *Let $P_{IAC}(n, 4)$ be the probability of the referendum paradox with four states of n voters (n odd) under the IAC model. Then,*

$$P_{IAC}(n, 4) = \frac{n^3 - n^2 - 9n + 9}{48(n+1)^3}$$

and the limit probability is:

$$P_{IAC}(\infty, 4) = \frac{1}{48} \approx 0.020833$$

Proof: see Annexe II.

3.2 The case $N = 5$

In the case of five states, there are two types of configurations that can give rise to the paradox. In the first type, four states prefer A to B whereas the total number of voters preferring B to A exceeds half of the total number of voters, in accordance with the following inequalities:

$$n_1 > n/2 \quad (3.11)$$
$$n_2 > n/2 \quad (3.12)$$
$$n_3 > n/2 \quad (3.13)$$
$$n_4 > n/2 \quad (3.14)$$
$$n_1 + n_2 + n_3 + n_4 + n_5 < 5n/2 \quad (3.15)$$

Notice that there are ten configurations of this type (two times the number of ways for choosing the single state preferring A to B).

In the second type of configuration, three states choose one of the candidate -say A-, two states choose B and B is the popular winner. This happens when inequalities (3.11), (3.12), (3.13) and (3.15) are satisfied together with:

$$n_4 < n/2 \tag{3.16}$$

$$n_5 < n/2 \tag{3.17}$$

There are twenty configurations of this type (two times the number of ways for choosing the two states preferring A to B among five states).

Proposition 3.3 *Let $P_{IC}(\infty, 5)$ be the probability of the referendum paradox for $N = 5$ and a large n under IC. Thus,*

$$P_{IC}(\infty, 5) \approx 0.181368$$

Proof: See Annexe III

Proposition 3.4 *Let $P_{IAC}(n, 5)$ be the probability of the referendum paradox for $N = 5$ and n odd under IAC. Thus,*

$$P_{IAC}(n, 5) = \frac{5(11n^4 + 44n^3 + 38n^2 - 12n - 81)}{384(n+1)^4}$$

and $P_{IAC}(\infty, 5)$ tends to $55/384 \approx 0.143229$.

Proof: See Annexe IV

Table 4 displays some values of $P_{IAC}(n, 3)$, $P_{IAC}(n, 4)$ and $P_{IAC}(n, 5)$ and shows that the probabilities tend quickly to their limiting values. For this reason, we will only consider the limiting case of a large n in our simulations.

3.3 Computer Simulations

We cannot compute anymore the exact figures when there are six states or more. Thus, we run computer simulations for N odd and even, and get the estimated values from 1 000 000 random draws in each case. Notice that these results are compatible with the exact values we get theoretically for the cases $N = 3$, $N = 4$ and $N = 5$. The main observation is that the values seems to tend rapidly to a limit value as N increases. The limit should be around 20.5% for the IC model, and around 16.5% for the IAC model for N odd. The figures are a bit lower for N even, but the probability of tied outcomes (i.e. exactly half of the states vote for A) is then significant, and tends very slowly to zero for $N \to \infty$. In fact, the curve is given by $\begin{pmatrix} N/2 \\ N \end{pmatrix} \frac{1}{2^N}$ which decreases as $\sqrt{\frac{2}{\pi}\frac{1}{N}}$ for large N.

Table 4: Probabilities as a function of n

n	$P_{IAC}(n,3)$	$P_{IAC}(n,4)$	$P_{IAC}(n,5)$
3	0.0938	0.0000	0.1172
5	0.1111	0.0062	0.1325
7	0.1172	0.0098	0.1373
9	0.1200	0.0120	0.1395
11	0.1215	0.0135	0.1407
13	0.1224	0.0148	0.1413
15	0.1230	0.0154	0.1418
17	0.1235	0.0160	0.1421
19	0.1238	0.0165	0.1423
21	0.1240	0.0169	0.1425
.	.	.	.
∞	0.1250	0.0208	0.1432

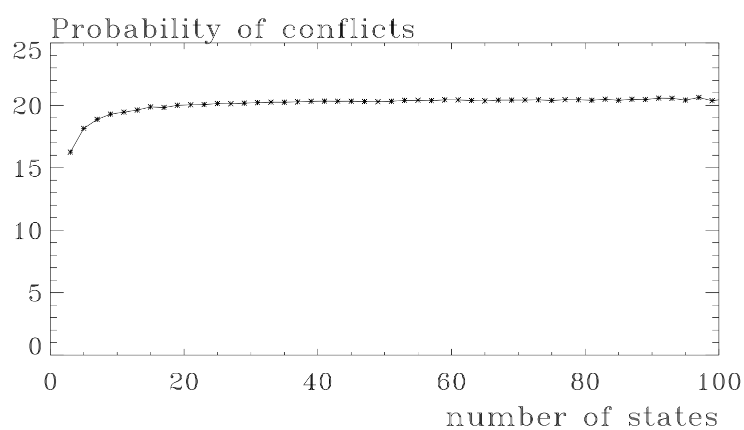

Figure 3: Probabilities under IC, N odd

4 Critics of the Classical Models and Introduction of a Bias

4.1 Classical Models Confronted to Reality

The question is, now, to appreciate the value of the results obtained above. Very likely, to improve our models we must make an adaptation to the type of election we consider. To allow some comparison with the reality, we compare our results to some stylised facts that can be observed in the case of the

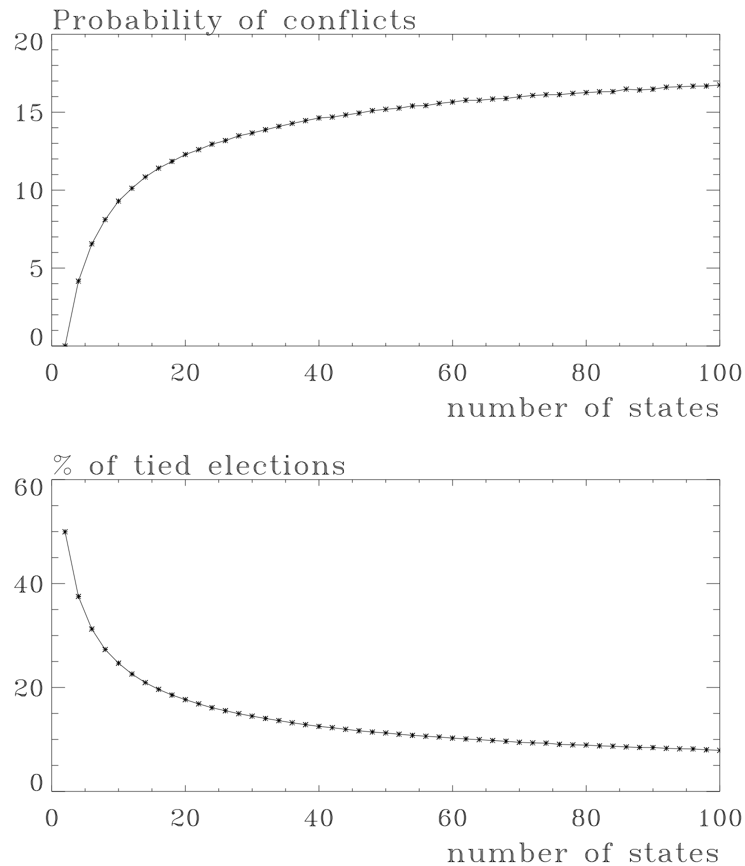

Figure 4: Probabilities under IC, N even

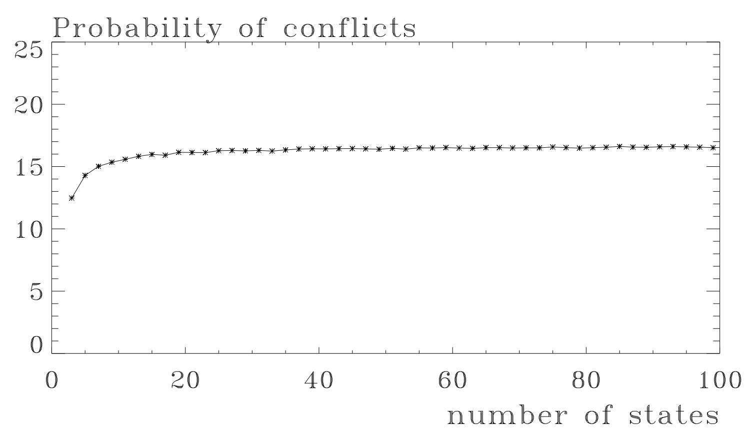

Figure 5: Probabilities under IAC, N odd

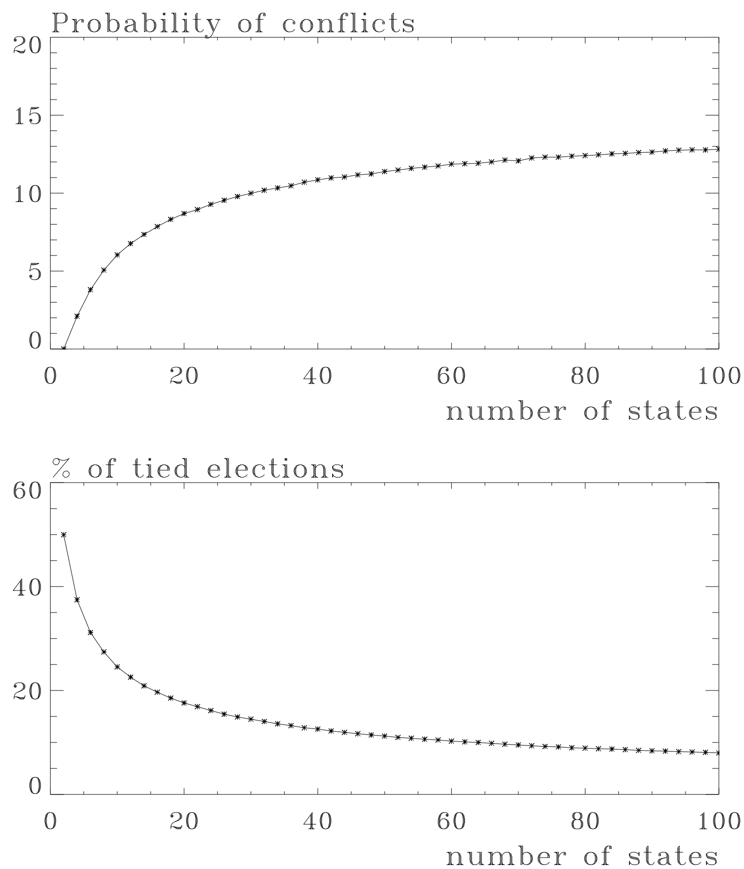

Figure 6: Probabilities under IAC, N even

USA presidential election. Nevertheless, we have to keep in mind that the situation we modelled is very different. All our states have equal population, and we cannot scope with important issues such as inequalities of population among states or malapportionement of the seats among the states. In the previously obtained results, two main points deserve discussion.

The good agreement between IC and IAC. This agreement is sometimes found in other problems : The Condorcet effect is an example (see Berg and Bjurulf [3]), but for the analysis of the propensity to manipulation, the results may be very different according to the assumption (see Lepelley, Pierron and Valognes [14]). Nevertheless, this agreeement is a priori strange when we consider the very different features of these two models. A positive aspect of this feature is that, a model that would produce a distribution of the states that is "in between" the IC and IAC case would not lead to drasticaly different results. Moreover, for such an hypothetical model, the distribution of

Table 5: Repartition of the Vote for Democrat in 2000

Percent	25-30	30-35	35-40	40-45	45-50	50-55	55-60	> 60
Number of States	4	3	4	11	14	7	4	3

the states between 0% support and 100 % would be more realistic (see Table 5 and [13]), in contrast with the flat distribution of IAC, and the extremely concentrated distribution of the IC case.

Is the frequency of the paradox too high ? This point is certainly controversial: It seems that the conflict's frequency (0.205 for IC, 0.165 for IAC) is too high, as only three conflicts have been detected in the 19^{th} century and one (November 2000) in the 20^{th} century. Examining the exact results of the US elections since 1824 (see [13]) , we can detect two cases:

- A clear election, when for example more than 52% of the population votes for one candidate. Usually, the state by state vote then amplifies the tendency.

- A tied results, when the margin between the two top candidates is less than 3%, with now a not negligible likelihood of conflict.

Obviously, the IC and IAC models describe only extremely tied elections, that is, the second case. This is clear for IC: Each elector selects with equal probability one of the two candidates. But this is also true for IAC. Indeed some states are strongly in favor of A and others strongly in favor of B, but there is no bias. The distribution of the total vote is not uniform on 0%-100%, but clearly present a maximum around 50% and becomes more and more peaked when the N increases. This perfect symetry between the two candidates, who have equal chance to win, also seems to be the reason of the existence of a non zero limiting value of the conflict frequency when the number of states increases, while intuitively we could have expected a decrease. To make a comparison with the reality, notice that, since 1824, the paradox appeared 3 times among the 10 closest US presidential elections (see Table 6, obtained from [13]). In view of these data, our models, which predicts around two conflicts seem quite reasonnable!

In view of these comments, two new issues arise. First of all, we assumed equal population states. This is clearly not the case neither in the electoral college, nor in the council of minister of the European Union. A precise analysis of the impact of difference in size would be a major task; it should

Table 6: The Ten Closest US Presidential Elections

Year	Popular Vote		Electoral College		Margin
	Dem.	Rep.	Dem.	Rep.	
1880	48.3 %	48.2 %	214	155	0.1 %
1884	48.5 %	48,3 %	219	182	0.2 %
1960	49.8 %	49.6 %	303	219	0.2 %
2000	48.4 %	47.9 %	267	271	0.5 %
1968	42,7 %	43,4 %	191	301	0.7 %
1888	48,6 %	47,8 %	168	233	0.8 %
1976	50.1 %	48,0 %	297	240	2.1 %
1876	51,0 %	48,0 %	184	185	3.0 %
1916	49.2 %	46.1 %	277	254	3.1 %
1892	46.1 %	43.0 %	277	145	3.1 %

also include issues on "malapportionment", that is the fact the electoral weight of certain states may be overestimated or underestimated (see for example the recent paper by Samuel and Snyder [21]). In the USA case the number of voters representing a state in the electoral college is equal to the sum of its number of senators (which is equal to two whatever the population of the state is) and its number of members in the House of Representatives (which is proportional to its population). If we remove this two seat premium to each state, Gore would have enjoyed a large victory in the new electoral college, by 225 votes to 211!

We keep the assumption of equal population states. Now, a way to obtain a more realistic model, and in particular to describe what happens when one candidate clearly wins, is to introduce a systematic bias in his favor. As a matter of fact this is exactly this bias that an election must exhibit. Let us first consider the IC case. Then an arbitrary small but finite bias, in the limit of a very large number of voters, implies that all the states cast a vote in favor of the same candidate. The fact that some states are traditionally democrats while others are republican appears only through fluctuations of the model which cannot accommodate any bias. Thus, we turn to the IAC model.

4.2 The Maximum Entropy Concept and the IAC Model

In the IAC model the probability of voting for A in state i is itself a random variable p_i drawn from $[0, 1]$ using an uniform distribution. Generalizing the uniform distribution of p_i in the IAC model, one can consider a density function $P(y)$ describing the distribution of the different $p_i = y = n_i/n$ that is, the a priori traditions of the different states to vote democrat or republican. If we take $P(y) = P(1-y)$, we have no bias in the whole. The bias is a consequence of the preference of one of the candidate by the nationwide electorate which modifies the distribution $P(y)$. We can directly modify $P(y)$ to take into account the fact that the mean of the y_i over all the states is not $\frac{1}{2}$. More precisely:

$$\int_0^1 yP(y)dy = a.$$

Then, $a - \frac{1}{2}$ represents the nationwide bias in favor of a candidate A. How to select $P(y)$ when a is known ? This is a classical problem in information theory, the key concept of which is given by the functional $S(P(y))$ defined by:

$$S(P(y)) = -\int_0^1 P(y)\log(P(y))dy$$

$S(P(y))$ is called entropy in physics, but here we would better call it "uncertainty". The idea is to compute $P(y)$ which maximize the uncertainty $S(P(y))$ compatible with the available information given here by the normalisation and the bias. This is a well posed mathematical problem (see solution in Annexe V). We obtain:

$$P(y) = \frac{\lambda}{1-e^{-\lambda}}e^{-\lambda y}$$

with λ given by the equation:

$$a = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1-e^{-\lambda})}$$

Note that $a < \frac{1}{2}$ gives $\lambda > 0$ and $a > \frac{1}{2}$ gives $\lambda < 0$. When $a \to \frac{1}{2}$, $\lambda \to 0$ and $P(y) \to 1$; In the no bias case we recuperate the usual IAC model. Moreover, we check that two values a and $\bar{a}$ such that $a+\bar{a} = 1$ give two opposite values for λ and $\bar{\lambda}$ with $P(y,\lambda) = \bar{P}(1-y,\bar{\lambda})$, preserving consequently the symmetry between A and B. Figure 7 displays the density function $P(y)$ for $a = 0.55$, $\lambda = -0.604$, and we present on Figure 8 the corresponding simulations. We see a slow decrease of the conflicts frequency with the number of states, an expected result. Nevertheless, the decrease is slow and the results are not very different from the no bias IAC model. Another critic comes from the fact that the density function $P(y)$ does not seem to be realistic when compared to Table 5.

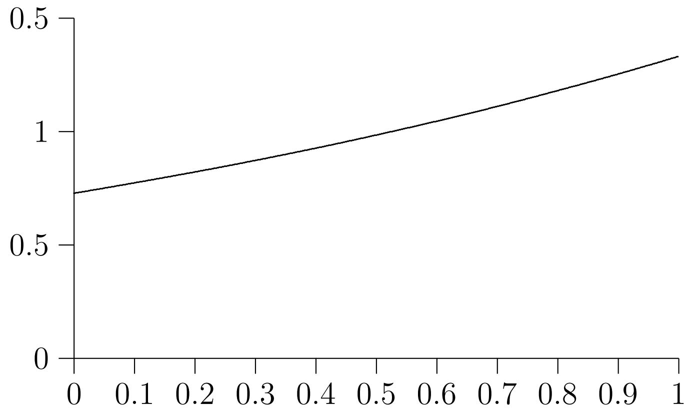

Figure 7: The density function $P(y)$ for $a = 0.55$, $\lambda = -0.604$.

4.3 The Rescaled IAC Model with Bias.

The last simulations point out the remaining problem. Although we have taken a strong bias, $P(0)$ and $P(1)$ are still important and our model exhibits a strong percentage of states voting B at more than, say 75%. Such a strong dichotomy (massively democrats or republican states) if it has ever existed in the beginning of the union, is certainly no more true today. There is probably a tendency to homogeneity in many states, and the range of variation of p_i from state to state is certainly narrower than 0%-100%. A measure of this dispersion must be introduced in our model in addition to the bias; A realistic model must be searched between the IC model, with no dispersion for the y and an y allowed to vary from 0 to 1 in the IAC model.

Among many possible models to answer the above critics, we propose the following one. $P(y)$ will be a constant (a property which simplifies considerably the computation in the N-dimension space of the x_i) between values:

$$y_m = \frac{1}{2} - D + E \text{ and } y_M = \frac{1}{2} + D + E$$

The normalization gives $\frac{1}{2D}$ for this constant. The parameter D is consequently a measure of the dispersion while E is a measure of the bias. D can take values between 0 and $\frac{1}{2}$, and $0 \leq E \leq 1 - D$. With these notations, we see that the frequency of conflicts will only depend on the ratio $p = E/D$. To prove it, we just consider how we proceed in the simulation. To obtain y_i, we draw a random number ϵ_i on $[0, 1]$ and have:

$$y_i = \frac{1}{2} - D + E + 2D\epsilon_i$$

This formula gives the density distribution $P(y)$ as desired. Now, the winner

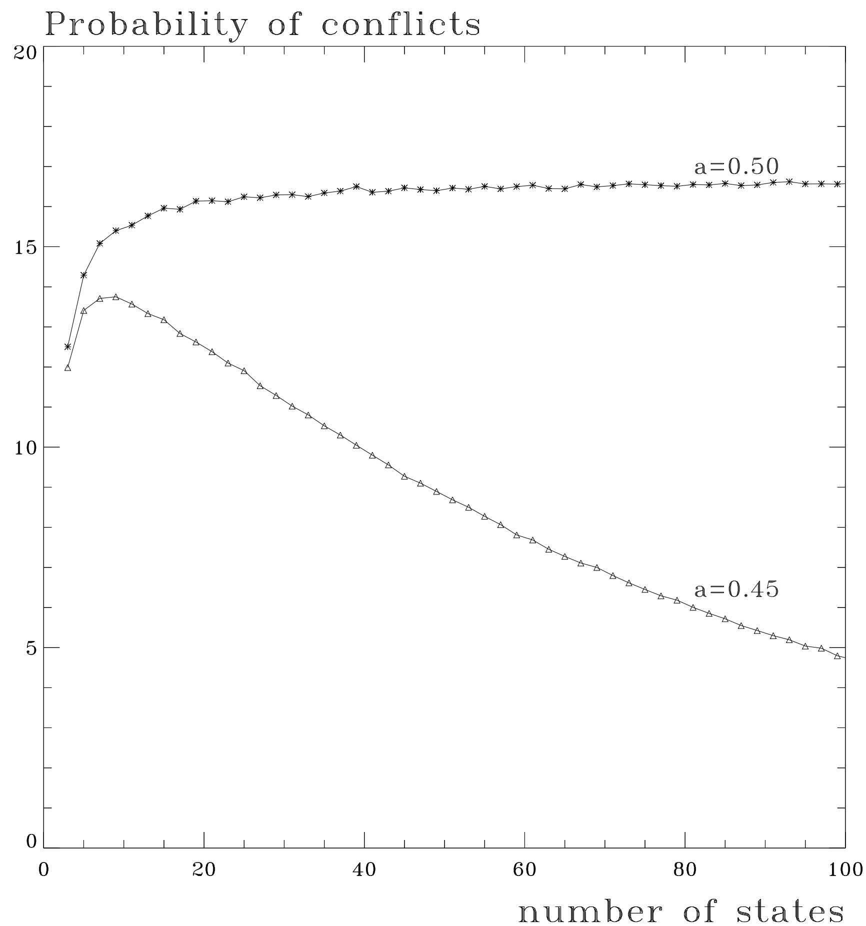

Figure 8: Conflicts with the maximum entropy distribution

on the states will be given by comparing y_i to $\frac{1}{2}$ in all the states:

$$y_i > \frac{1}{2} \Leftrightarrow \epsilon_i > \frac{1}{2} - \frac{E}{2D}$$

Similarly, candidate A is the popular winner if

$$\frac{1}{N}\sum_{i=1}^{N} \epsilon_i > \frac{1}{2} - \frac{E}{2D}$$

In any case, only the ratio $p = \frac{E}{D}$ matters. For $E = 0$, the results are independent of the value of D and the support of p_i can be either $[0, 1]$ (as in the classical IAC model), or $[\frac{1}{2} - D, \frac{1}{2} + D]$, D being as small as we like. Consequently a rescaling may be introduced. In a similar way, the IC model introduced a rescaling of the fluctuation varying as $\sqrt{n}$. In the IAC model a bias may be introduced, which is not the case in the IC model for $n \to \infty$. This explains why in the absence of bias, the two models give quite similar results and justifies the name given to this model: Rescaled IAC model with bias.

Proposition 4.1 *Let $P_{RB}(\infty, 3)$ be the probability of conflicts in two-level elections for $N = 3$ and $n \to \infty$ under the rescaled IAC model with bias, D being the dispersion parameter, E the bias and $p = E/D$. Thus,*

$$P_{RB}(\infty, 3) = \frac{1}{8}(1 + 3p^2 - 8p^3) \; if \; p \in [0, \frac{1}{3}]$$

$$P_{RB}(\infty, 3) = \frac{1}{16}(1 + 9p - 21p^2 + 11p^3) \; if \; p \in [\frac{1}{3}, 1]$$

Proof: See annexe VI.

We turn now to computer simulations with this model, which are presented on Figure 9 for N odd. For small values of p, the begining of the curves are quite similar to the case $p = 0$ and it is only for sufficiently large N that the bias decreases more and more strongly the conflict probability. Table 9 combined with the curves of Figure 9 also show that the probability of conflicts is very sensitive to the small bias values. For example, a dispersion of around 10% with a bias of 52% ($p = 0.2$) gives 7% of conflicts for $N = 51$. Now, for the same dispersion and a bias of 53% ($p = 0.3$) the probability of conflicts fall to arround 1.5%. For values of p larger than 0.3, the probability of conflict pratically disppear for 51 states (or more).

5 Conclusion and open problems

In this paper, we have studied the possible differences in the results between a direct election and a two-level one with a first election at the level of a district (or state), which, subsequently cast all its votes in favor of the candidate who won this district (or state). Typical of this problem is the US presidential elections with an elected president possibly loosing the popular vote (a rather rare issue). To evaluate the frequency of such conflicts in a simplified model (all the states have the same population), we first use the usual statistical models describing the behavior of the electoral body for a simple two-candidate election.

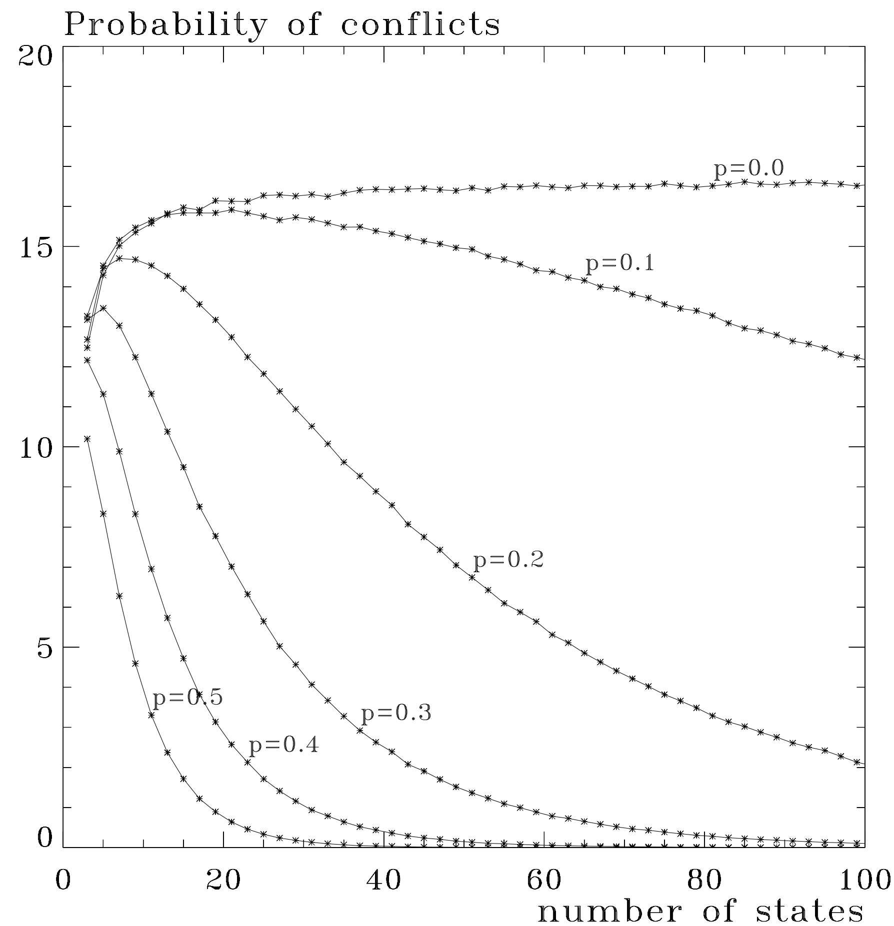

Figure 9: Probability of the paradox with bias

The IC model assumes, for each voter a probability $\frac{1}{2}$ to cast a vote in favor of a candidate, and the sole fluctuations decide of the issues. The absolute size of these fluctuations is irrelevant.

The IAC model is a priori quite different. For each state i, a probability p_i of a vote for candidate A is introduced with equirepartition of p_i on $[0, 1]$. In fact the two models give quite similar results with a conflict frequency reaching a limit when the number of states increases, with a rather high rate of conflict (16.5% to 20.5%). Actually the two classical models, IC and IAC, describe only completely tied up elections when no systematic tendency in favor of a candidate could be detected. We conjecture that similar results

Table 7: The relations among E,D and p

$\frac{1}{2}+E$		51%	52 %	53%	54%	55%	60%
$p=0.1$	y_m	41%	32%	23%	14%		
	y_M	61%	72%	83%	94%		
$p=0.2$	y_m	46%	42%	38%	34%	30%	
	y_M	56%	62%	68%	74%	80%	
$p=0.3$	y_m	47.7%	45.3%	43%	40.7%	38.3%	26.7%
	y_M	54.3%	58.7%	63%	67.3%	71.7%	93.3%
$p=0.4$	y_m	48.5%	47%	45.5%	44%	42.5%	35%
	y_M	53.5%	57%	60.5%	64%	67.5%	85%
$p=0.5$	y_m	49%	48%	47%	46%	45%	40%
	y_M	53%	56%	59%	62%	65%	80%

could be obtained with the use of more subtle models derived from the Pólya Eggenberger family.

New models must be introduced taking the bias into account. But in that case, the IC model looses its meaning. We have introduced two new models which both can be considered as a generalization of the IAC model. The first one takes into account the average value of p_i (the probability of state i to vote for candidate A) i.e. the global bias in the election and use the maximum entropy concept to derive the p_i probability distribution. In the no bias case, we recover the classical IAC model. But this model introduces too many states voting quasi unanimously republican or democrate. In fact, a realistic model will reduce the support of p_i. Nevertheless, the maximum entropy concept may be fruitfull in models where not much is known about the preferences of the voters except the results of the vote; this is not the case when there are only two candidates.

If p_i has no dispersion in the IC model, this dispersion is much too large in the IAC model. A new model (rescaled IAC with bias) is introduced with equirepartition of p_i on the interval $[\frac{1}{2}-D+E, \frac{1}{2}+D+E]$. The bias is measured by E and the dispersion by D. It has been proven that the only parameter is $p = E/D$. For $E/D > 0.3$ and $N > 40$ the rate of conflict becomes completely negligible.

Of course, all these models assume a "veil of ignorance" assumption: Each state or voter has the same behavior. The idea is that, even if during some period of time, one state can be considered as rightist or leftist, in the very long run, we cannot a priori identify a state with a particuliar behavior. We only study the technical aspects of the “paradox” due to the voting rule itself, and not the sociological causes or particular political configurations that lead to the occurence of the paradox. This is why we keep with a probability distribution function as general as possible. In further studies, we hope to be able to identify the parameters of the voting rule (number of states, repartition of the population among the states, malapportionement of the seats) that have an influence on the propability of conflicts.

6 Annexe I: Proof of Proposition 3.1

Let $x_i = \frac{1}{\sigma\sqrt{n}}(n_i - n/2)$. The configurations implying the occurrence of the paradox are as follows:

$$\begin{aligned} x_1 &> 0 && (6.1)\\ x_2 &> 0 && (6.2)\\ x_3 &> 0 && (6.3)\\ -x_1 - x_2 - x_3 - tx_4 &> 0 && (6.4) \end{aligned}$$

with $t = 1$. When $t = 0$, the paradox cannot occur. The joint distribution of x_1, x_2, x_3 and x_4 is given by:

$$P(x_1, x_2, x_3, x_4) = \frac{1}{(\sqrt{2\pi})^4} e^{\frac{-|\mathbf{x}|^2}{2}}$$

and

$$P(x_1,\, x_2,\, x_3,\, x_4,\ \text{satisfy (6.1), to (6.4)}) \longmapsto \frac{1}{(\sqrt{2\pi})^4} \int_{C_2} e^{\frac{-|\mathbf{x}|^2}{2}} dx_1 dx_2 dx_3 dx_4$$

where $C_2 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ which satisfy (7.1), (7.2), (7.3), (7.4)$\}$ and $|\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Using the same arguments as in section 2, we can prove that the computation of the probability reduces to finding the measure of the cone C_2, that is the measure of the spherical simplex defined by (6.1) to (6.4) on the surface the unit hypersphere in $\mathbb{R}^4$. In order to compute this area, we use the same technique as Saari and Tataru [19]. By Schläfli [20], Coxeter [4] and Kellerhals [11], the differential volume on the set of spherical p-simpleces is given by:

$$dvol_p(C_2) = \frac{1}{p-1} \sum_{1 \leq j < k \leq 4} vol_{p-2}(S_j \cap S_k) d\alpha_{jk}$$

with S_j the hyperplane (or facet) corresponding to inequality (6.j) and α_{jk} the angle between hyperplanes S_j and S_k. As t will be the parameter of integration, $vol(C_2) = 0$ when $t = 0$. Let W_j be a normal vector to S_j pointing inside the simplex:

$$\begin{aligned} W_1 &= (1, 0, 0, 0)\\ W_2 &= (0, 1, 0, 0)\\ W_3 &= (0, 0, 1, 0)\\ W_4 &= (-1, -1, -1, -t) \end{aligned}$$

Thus

$$\alpha_{12} = \arccos\left(\frac{-W_1 W_2}{||W_1||\ ||W_2||}\right) = \frac{\pi}{2}$$

$$\alpha_{14} = \arccos\left(\frac{-W_1 W_4}{||W_1||\ ||W_4||}\right) = \arccos\left(\frac{1}{\sqrt{3+t^2}}\right)$$

Note that $\alpha_{12} = \alpha_{13} = \alpha_{23}$ and $\alpha_{14} = \alpha_{24} = \alpha_{34}$. This implies:

$$d\alpha_{12} = d\alpha_{13} = d\alpha_{23} = 0$$

$$d\alpha_{14} = d\alpha_{24} = d\alpha_{34} = \frac{t}{\sqrt{3+t^2}\sqrt{6+5\,t^2+t^4}}$$

Finding the vertex $P_{123} = S_1 \cap S_2 \cap S_3$ reduces to solving the system:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ -x_1 - x_2 - x_3 - t x_4 > 0 \end{cases}$$

By solving similar linear systems, we obtain directions of the vertices of the cone C_2 to be equal to :

$$\begin{aligned} P_{123} &= (0,0,0,-1) \\ P_{124} &= (0,0,t,-1) \\ P_{134} &= (0,t,0,-1) \\ P_{234} &= (t,0,0,-1) \end{aligned}$$

Also we compute:

$$vol(S_1 \cap S_4) = vol(S_2 \cap S_4) = vol(S_3 \cap S_4) = (\widehat{P_{123}, P_{134}}) = \arccos\left(\frac{1}{1+t^2}\right)$$

Thus:

$$\frac{dvol(C_2)}{dt} = \frac{3\ \arccos((t^2+1)^{-1})t}{2\sqrt{3+t^2}\sqrt{6+5\,t^2+t^4}}$$

We have to integrate this differential volume between 0 and 1, to multiply this value by 8, and to divide it by $\omega^4 = 2\pi^2$, the area of the unit sphere in $\mathbb{R}^4$ to get the probability of the paradox for $N = 4$:

$$P_{IC}(\infty, 4) = \int_0^1 \frac{6\ \arccos((t^2+1)^{-1})t}{\sqrt{3+t^2}\sqrt{6+5\,t^2+t^4}\pi^2} dt = \frac{1}{24} \approx 0.041666$$

7 Annexe II: Proof of Proposition 3.2

The only type of configuration implying the occurrence of the paradox is as follows:

$n_1 < n/2$, $n_2 < n/2$, $n_3 < n/2$ and $n_1 + n_2 + n_3 + n_4 > 2n$.

In this configuration, the three first states vote for B and A obtains a majority of votes at the global level. This implies for n odd:

$$2 \leq n_1 \leq \frac{n-1}{2}, \quad \frac{n+3}{2} - n_1 \leq n_2 \leq \frac{n-1}{2},$$

$$n + 1 - n_1 - n_2 \leq n_3 \leq \frac{n-1}{2}, \quad 2n + 1 - n_1 - n_2 - n_3 \leq n_4 \leq n.$$

From these inequalities, it can be obtained that the number of voting situations giving rise to the paradox in the configuration under consideration is equal to:

$$\frac{(n+1)(n^3 - n^2 - 9n + 9)}{384}.$$

To obtain the probability of the paradox, we multiply this number by eight (as there are eight configurations similar to the one we have analyzed) and we divide the product by $(n+1)^4$:

$$P_{IAC}(n, 4) = \frac{n^3 - n^2 - 9n + 9}{48(n+1)^3}.$$

From this representation, it follows that $P_{IAC}(\infty, 4) = 1/48$.

8 Annexe III: Proof of Proposition 3.3

We assume that $n \to \infty$ in the five states under IC. Thus, the joint distribution of x_1, x_2, x_3, x_4 and x_5 is given by

$$P(x_1, x_2, x_3, x_4, x_5) = \frac{1}{(\sqrt{2\pi})^5} e^{\frac{-|\mathbf{x}|^2}{2}}$$

where $|\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$. Consider the first type of configurations that lead to the referendum paradox for $N = 5$. They are characterized by situations similar to the ones described by equations (3.11) to (3.15). It is equivalent to estimate the probability that x_1, x_2, x_3, x_4, x_5 satisfy:

$$x_1 > 0 \quad (8.1)$$

$$x_2 > 0 \quad (8.2)$$

$$x_3 > 0 \quad (8.3)$$

$$x_4 > 0 \quad (8.4)$$

$$-x_1 - x_2 - x_3 - x_4 - t x_5 > 0 \quad (8.5)$$

with $t \in [0,1]$. Again, these inequalities define a cone, C_3, which intersects the unit hypersphere in $\mathbb{R}^5$. Computing the probability of these events reduces to finding the measure of C_3 on the surface of the hypersphere. By Schläfli's formula:

$$dvol_p(C_3) = \frac{1}{p-1} \sum_{1 \leq j < k \leq 5} vol_{p-2}(S_j \cap S_k) d\alpha_{jk}$$

where S_j is the facet defined by equation (8.j), and α_{jk} the angle between facets S_j and S_k. When $t = 0$, $vol(C_3) = 0$.

Computation of the dihedral angles.

Let U_j be a vector normal to facet S_j, pointing inside the cone:

$$\begin{array}{rcl} U_1 & = & (1,0,0,0,0) \\ U_2 & = & (0,1,0,0,0) \\ U_3 & = & (0,0,1,0,0) \\ U_4 & = & (0,0,0,1,0) \\ U_5 & = & (-1,-1,-1,-1,-t) \end{array}$$

In deriving the dihedral angles, we are only interested in the angles which depends upon t. There are four of them:

$$\alpha_{15} = \alpha_{25} = \alpha_{35} = \alpha_{45} = \arccos\left(\tfrac{1}{\sqrt{4+t^2}}\right)$$

Hence:

$$d\alpha_{15} = d\alpha_{25} = d\alpha_{35} = d\alpha_{45} = \frac{t}{\sqrt{3+t^2}\,(4+t^2)}$$

The vertices of the dihedral volumes.

In order to evaluate the surfaces of the remaining dihedral volumes, first find their vertices. A direction in $\mathbb{R}^5$ is given by solving a system of four linear equations. We find the coordinates of the vertices by solving systems similar to the following one:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \\ -x_1 - x_2 - x_3 - x_4 - tx_5 > 0 \end{cases}$$

The solutions of this system gives the direction $P_{1234} = S_1 \cap S_2 \cap S_3 \cap S_4$. There are 5 such linear systems, which give five directions:

$$\begin{aligned} P_{1234} &= (0,0,0,0,-1) \\ P_{1235} &= (0,0,0,t,-1) \\ P_{1245} &= (0,0,-t,0,-1) \\ P_{1345} &= (0,-t,0,0,-1) \\ P_{2345} &= (-t,0,0,0,-1) \end{aligned}$$

The vertices of $(S_j \cap S_k)$ are the directions P_{abcd} where both j and k appear as indices. We shall now detail the computations for the volume of $S_1 \cap S_5$. This volume is the triangle on the sphere in $\mathbb{R}^3$ defined by the three direction P_{1235}, P_{1245} and P_{1345}. The area of this triangle is equal to the sum of the angles on its surface minus π (see equation 2.4) Let β_{23}, β_{24} and β_{34} be the angles on the surface of the triangle, respectively defined by the vertices P_{1235}, P_{1245} and P_{1345}; δ_2, δ_3 and δ_4 are respectively the angles $\widehat{P_{1235}, P_{1245}}$, $\widehat{P_{1235}, P_{1345}}$ and $\widehat{P_{1245}, P_{1345}}$. Hence, using again equation 2.5:

$$\cos(\beta_{23}) = \frac{\cos(\delta_4) - \cos(\delta_2)\cos(\delta_3)}{\sin(\delta_2)\sin(\delta_3)}$$

In our case:

$$\delta_2 = \delta_3 = \delta_4 = \arccos\left(\frac{1}{t^2+1}\right)$$

In turns,

$$\beta_{23} = \beta_{24} = \beta_{34} = \arccos\left(\frac{1}{t^2+2}\right)$$

$$Vol_{p-2}(S_1 \cap S_5) = \beta_{23} + \beta_{24} + \beta_{34} - \pi = 3 \arccos\left(\frac{1}{t^2+2}\right) - \pi$$

In fact, by symmetry, we also obtain:

$$Vol_{p-2}(S_1 \cap S_5) = Vol_{p-2}(S_2 \cap S_5) = Vol_{p-2}(S_3 \cap S_5) = Vol_{p-2}(S_4 \cap S_5)$$

As $p = 4$, by Schläfli's formula,

$$dvol_p(C_3) = I_1(t) = \frac{\left(12 \arccos\left(\frac{1}{t^2+2}\right) - 4\pi\right) t}{3\sqrt{3+t^2}\,(4+t^2)}$$

We have to integrate the differential volume between 0 and 1, to multiply this value by 10 and to divide it by the surface of the hypersphere in $\mathbb{R}^5$, $\omega_5 = (8\pi^2)/3$. Thus

$$\frac{30}{8\pi^2}\int_0^1 I_1(t)dt \approx 0.009106$$

is the likelihood of the first type of configurations.

The strategy is similar for the computation of the likelihood of the second configuration of the paradox. Together with equations (8.1), (8.2) and (8.3), also consider:

$$\begin{aligned} -x_4 &> 0 && (8.6)\\ -x_5 &> 0 && (8.7)\\ -x_1 - x_2 - x_3 - tx_4 - tx_5 &> 0 && (8.8) \end{aligned}$$

with $t \in [0,1]$. Again, these equations define a cone C_4 which intersects the unit hypersphere $\mathbb{R}^5$. Computing the probability of these events reduces to finding the measure of C_4 on the surface of the hypersphere. Again, we can use the Schläfli formula

$$dvol_p(C_4) = \frac{1}{p-1} \sum_{j<k} vol_{p-2}(S_j \cap S_k) d\alpha_{jk}$$

where S_j is the facet defined by equation $(8.j)$, $j = 1, 2, 3, 6, 7, 8$.

Computation of the dihedral angles.

Let U_j be a vector normal to facet S_j, pointing inside the cone:

$$\begin{aligned} U_1 &= (1,0,0,0,0)\\ U_2 &= (0,1,0,0,0)\\ U_3 &= (0,0,1,0,0)\\ U_6 &= (0,0,0,-1,0)\\ U_7 &= (0,0,0,0,-1)\\ U_8 &= (-1,-1,-1,-t,-t) \end{aligned}$$

In deriving the dihedral angles, we are only interested in the angles which depends upon t. There are five of them:

$$\begin{aligned} &\alpha_{18} = \alpha_{28} = \alpha_{38} = \arccos\left(\tfrac{1}{\sqrt{3+t^2}}\right)\\ &\alpha_{68} = \alpha_{78} = \arccos\left(\tfrac{-t}{\sqrt{3+2t^2}}\right) \end{aligned}$$

Hence:

$$\begin{aligned} &d\alpha_{18} = d\alpha_{28} = d\alpha_{38} = \tfrac{\sqrt{2}t}{\sqrt{1+t^2}(3+2\,t^2)}\\ &d\alpha_{68} = d\alpha_{78} = \tfrac{3}{\sqrt{3+t^2}(3+2\,t^2)} \end{aligned}$$

Table 8: Vertices of the simplices

Volumes	*Directions*
$S_1 \bigcap S_8$	Q_{1268}, Q_{1368}, Q_{1278}, Q_{1378}
$S_6 \bigcap S_8$	Q_{1268}, Q_{1368}, Q_{2368}

The vertices of the dihedral volumes.

We find the coordinates of the vertices by solving systems similar to the following one:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ -x_4 = 0 \\ -x_5 > 0 \\ -x_1 - x_2 - x_3 - tx_4 - tx_5 > 0 \end{cases}$$

The solution of this system gives the direction Q_{1234}. There are 15 such linear systems, but *only 8 of them will have a solution.*

$$\begin{array}{rcl} Q_{1236} & = & (0,0,0,0,-1) \\ Q_{1268} & = & (0,0,t,0,-1) \\ Q_{1368} & = & (0,t,0,0,-1) \\ Q_{2368} & = & (t,0,0,0,-1) \\ Q_{1237} & = & (0,0,0,-1,0) \\ Q_{1278} & = & (0,0,-t,-1,0) \\ Q_{1378} & = & (0,-t,0,-1,0) \\ Q_{2378} & = & (-t,0,0,-1,0) \end{array}$$

The vertices of $(S_j \cap S_k)$ are the directions Q_{abcd} where both j and k appear as indices. According to the situation, there will three or four of them (see Table 8). By symmetry, the volumes $(S_j \cap S_8)$, $j = 1, 2, 3$ are equivalent, and the volume $(S_k \bigcap S_8)$, $k = 6, 7$ are equivalent too. This leaves only two volumes to compute.

We shall now detail the computations for the volume of $S_1 \bigcap S_8$. This volume is petrulater on the sphere in $\mathbb{R}^3$ defined by the four directions Q_{1268}, Q_{1368}, Q_{1278} and Q_{1378}. We can divide this petulater into two spherical triangles and use again the Gauss-Bonnet theorem. After several computations, we prove that the area of the petrulater is:

$$\begin{aligned} Vol_{p-2}(S_1 \cap S_8) &= 2\left(\arccos(\frac{1}{\sqrt{1+2\,t^2}}) + \arccos(\frac{t}{\sqrt{2+t^2}})\right) \\ &\quad -2\left(\arccos(\frac{t}{\sqrt{1+2\,t^2}\sqrt{2+t^2}})\right) \end{aligned}$$

Similarly, we get:

$$Vol_{p-2}(S_6 \cap S_8) = 3\,\arccos(\left(2+t^2\right)^{-1}) - \pi$$

As $p = 4$, by Schläfli's formula,

$$\begin{aligned} dvol_p(C_4) &= I_2(t) \\ &= \frac{6\sqrt{2}t\left(\arccos(\frac{1}{\sqrt{1+2\,t^2}}) + \arccos(\frac{t}{\sqrt{2+t^2}}) - \arccos(\frac{t}{\sqrt{1+2\,t^2}\sqrt{2+t^2}})\right)}{\sqrt{1+t^2}(3+2\,t^2)} \\ &\quad + \frac{18\arccos\left(\frac{1}{2+2t^2}\right) - 6\pi}{\sqrt{3+t^2}(3+2t^2)} \end{aligned}$$

We have to multiply this value by 20 and divide it by the surface of the hypersphere in $\mathbb{R}^5$, $\omega_5 = (8\pi^2)/3$.Thus

$$\frac{60}{8\pi^2}\int_0^1 I_2(t)dt \approx 0.172262$$

is the likelihood of the second configuration of the paradox.

9 Annexe IV: Proof of Proposition 3.4

In the case of five states, there are two types of configuration that can give rise to the paradox. In the first type, four states prefer, for example, B to A whereas the total number of voters preferring A to B is greater than one half of the voters, in accordance with the following inequalities:

$n_1 < n/2$, $n_2 < n/2$, $n_3 < n/2$, $n_4 < n/2$ and $n_1+n_2+n_3+n_4+n_5 > 5n/2$.

Let Y be the set of voting situations satisfying the above inequalities. There are ten configurations of this type (two times the number of ways for choosing the single state preferring A to B).

In the second type of configuration, three states choose one of the candidate -say B-, two states choose A and A is the popular winner:

$n_1 < n/2$, $n_2 < n/2$, $n_3 < n/2$, $n_4 > n/2$, $n_5 > n/2$ and $n_1 + n_2 + n_3 + n_4 + n_5 > 5n/2$.

Let Z be the set of voting situations consistent with these inequalities. There are twenty configurations of this type (two times the number of ways for choosing the two states preferring A to B among the five states). Hence, the cardinality of the set X of voting situations giving rise to the paradox is given by:

$$|X| = 10\ |Y|\ +\ 20\ |Z|.$$

It can be checked that a voting situation belongs to Y if and only if

$$2 \leq n_1 \leq \frac{n-1}{2}, \quad \frac{n+3}{2} - n_1 \leq n_2 \leq \frac{n-1}{2}, \quad n+1-n_1-n_2 \leq n_3 \leq \frac{n-1}{2},$$

$$\frac{3n+1}{2} - n_1 - n_2 - n_3 \leq n_4 \leq \frac{n-1}{2}, \quad \frac{5n+1}{2} - n_1 - n_2 - n_3 - n_4 \leq n_5 \leq n.$$

The cardinality of Y directly follows:

$$|Y| = \frac{(n+1)(n^4 + 4n^3 - 14n^2 - 36n + 45)}{3840}.$$

In order to compute the cardinality of Z, we begin by evaluating the cardinality of the set W defined by the following inequalities:

$n_1 < n/2$, $n_2 < n/2$, $n_3 < n/2$, and $n_1 + n_2 + n_3 + n_4 + n_5 > 5n/2$.

Clearly, W contains Z and it turns out that its cardinality is easier to evaluate than the one of Z. This evaluation can be obtained by partitioning W in three subsets defined in the following way:

- Subset 1:

$$n_1 = 0, \quad 1 \leq n_2 \leq \frac{n-1}{2}, \quad \frac{n+1}{2} - n_1 - n_2 \leq n_3 \leq \frac{n-1}{2},$$

$$\frac{3n+1}{2} - n_1 - n_2 - n_3 \leq n_4 \leq n, \quad \frac{5n+1}{2} - n_1 - n_2 - n_3 - n_4 \leq n_5 \leq n;$$

- Subset 2:

$$1 \leq n_1 \leq \frac{n-1}{2}, \quad 0 \leq n_2 \leq \frac{n+1}{2} - n_1, \quad \frac{n+1}{2} - n_1 - n_2 \leq n_3 \leq \frac{n-1}{2},$$

$$\frac{3n+1}{2} - n_1 - n_2 - n_3 \leq n_4 \leq n, \quad \frac{5n+1}{2} - n_1 - n_2 - n_3 - n_4 \leq n_5 \leq n;$$

- Subset 3:

$$1 \leq n_1 \leq \frac{n-1}{2}, \quad \frac{n+1}{2} - n_1 + 1 \leq n_2 \leq \frac{n-1}{2}, \quad 0 \leq n_3 \leq \frac{n-1}{2},$$

$$\frac{3n+1}{2} - n_1 - n_2 - n_3 \leq n_4 \leq n, \quad \frac{5n+1}{2} - n_1 - n_2 - n_3 - n_4 \leq n_5 \leq n.$$

After evaluating each of these subsets, we obtain:

$$|W| = \frac{29n^5 + 145n^4 + 190n^3 - 10n^2 - 219n - 135}{3840}.$$

Now, it can be observed that W can be written as : $W = Z \cup Y \cup T$ where T is defined by:

$n_1 < n/2$, $n_2 < n/2$, $n_3 < n/2$, $n_5 < n/2$ and $n_1 + n_2 + n_3 + n_4 > 5n/2$.
By symmetry, we have $|T| = |Y|$ and, as Z, Y and T are disjointed, we conclude that:

$$|Z| = |W| - 2|Y| = \frac{9n^5 + 45n^2 + 70n^3 + 30n^2 - 79n - 75}{960}.$$

The desired probability then easily follows as:

$$P_{IAC}(n, 5) = \frac{10|Y| + 20|Z|}{(n+1)^5} = \frac{5(11n^4 + 44n^3 + 38n^2 - 12n - 81)}{384(n+1)^4}$$

and $P_{IAC}(\infty, 5)$ tends to 55/384.

10 Annexe V: Lagrange Multipliers and Maximum Entropy

Find $P(y)$, such that $y \in [0, 1]$, which maximizes

$$S = -\int_0^1 P(y) \log P(y) dy$$

under the constraints

$$\begin{aligned} \int_0^1 P(y) dy &= 1 \\ \int_0^1 y P(y) dy &= a \end{aligned}$$

Change $P(y)$ in $P(y) + \delta P$ and take the functional derivative. We get:

$$\begin{aligned} \int_0^1 \delta P(\log P + 1) dy &= 0 \\ \int_0^1 \delta P dy &= 0 \\ \int_0^1 \delta P y dy &= 0 \end{aligned}$$

Taking into account the second relation the first can be written:

$$\int_0^1 \delta P y dy \quad = \quad 0$$

Multiplying the three last relations by respectively α, $-\lambda$ and -1 and adding them together, we get:

$$\int_0^1 (\alpha - \lambda x - \log P)\delta P dx$$

a relation which must hold $\forall \delta P$. Consequently,

$$P = e^{\alpha} e^{-\lambda x} = A e^{-\lambda x}$$

A and λ are given by the normalisation relation and the value of $\int_0^1 xP(x)dx = a$

11 Annexe VI: Proof of Proposition 4.1

First consider the case $p \leq \frac{1}{3}$. A graphic interpretation is provided on Figure 10, which has to be compared to Figure 2. The density is still uniform on a cube, but this cube has been reduced, and the origin of the axis is no longer the center of the cube. The new center of the cube is the point (E, E, E), which indicates that there is a bias in favor of candidate A. The hyperplane $z_1 + z_2 + z_3 = 0$ (represented with dashed lines) still cuts the cube in two parts, but no longer passes through the center of the cube. The darker pyramid correponds to situations where candidate A wins in States 1 and 2, but gets less vote than candidate B. The lighter polyhedron describes the case where candidate A losses in States 2 and 3, but enjoys a victory in the whole country. The volume of the small dark pyramid is $\frac{1}{6}(D-E)^3$. There are 3 similar volumes, and we have to divide this volume by the total volume of the cube, $(2D)^3$. Now, we perform the volume of the lightly shaded area. It is equal the volume of a bigger pyramid minus the volume of two dashed cones, that is :

$$\frac{1}{6}[(D+E)^3 - 2(2E)^3)]$$

Similarly, we mutiply this volume by three and divide it by $(2D)^3$. After simplification, we obtain the first formula of Prop 4.1.

Consider now the case $p \in [\frac{1}{3}, 1]$. A graphic interpretation is provided on Figure 11, for the case $E = 0.04$, $D = 0.1$ and $p = 0.4$. We get the same formula for the small dark pyramid. The volume of the lighter polyhedron

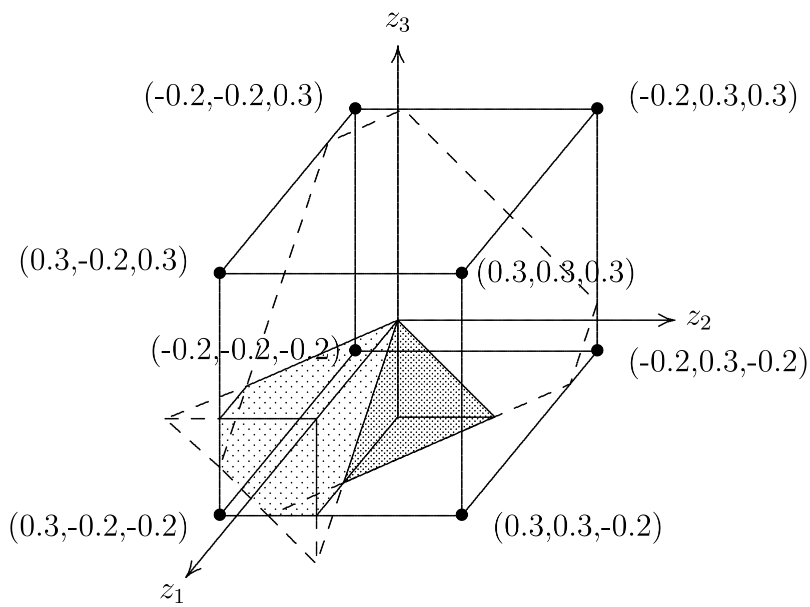

Figure 10: The volumes for $D = 0.25$, $E = 0.05$, and $p = 0.2$

is given by: $2E(D - E)^2$. We multiply the sum of these to volumes by three and divide it by $(2D)^3$ to obtain, after simplification, the second formula of Prop 4.1.

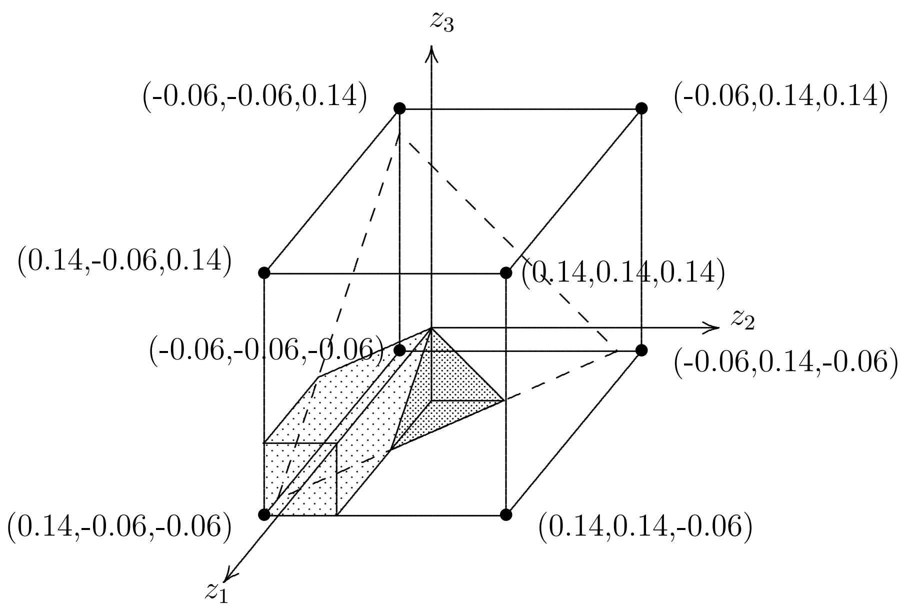

Figure 11: The volumes for $D = 0.1$, $E = 0.04$ and $p = 0.4$

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