Positive and normative assessment of voting situations*

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Abstract

This paper provides a rigourous formulation as well as a positive/descriptive extension of several notions relative to the role of voters in a voting situation. Success, decisiveness, and luck, as well as some variations of these notions, are precisely formulated as objective measures of different features present in such situations. The formulations provided are based on the two inputs that enter any real-world voting situation: the voting procedure and the voters' voting behavior, where the latter is summarized by a probability distribution over all feasible vote configurations. The purely normative value of some classical measures is also discussed.

Key words: Voting, power, success, decisiveness, luck, power measurement.

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1 Introduction

This paper is concerned with the analysis of voting situations. By a voting situation, we mean a situation in which a set of voters faces collective decision-making by means of a procedure that specifies when a proposal is to be accepted and when is to be rejected after a vote is cast. More specifically we are interested in the accurate formulation and quantification of several features involved in these situations, as the capacity of different voters to get the result they want, be it 'by luck' or being influential or 'decisive' for it. We will show how these notions, which can be traced a long way back in the literature (among others, Penrose (1946), Shapley-Shubik (1954), Banzhaf (1965), Rae (1969), Coleman ([1971] 1986), and Barry (1980)), can be formulated in very precise and general terms, taking their meaning far beyond the purely normative terms in previous formulations. To do so we will include in the model the two separate ingredients in any real-world voting situation: the voting rule and the voters' behavior.

Traditionally voting situations have been modelled by simple superadditive games. But in spite of the deceivingly suggestive game-theoretic terminology, the only objective information which is embodied in this traditional model is the voting rule: any 'coalition' that can pass a proposal is assigned the worth 1, while 0 is assigned to coalitions that cannot. No information about the voters is included in this model. In this context several measures or 'power indices' have been proposed in order to assess the voters' a priori 'decisiveness' or 'power'. The main power indices to be found in the literature are the Shapley-Shubik (1954) index, the Banzhaf (1965) index and the Coleman ([1971] 1986) indices[1].

This classical approach can be criticized for several reasons. First, the game-theoretical framework and terminology is inadequate and misleading. Even though a voting rule can be modelled by a simple game, the associated vocabulary does not suit the voting *rule* analysis. There is no transferable utility involved, there are not really 'players' nor 'voters,' for the voting rule only specifies when a proposal will pass: to describe a voting rule one should more properly speak just of 'seats.' There are no 'coalitions' -a term that suggests cooperation with a purpose where there is only coincidence of vote. There are no 'marginal contributions' where there is no cake to share. Moreover, as a rule the axiomatic characterizations of power indices either do not fit the specificity of the context, or lack compellingness or motivation from a positive or a normative point of view. Second, the existence and misuse of several indices without a clear interpretation is confusing and does not contribute to their credit. Finally, power indices are often criticized (see, e.g., Garrett

[1]Some other indices that have been proposed are the Johnston index (1978), the Deegan-Packel index (1978) and the Holler-Packel index (1983) (see Felsenthal and Machover (1998) for a recent critical review).

and Tsebelis (1999)) on the basis that the only information they take into account is the voting rule, while the voters' preferences, which clearly influence their capacity of being successful, decisive or lucky, are ignored.

In this paper we propose a more general model which includes the two separate ingredients in a voting situation: the voting rule and the voters. The voting rule, specifies for a given set of seats when a proposal is to be accepted or rejected depending on the resulting vote configuration. Voters, the second ingredient in a voting situation, are included via their voting behavior, which is summarized by a distribution of probability over the vote configurations. This distribution of probability depends on the preferences of the actual voters over the issues they will have to decide upon, the likelihood of these issues being proposed, the agenda-setting issue, etc. In each real-world voting situation these probabilities must be approximated from the available data. Thus this general model, unlike the traditional one, is apt for positive or descriptive purposes.

Within this general framework we re-examine the concepts of 'success,' 'decisiveness' and 'luck' that in a more or less clear formulation can be traced back in the literature. This more general setting allows a simple and precise reformulation of these concepts as probabilities which depend on the voting rule and the voters' voting behavior. Moreover, our formulations extend previous purely normative notions to more general positive/descriptive concepts, which can be particularized into some familiar but not always well-understood notions, shedding new light on their meaning and relations.

The rest of the paper is organized as follows. Section 2 contains the basic model of the first ingredient in any voting situation: the voting rule. In section 3, we define the primitive a posteriori versions of the concepts of success, decisiveness and luck. Section 4 incorporates to the model the second ingredient in any voting situation: the voters' behavior. Section 5 provides the general and positive a priori extension of the concepts introduced in section 3, as well as some conditional variations of these concepts. Section 6 treats the special case in which all vote configurations are considered equiprobable with normative purposes, showing how some classical measures emerge as particular cases of the general notions introduced in section 5. Section 7 summarizes the main conclusions of the paper and points out some lines of further research.

2 Voting rules

A voting situation is a situation in which a set of voters faces decision-making by means of casting a vote on a proposal and passing or rejecting it according to the specifications of a voting procedure. Thus, there are two separate ingredients: the voters and what we will call the voting rule. In this section we concentrate in this second element.

A voting rule is a well-specified procedure to make decisions by the vote of any kind of committee of a certain number of members. If the number of voters is n, the different *seats* will be labelled, and N will denote the set of labels, where usually $N = \{1, ..., n\}$. Voters will be labelled by their seat's labels. Once a proposal is submitted to the committee, voters will cast votes. A *vote configuration* is a possible or conceivable result of a vote, that lists the vote cast by the voter occupying each seat. We will consider only the case where voters do not abstain: every voter will be assumed to vote either 'yes' or 'no'. Under this assumption there are 2^n possible configurations of votes, and each configuration can be represented by the set of labels of the 'yes'-voters' seats. So, for each $S \subset N$, we refer as the *vote configuration* S to the result of a vote where the voters in S vote 'yes' while the voters outside S vote 'no'. The number of 'yes'-voters in the configuration S, i.e., the cardinal of S, will be denoted by s.

An N*-voting rule* specifies which vote configurations lead to the passage of a proposal and which ones to its rejection. Thus an N-voting rule can be represented by the set of configurations of votes (i.e., subsets of N) that would lead to the passage of a proposal. These configurations will be called *winning configurations*. In what follows $\mathcal{W}_N$ (or $\mathcal{W}$ when N is clear from the context) will denote the set of winning configurations representing an N-voting rule.

It will be assumed that a voting procedure satisfies these requirements. 1: The unanimous 'yes' configuration leads to the passage of the proposal: $N \in \mathcal{W}$. 2: The unanimous 'no' configuration leads to the rejection of the proposal: $\emptyset \notin \mathcal{W}$. 3: If a configuration of votes is winning, then any other configuration containing it is also winning: If $S \in \mathcal{W}$, then $T \in \mathcal{W}$ for any T containing S. 4: If a configuration of votes leads to the passage of a proposal, the configuration $N \backslash S$ will not: If $S \in \mathcal{W}$ then $N \backslash S \notin \mathcal{W}$. The last condition prevents the passage of a proposal and its negation if they were supported by S and $N \backslash S$, respectively.

Let VR_N denote the set of all such N-voting rules, each of them identified with the set $\mathcal{W}$ of winning configurations that specifies it. Some particular voting procedures that will be alluded later are the following. In the *simple majority rule*, a proposal is passed if the number of votes in favor of the proposal is strictly greater than half the total number of votes. That is, denoting $\mathcal{W}^{SM}$ the simple majority rule,

$$\mathcal{W}^{SM} = \{S \subseteq N : s > n/2\}.$$

In the *unanimity rule*, the only configuration of votes that can pass the proposal is the one where all voters are in favor of the proposal. Denoting $\mathcal{W}^U$ this rule, we have

$$\mathcal{W}^U = \{N\}.$$

Seat *i's dictatorship* is the voting rule in which the decision always coincides with voter in seat i's vote: $\mathcal{W}^{D_i} = \{S \subseteq N : i \in S\}$. We will refer to the voter sitting in that seat as the *dictator*.

3 A posteriori success, decisiveness and luck

To speak of success, decisiveness and luck, or any other feature concerning the role played in a voting situation requires voters. Let the voters enter the scene and vote in favor or against a proposal. A vote configuration emerges, and the voting rule prescribes the final outcome, passage or rejection of the proposal. Now the basic concepts relative to each voter's role in the decision made can be defined in *a posteriori* terms, that is, once the result of the vote is known. If the proposal is accepted, only the voters who have voted in favor have got the outcome that they voted for: only they have had success[2]. Similarly if the proposal is rejected, only the voters who voted against it have had success. Thus, being *successful* means having the outcome -acceptance or rejection- one voted for. But being successful does not mean being influential or decisive. We will say that a successful voter has been *decisive* in a vote if her vote was crucial for her success. Thus, to be *decisive* means to have success and to be able to reverse the outcome -acceptance or rejection- by changing one's vote. This is the basic notion behind almost any concept of 'voting power.' Finally, a successful voter who has not been decisive will be said to have been *lucky*. It just means that even if by mistake the voter voted the opposite the result would have not changed. Although it is questionable the suitability of the word 'lucky' for this situation, we will use it following Barry (1980), as we do with his 'success' and 'decisiveness.'

Formally we have the following *a posteriori boolean* notions. 'A posteriori' as dependent on the voting rule used to make decisions and the resulting configuration of votes *after* a vote is cast; and 'boolean' in the sense that there is *no quantification* in these notions, a voter just may or may not be successful, decisive or lucky.

Definition 1 *After a decision is made according to an N-voting rule $\mathcal{W}$, if the resulting configuration of votes is S, and $i \in N$,*

(i) Voter i is said to have been (a posteriori) successful (for brief, i is successful in $(\mathcal{W}, S)$), if the decision coincides with voter i's vote, that is, iff

$$(i \in S \in \mathcal{W}) \text{ or } (i \notin S \notin \mathcal{W}). \tag{1}$$

(ii) Voter i is said to have been (a posteriori) decisive (for brief, i is decisive in $(\mathcal{W}, S)$),

[2]The expression is due to Barry (1980), but the notion can be traced back at least to Rae (1969).

if voter i was successful and i's vote was critical for it, that is, iff

$$(i \in S \in \mathcal{W} \text{ and } S\backslash\{i\} \notin \mathcal{W}) \text{ or } (i \notin S \notin \mathcal{W} \text{ and } S \cup \{i\} \in \mathcal{W}). \tag{2}$$

(iii) Voter i is said to have been (a posteriori) lucky (for brief, i is lucky in $(\mathcal{W}, S)$), if voter i was successful but i's vote was not critical for it, that is, iff

$$(i \in S \in \mathcal{W} \text{ and } S\backslash\{i\} \in \mathcal{W}) \text{ or } (i \notin S \notin \mathcal{W} \text{ and } S \cup \{i\} \notin \mathcal{W}). \tag{3}$$

The three concepts are obviously related: if a voter has success, she or he must be either decisive or lucky:

$$i \text{ is successful in } (\mathcal{W}, S) \Leftrightarrow [(i \text{ is decisive in } (\mathcal{W}, S)) \veebar (i \text{ is lucky in } (\mathcal{W}, S))],$$

where '$\veebar$' is an exclusive 'or': if a voter has success, then she must be either decisive or lucky, but cannot be both.

All the three a posteriori concepts introduced depend on the resulting vote configuration and the voting rule which prescribes whether such a configuration is winning or not. Can these concepts be defined *a priori,* that is, before voters cast their vote?[3] If what the voters will vote is known with certainty, the answer is obvious. Otherwise, only in a few cases a partial answer is possible. For instance, in a dictatorship the dictator will surely be successful and decisive, while none of the other voters will be decisive, although those whose vote coincide with that of the dictator will be lucky. In a unanimity rule all voters are successful and decisive when a proposal is passed.

But in general, the knowledge of the voting rule is insufficient to determine whether a voter will have or not success, or whether, in the latter case, he will be decisive or lucky. Indeed a voter's success, decisiveness and luck depend on the voting rule but also on how she or he and the other voters will vote. For instance, in a simple majority, a voter will have success if at least half the other voters vote as she or he does. In this case the larger the probability that at least half the voters vote as voter i does, the larger the probability that voter i will have success. If the probability of the different vote configurations were known, a priori success, decisiveness and luck could be defined as the probabilities of having success, or being decisive or lucky.

4 Voting behavior

If what voters are going to vote is known in advance with certainty, the corresponding configuration of votes has a probability 1 of emerging, while the other configurations have

[3] Mind by 'a priori' here we mean prior to the vote is cast but once the voters occupy their seats. Not in the more radical sense in which this is often understood, meaning prior to both things.

a probability 0. In the absence of such an information, we can work with the *probabilities* of the different possible vote configurations. We assume that for any vote configuration S that may arise we know -or at least have an estimate of- the probability $p(S)$ that voters vote in such a way that S emerges. In this way we incorporate into the model the voters' voting behavior via the distribution of probability over all possible vote configurations.

Formally, $p : 2^N \rightarrow R$ will denote a distribution of probability that associates with each configuration of votes S its probability of occurrence $p(S)$. That is, $p(S)$ gives the probability that voters whose labels are within S vote 'yes', while voters whose labels are outside S vote 'no'. We have of course $0 \leq p(S) \leq 1$ for any $S \subseteq N$, and $\sum_{S \subseteq N} p(S) = 1$. Let $\mathfrak{P}_N$ denote the set of all such distributions of probability over 2^N. This set can be interpreted as the set of all conceivable voting behaviors of N-voters (yes/no voters, in fact, as we assume that there is no abstention). It is worth noting that in this model the event 'voter i votes "yes"' is not necessarily independent of the other voters' vote, as in most of previous probabilistic models. But mind this is not a loss of generality but the opposite, for such independence is only a particular case within our more general model. In fact this generalization was already considered as desirable (because more realistic) by Niemi and Weisberg (1972) in conclusion of the series of papers collected under the title "Probability models of collective decision making".

These probabilities permit to reflect the relative proximity of voters' preferences, summarizing their voting behavior. At the theoretical level these probabilities open the door to connection with models involving the voters' preferences, as spatial models. The distribution of probability can also reflect dependence between some voting behaviors. For instance, if there is a strict discipline of vote within a party, all voters from the party will always vote the same way. This can be easily modelled by assigning a probability 0 to any configuration of votes where this party's members vote in a different way. At the empirical level such probabilities could be derived from previous votes, possibly based on the frequencies of configurations in previous votes, or whatever available data. Although these data may be difficult to collect it is clear that the better the probabilities are approximated, the better the results that can be obtained.

Now the model is complete: an N-voting situation is fully specified by the two ingredients in a pair $(\mathcal{W}, p)$, that is, the voting rule $\mathcal{W} \in VR_N$ and the probability distribution over vote configurations $p \in \mathfrak{P}_N$ that summarizes the voters' voting behavior.

5 A priori success, power and luck

The *a priori* version in a voting situation of the concepts introduced in section 4 in their primitive a posteriori version is now possible. A priori success, decisiveness and luck can be defined as the probability of being successful, decisive or lucky, respectively. These probabilities depend on the voting rule and on the probability distribution over vote configurations, the two ingredients that specify a voting situation. It suffices to replace in the a posteriori definitions (1), (2) and (3) the sure configuration S by the distribution of probability over the vote configurations p. This yields the following extension of these concepts. In formulae we will drop i's brackets in $S\backslash\{i\}$ or $S \cup \{i\}$.

Definition 2 *Let $(\mathcal{W}, p)$ be an N-voting situation, where $\mathcal{W}$ is the voting rule to be used and p is the probability distribution over vote configurations, and let $i \in N$:*

(i) Voter i's (a priori) success is the probability that i is successful:

$$\Omega_i(\mathcal{W}, p) := \sum_{\substack{S:S\ni i \\ S\in\mathcal{W}}} p(S) + \sum_{\substack{S:i\notin S \\ S\notin\mathcal{W}}} p(S). \tag{4}$$

(ii) Voter i's (a priori) decisiveness is the probability that i is decisive:

$$\Phi_i(\mathcal{W}, p) := \sum_{\substack{S:S\ni i \\ S\in\mathcal{W} \\ S\backslash i\notin\mathcal{W}}} p(S) + \sum_{\substack{S:i\notin S \\ S\notin\mathcal{W} \\ S\cup i\in\mathcal{W}}} p(S). \tag{5}$$

(iii) Voter i's (a priori) luck is the probability that i is lucky:

$$\Upsilon_i(\mathcal{W}, p) := \sum_{\substack{S:S\ni i \\ S\backslash i\in\mathcal{W}}} p(S) + \sum_{\substack{S:i\notin S \\ S\cup i\notin\mathcal{W}}} p(S). \tag{6}$$

These measures provide a precise and very general formulation of Barry's (1980) notions of success, decisiveness and luck. In particular, Barry's equation: 'Success' = 'Decisiveness' + 'Luck,' remains valid in a much more precise and general version. Namely, for any voting rule $\mathcal{W}$, any distribution of probability p, and any voter i, we have

$$\Omega_i(\mathcal{W}, p) = \Phi_i(\mathcal{W}, p) + \Upsilon_i(\mathcal{W}, p).$$

These accurate definitions have some virtues that deserve to be stressed. First, they permit to distinguish neatly between the 'a posteriori' and 'a priori' concepts. The a priori measures are based on the primitive a posteriori notions, with which they are consistent: if p is the trivial distribution in which a vote configuration S is sure, the a priori measure yields the corresponding a posteriori concept for that configuration of votes. Second, these measures take the voters' voting behavior as an explicit input necessary for the descriptive assessment of real world-voting situations. If restricted versions of all these measures can be traced a long way back in the literature on collective decision-making, in connection with different notions related to 'voting power,' so far the probability distribution over the vote configurations has not been considered as an (in general) independent input. Often such a distribution of probability was hidden or only implicit in the definition of some measures related with 'power'. Third, these formulations yield what can be interpreted as descriptive/positive measures, taking previous formulations far beyond their not always clear normative meaning. At the same time, as will be discussed in more detail in the next section, these general formulations provide a point of view which will permit a clear understanding of the normative (and only normative) value of some traditional 'power indices'.

The precise probabilistic framework in which all the three notions stand permits to address the accurate formulation of further specific notions for a given voting situation $(\mathcal{W}, p)$. For instance, if voter i is sure to vote in favor of the proposal, the *conditional* probabilities of success, decisiveness or luck can be calculated. Similarly if the voter is known to vote against the proposal. Alternatively, conditional success, decisiveness and luck can be defined conditionally to the acceptance or to the rejection of the proposal. The corresponding conditional probability gives the answer to each of the following questions:

Q.1: Which is voter i's conditional probability of success (resp., decisiveness or luck), given that voter i votes in favor of the proposal?

Q.2: Which is voter i's conditional probability of success (resp., decisiveness or luck), given that voter i votes against the proposal?

Q.3: Which is voter i's conditional probability of success (resp., decisiveness or luck), given that given that the proposal is accepted?

Q.4: Which is voter i's conditional probability of success (resp., decisiveness or luck), given that given that the proposal is rejected?

The conditional probabilities which answer any of these questions are given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \tag{7}$$

where A may stand for 'voter i is successful/decisive/lucky' and B stands either for 'voter i votes "yes"/"no"', or 'the proposal was passed/rejected.' This makes twelve possible conditional probabilities[4] which answer the previous questions. Of course the framework

[4] Of course, conditional probabilities only make sense if $p(B) \neq 0$.

allows for other alternative questions. We restrict to these ones because, as we will see in the next section, some power measures proposed in the literature can be reinterpreted as one of these conditional probabilities for a particular probability distribution.

Here a bit of notation is necessary. We will superindex the measures -Ω_i, Φ_i or Υ_i- when they represent conditional probabilities. The superindex '+' will refer to the condition 'given that i votes "yes"'. So the answer to the first question will be given by $\Omega_i^+(\mathcal{W}, p)$, $\Phi_i^+(\mathcal{W}, p)$, and $\Upsilon_i^+(\mathcal{W}, p)$, respectively. The superindex '$-$' will refer to the condition 'given that i votes 'no', the answer to the second question being given by $\Omega_i^-(\mathcal{W}, p)$, $\Phi_i^-(\mathcal{W}, p)$, and $\Upsilon_i^-(\mathcal{W}, p)$, respectively. The superindex 'Acc' will refer to the condition 'given that the proposal is accepted', the answer to the third question being given by $\Omega_i^{Acc}(\mathcal{W}, p)$, $\Phi_i^{Acc}(\mathcal{W}, p)$, and $\Upsilon_i^{Acc}(\mathcal{W}, p)$, respectively. And, finally, the superindex 'Rej' will refer to the condition 'given that the proposal is rejected'. So the answer to the fourth question will be given by $\Omega_i^{Rej}(\mathcal{W}, p)$, $\Phi_i^{Rej}(\mathcal{W}, p)$, and $\Upsilon_i^{Rej}(\mathcal{W}, p)$, respectively. As an illustration, we formulate two of them, the other variations being obtained in the same way.

Voter i's conditional probability of success given that the proposal is accepted, is given by:

$$\Omega_i^{Acc}(\mathcal{W}, p) = P(i \text{ is successful} \mid \text{the proposal is accepted}) = \frac{\sum\limits_{\substack{S:i\in S\\ S\in\mathcal{W}}} p(S)}{\sum\limits_{T\in\mathcal{W}} p(T)}.$$

Voter i's conditional probability of being decisive given that voter i votes against the proposal, is given by:

$$\Phi_i^-(\mathcal{W}, p) = P(i \text{ is decisive} \mid i \text{ votes against the proposal}) = \frac{\sum\limits_{\substack{S:i\notin S\\ S\notin\mathcal{W}\\ S\cup i\in\mathcal{W}}} p(S)}{\sum\limits_{T:i\notin T} p(T)}.$$

6 The normative point of view

All three basic concepts given by (4), (5) and (6) in Definition 2, as well as the conditional variations of them considered, are *positive* or *descriptive* notions based on the two separated data which specify a voting situation: the voting rule and the voters' voting behavior summarized by the probability distribution over vote configurations. Thus, for an accurate positive assessment of any of these features concerning the relevance of the role played by a voter in a voting situation, the voting rule is not enough, the probability of different vote configurations must be known or estimated from the available data. Therefore there is not such a general thing as 'the best positive measure' in any of the senses specified so far beyond the general formulae based on the two inputs. In every particular case only the probability distribution that best suits the case will provide the right measure.

In opposition to the positive/descriptive point of view considered so far, there is the *normative* point of view. This is the case when one is only concerned with the purely normative issues that arise in the design and the assessment of the voting rule *itself*, irrespective of which voters occupy the seats. For this purpose, the probabilities of the vote configurations, dependent on the particular voters' preferences, *should not* be taken into account, even if they were known[5]. In this case we are at a logical deadlock: no measurement seems possible without a probability distribution over vote configurations, but the particular probability distribution has to be ignored for normative purposes. What can be done? A way out of this difficulty consists of assuming equally probable all configurations of votes, the natural starting point in case of actual ignorance about the voters[6]. In other words, replacing p in formulae (4), (5) and (6), or in any of the conditional alluded variations, by the probability distribution that assigns the same probability to all configurations, we obtain *normative* assessments of a priori success, decisiveness, and luck (conditioned or not) attached to a seat in a voting rule.

In fact, as we will presently see, some classical measures or 'power indices' to be found in the literature are but the particularization of some of the measures introduced in the previous section for this particular probability distribution. This is the case of Rae's (1969) 'expected correspondence between individual values and collective choices', the (non normalized) Banzhaf index and the Coleman indices, which can be formulated as follows.

Rae (1969) studies the anonymous decision-rule that maximizes the correspondence between a single anonymous individual and those expressed by collective policy. He makes the three following assumptions: The probability that one member will support (or oppose) a proposal is independent of that probability for any other member (assumption I). The probability that each member will support any proposal is exactly one-half, and since he must either support or oppose each proposal, the probability that he will oppose any proposal is also exactly one-half (assumption II). The probability that no member supports the proposal is zero (assumption III). Dropping assumption III (which is incompatible with assumptions I and II!)[7], and extending the 'index' giving the probability of coincidence of voter i's vote and the outcome to any voting rule (as already suggested by Dubey and

[5] In Straffin's (1988, pp. 77-78) words: 'The fairness of the structure we design should not depend on the particular voters who will fill positions in that structure. As John Rawls might put it, formal justice should be designed behind a veil of ignorance.'

[6] In similar terms Felsenthal and Machover (1998, p. 38) justify the use of this distribution of probability by the so-called 'Principle of Insufficient Reason.'

[7] Under assumptions I and II, the probability that no one supports the proposal is $1/2^n$.

Shapley, 1979), we obtain:

$$Rae_i(\mathcal{W}) = \sum_{\substack{S:S\ni i\\ S\in\mathcal{W}}} \frac{1}{2^n} + \sum_{\substack{S:i\notin S\\ S\notin\mathcal{W}}} \frac{1}{2^n}.$$

Voter i's (non normalized) Banzhaf index[8] in a voting rule $\mathcal{W}$ is given by:

$$Bz_i(\mathcal{W}) = \sum_{\substack{S:S\ni i\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^{n-1}}.$$

Coleman ([1971] 1986) defines three different indices. The *power of a collectivity to act,* that measures the easiness to make decisions by means of a voting rule $\mathcal{W}$, and is given by

$$A(\mathcal{W}) = \sum_{S\in\mathcal{W}} \frac{1}{2^n}.$$

Voter i's *Coleman index to prevent action* (Col_i^P) is given by

$$Col_i^P(\mathcal{W}) = \frac{\sum\limits_{\substack{S:i\in S\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} 1}{\sum\limits_{S\in\mathcal{W}} 1},$$

while voter i's *Coleman index to initiate action* (Col_i^I) are given by

$$Col_i^I(\mathcal{W}) = \frac{\sum\limits_{\substack{S:i\notin S\\ S\notin\mathcal{W}\\ S\cup i\in\mathcal{W}}} 1}{\sum\limits_{S\notin\mathcal{W}} 1}.$$

Observe that the only input necessary to determine any of these indices is the voting rule: no distribution of probability enters explicitly their definitions. But reinterpreting them in probabilistic terms, the implicit assumption behind these indices is that all vote configurations are equally probable. This is in fact equivalent to assume that each voter, independently from the others, votes 'yes' with probability 1/2, and votes 'no' with probability 1/2. Indeed assuming that each voter i (independently from any other) votes with a certain probability α_i in favor of the proposal and with probability $1-\alpha_i$ against it, we have

$$p(S) = \prod_{i\in S}\alpha_i \prod_{j\in N\backslash S}(1-\alpha_j). \tag{8}$$

[8]Notice this is Owen's (1975) non normalized version of the Banzhaf index. See also Dubey and Shapley (1979).

Choosing $\alpha_i = 1/2$ for any voter and denoting $\overline{p}$ the resulting distribution of probability, have

$$\overline{p}(S) = \frac{1}{2^n} \quad \text{for any } S.$$

That is, all vote configurations have the same probability. Then we have the following conclusions about the meaning of the Rae index, the Banzhaf index and the Coleman indices:

Proposition 1 *Rae's correspondence between a voter and the collective policy is the voter's probability of success when all vote configurations have the same probability ($p = \overline{p}$), namely:*

$$Rae_i(\mathcal{W}) = \Omega_i(\mathcal{W}, \overline{p}).$$

Rae (1969) shows that among the anonymous voting rules, the simple majority rule maximizes this index, that is, any voter's a priori success if $p = \overline{p}$.

Proposition 2 *The Banzhaf index answers the following questions about a voting situation in which all vote configurations have the same probability ($p = \overline{p}$):*

(i) 'Which is voter i's probability of being decisive?' That is,

$$Bz_i(\mathcal{W}) = \Phi_i(\mathcal{W}, \overline{p}).$$

(ii) 'Which is voter i's conditional probability of being decisive, given that voter i votes in favor of the proposal?' That is,

$$Bz_i(\mathcal{W}) = \Phi_i^+(\mathcal{W}, \overline{p}).$$

(iii) 'Which is voter i's conditional probability of being decisive, given that voter i votes against the proposal?' That is,

$$Bz_i(\mathcal{W}) = \Phi_i^-(\mathcal{W}, \overline{p}).$$

Proposition 3 *The Coleman indices answer each of the following questions about a voting situation in which all vote configurations have the same probability ($p = \overline{p}$):*

(i) The power of a collectivity to act ($A(\mathcal{W})$) answers the question: 'Which is the probability of passing a proposal?' That is,

$$A(\mathcal{W}) = P(\text{the proposal is accepted}).$$

(ii) Voter i's power to prevent action ($Col_i^P(\mathcal{W})$) answers the question: 'Which is voter i's conditional probability of being decisive, given that the proposal is accepted?' That is,

$$Col_i^P(\mathcal{W}) = \Phi_i^{Acc}(\mathcal{W}, \overline{p}).$$

(iii) Voter i's power to initiate action ($Col_i^I(\mathcal{W})$) answers the question: 'Which is voter i's conditional probability of being decisive, given that the proposal is rejected?' That is,

$$Col_i^I(\mathcal{W}) = \Phi_i^{Rej}(\mathcal{W}, \overline{p}).$$

The precise equalities in the previous propositions show in particular and very clearly the difference between the Coleman indices and the Banzhaf index, often mistakenly confused. Both measure decisiveness, (i.e., the probability of being decisive) assuming all vote configurations equally probable. But the difference is that the Banzhaf index measures decisiveness (non conditional, or conditionally to i's positive or negative vote indistinctly), while Coleman indices measure decisiveness conditionally to the acceptance ($Col_i^P(\mathcal{W})$) or the rejection ($Col_i^I(\mathcal{W})$) of the proposal. The origin of the confusion between the Banzhaf index and the Coleman indices is due to the fact that their normalizations coincide, giving rise to the so-called 'Banzhaf-Coleman' index. In formula, denoting $\widetilde{x}$ the normalization of any vector $x \in R^n$, that is,

$$\widetilde{x} := \frac{x}{\sum_{i \in N} x_i},$$

we have the following result:

Proposition 4 *For any voting rule $\mathcal{W}$, $\widetilde{Bz}_i(\mathcal{W}) = \widetilde{Col}_i^P(\mathcal{W}) = \widetilde{Col}_i^I(\mathcal{W})$.*

This coincidence only advocates against the common practice of normalizing these indices, for, along with the loss of information this normalization entails, it makes them lose their probabilistic interpretation. Mind that in general the normalizations of the conditional probabilities do not coincide for arbitrary probability distributions.

Inverting the point of view, Propositions 2 and 3 can be reinterpreted as supporting $\Phi_i(\mathcal{W}, p)$, $\Phi_i^+(\mathcal{W}, p)$, $\Phi_i^-(\mathcal{W}, p)$, $\Phi_i^{Acc}(\mathcal{W}, p)$, $\Phi_i^{Rej}(\mathcal{W}, p)$ as natural positive/descriptive generalizations of the purely normative Banzhaf and Coleman indices. Similarly, Proposition 1 supports $\Omega_i(\mathcal{W}, p)$ as the natural positive/descriptive generalization of Rae's index.

7 Conclusion

In this paper we have laid the basis for a positive/descriptive recuperation of the best ideas behind old 'power indices,' so often misunderstood and misused. To this end we have provided a basic model in which the two ingredients in any real-world voting situation, the voting rule and the voters' behavior, often confused in the literature, are clearly separated and included. This framework allows a unified, precise and generalized reformulation of old power measures, taking them far beyond their purely normative meaning. That is to say, the measures proposed in this paper, based on the two elements in any voting situation, voters and voting rule, have a descriptive value for the assessment of different aspects of the role of voters in such situations.

On the other hand, this formulation gives at once a clear foundation to the purely normative use of some classical measures, and a clear understanding of their obvious lack of descriptive value. At the same time, as a result of all these measures fitting a same general model, a better understanding of all them is reached. It shows the sterility of the never ending argument about 'the best power index,' and in particular the futility of the 'axiomatic contest' between them, or rather between their partisans, to settle the question. In this sense the conclusion of the paper is clear. For *purely normative purposes,* and depending on the exact question one wants to answer, Banzhaf and Coleman indices, as well as Rae's index, make sense. But for descriptive purposes no power index is of any use.

Coming back to the positive/descriptive worth of the general measures provided in this paper, there are several lines of further work. First, the direct application of these measures in real-world voting situations. This entails the search of data for an assessment of the probability distribution over vote configurations that better summarizes the voters' behavior. It seems an interesting approach using empiric probabilities based on the frequencies of vote configurations in committees. For instance, for a parliament, based on the record of a past legislature. In this respect, Dahl's (1957) paper contains interesting comments on the comparability of the data that are collected. On the theoretical level, it seems promising to investigate the connection of the probability over vote configurations with the voters' preferences, be it by enriching the model or by connecting it with other models incorporating voters' preferences, as, for instance, spatial models. A different line of enrichment of the model would be widening the class of rules so as to include voting rules in which abstention is not treated as a negative vote. For instance, Felsenthal and Machover (1998) 'ternary voting rules' can be treated similarly, just extending the class of admissible vote configurations to ternary ones.

References

[1] Banzhaf, J. F., 1965, Weighted Voting Doesn't Work: A Mathematical Analysis. *Rutgers Law Review* **19**, 317-34

[2] Barry, B., 1980a, Is it Better to Be Powerful or Lucky?, Part I, *Political Studies* **28**, 183-194.

[3] Barry, B., 1980b, Is it Better to Be Powerful or Lucky?, Part II, *Political Studies* **28**, 338-352.

[4] Coleman, J. S., 1971, Control of Collectivities and the Power of a Collectivity to Act. In: *Social Choice*, Lieberman, B., ed., Gordon and Breach, London.

[5] Coleman, J. S., 1986, *Individual Interests and Collective Action: Selected Essays*, Cambridge University Press, Chapter 8: Control of collectivities and the power of a collectivity to act, 192-255.

[6] Dahl, R., 1957, The concept of power, *Behavioral Science* **2**, 201-215.

[7] Deegan, J., and E. W. Packel, 1978, A New Index of Power for Simple n-Person Games, *International Journal of Game Theory* **7**, 113-123.

[8] Dubey, P., and L. S. Shapley, 1979, Mathematical Properties of the Banzhaf Power Index, *Mathematics of Operations Research* **4**, 99-131.

[9] Felsenthal, D. S., and M. Machover, 1998, *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*, Edward Elgar Publishers, London.

[10] Garrett, G., and G. Tsebelis, 1999, Why Resist the Temptation to Apply Power Indices to the European Union?, *Journal of Theoretical Politics* **11**, 291-308.

[11] Holler, M. J., and E. W. Packel, 1983, Power, Luck and the Right Index, *Journal of Economics* **43**, 21-29.

[12] Johnston, R. J., 1978, On the Measurement of Power: Some Reactions to Laver, *Environment and Planning A* **10**, 907-914.

[13] Laruelle, A., and F. Valenciano, 2001, A probabilistic re-foundation of voting power measures, *Discussion Paper* **14/2001**, Departamento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.

[14] Niemi, R. G., and H. F.Weisberg, eds, 1972, *Probability Models of Collective Decision Making,* Merrill Publishing: Columbus, Ohio.

[15] Owen, G., 1975, Multilinear extensions and the Banzhaf value, *Naval Research Logistic Quarterly* **22**, 741-750.

[16] Penrose, L. S., 1946, The Elementary Statistics of Majority Voting, *Journal of the Royal Statistical Society* **109**, 53-57.

[17] Rae, D., 1969, Decision Rules and Individual Values in Constitutional Choice, *American Political Science Review,* **63**, 40-56.

[18] Shapley, L. S., and M. Shubik, 1954, A Method for Evaluating the Distribution of Power in a Committee System , *American Political Science Review* **48**, 787 -792.

[19] Straffin, P. D., 1988, The Shapley-Shubik and Banzhaf Power Indices as Probabilities. In: *The Shapley Value. Essays in Honor of Lloyd S. Shapley,* Roth, A. E., ed., Cambridge University Press, Cambridge, 71-81.

8 Appendix

Proof of Proposition 1: It follows from the definition of $Rae_i(\mathcal{W})$ and substituting $\overline{p}$ for p in formula (4). ■

Proof of Proposition 2: To derive the results, first recall that if the distribution of probability is $\overline{p}$, each voter votes with probability 1/2 'yes' and probability 1/2 'no'. Second, note any time voter i is decisive in a winning configuration S, voter i is also decisive in the vote configuration $S\backslash\{i\}$, that is, $\sum_{\substack{S:S\ni i\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} 1 = \sum_{\substack{S:i\notin S\\ S\notin\mathcal{W}\\ S\cup i\in\mathcal{W}}} 1$. Then the result follows easily:

$$\Phi_i(\mathcal{W},\overline{p}) = \sum_{\substack{S:S\ni i\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^n} + \sum_{\substack{S:i\notin S\\ S\notin\mathcal{W}\\ S\cup i\in\mathcal{W}}} \frac{1}{2^n} = \sum_{\substack{S:S\ni i\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^{n-1}} = Bz_i(\mathcal{W}),$$

$$\Phi_i^+(\mathcal{W},\overline{p}) = \frac{\sum_{\substack{S:S\ni i\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^n}}{\frac{1}{2}} = \sum_{\substack{S:S\ni i\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^{n-1}} = Bz_i(\mathcal{W}),$$

$$\Phi_i^-(\mathcal{W}_n,\overline{p}) = \frac{\sum_{\substack{S:i\notin S\\ S\notin\mathcal{W}\\ S\cup i\in\mathcal{W}}} \frac{1}{2^n}}{\frac{1}{2}} = \sum_{\substack{S:i\notin S\\ S\notin\mathcal{W}\\ S\cup i\in\mathcal{W}}} \frac{1}{2^{n-1}} = \sum_{\substack{S:S\ni i\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^{n-1}} = Bz_i(\mathcal{W}).$$

■

Proof of Proposition 3: First, (i) immediate. Second, (ii) and (iii) follow directly from the conditional probabilities that yield $\Phi_i^+(\mathcal{W},\overline{p})$ and $\Phi_i^-(\mathcal{W},\overline{p})$. ■

Proof of Proposition 4: In view of Propositions 2 and 3, it is enough to prove that $\widetilde{\Phi}_i^+(\mathcal{W},\overline{p}) = \widetilde{\Phi}_i^{Acc}(\mathcal{W},\overline{p})$ and $\widetilde{\Phi}_i^-(\mathcal{W},\overline{p}) = \widetilde{\Phi}_i^{Rej}(\mathcal{W},\overline{p})$. Let us see the first equality. For it observe that previous to normalization, the two conditional probabilities are variants of formula (7) in which the event A ('i is decisive') coincides, while B is different. Nevertheless, the event $A\cap B$ is the same in both cases, because the event 'i is decisive and votes "yes"' is equivalent to the event 'i is decisive and the proposal is accepted'. In formula we have

$$\Phi_i^+(\mathcal{W},\overline{p}) = \frac{\sum_{\substack{S:i\in S\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^n}}{\sum_{S:i\in S} \frac{1}{2^n}} \quad \text{and} \quad \Phi_i^{Acc}(\mathcal{W},\overline{p}) = \frac{\sum_{\substack{S:i\in S\\ S\in\mathcal{W}\\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^n}}{\sum_{S\in\mathcal{W}} \frac{1}{2^n}}.$$

The numerator in both expressions is the same, while he denominator is different, but in either case the same for every $i \in N$. In other words, vectors $\Phi^{+}(\mathcal{W}, \overline{p})$ and $\Phi^{Acc}(\mathcal{W}, \overline{p})$ differ only in a proportionality constant, and consequently their normalization coincides. The other equality follows easily in the same way. ■