Public Choice (2007) 133: 25–29
DOI 10.1007/s11127-007-9212-7

Individual rationality and bargaining

Abraham Diskin · Dan S. Felsenthal

Received: 15 June 2007 / Accepted: 18 June 2007 / Published online: 8 August 2007

Abstract We argue that Nash's solution to the bargaining problem should be modified such that it will be based on a New Reference Point (NRP). Such a point is needed so that a player is not considered 'individually rational' if he accepts an agreement that provides him with a utility lower than the minimal utility he can derive from any Pareto optimal agreement, or if he accepts an agreement that provides him a utility lower than the one he can obtain by unilateral action. The employment of such NRP requires modifying two axioms and hence leads to a new proposed solution.

Keywords Bargaining problem · Individual rationality · Minimal utility · Minimax point · Nash's bargaining solution · Pareto optimality · Reference point

1 Introduction

One of the most famous contributions to game theory was Nash's (1950) solution to the (two-person) bargaining problem. In this article Nash described the two-person bargaining problem, proposed a number of axioms that, in his opinion, a rational and fair solution of the problem should satisfy, and proved that there always exists a unique solution point that satisfies these axioms.

The two-person bargaining set is a set of points, S, denoting the utilities a and u derived by two bargainers, A and U, from all possible trades between them. The set S is bounded, convex and closed, and contains the status quo (aka *maximin*) *point* (a_0, u_0), denoting the utility that each of the bargainers can obtain by unilateral action. The bargaining problem is determining which (unique) point (a^*, u^*) ought to be selected as the most reasonable

A. Diskin
Department of Political Science, The Hebrew University of Jerusalem, Jerusalem 91905, Israel
e-mail: mshanna@pluto.huji.ac.il

D.S. Felsenthal (✉)
Department of Political Science, University of Haifa, Haifa 31905, Israel
e-mail: msdanfl@mscc.huji.ac.il

agreement between the two bargainers. Obviously the selected point depends, *inter alia*, on the particular properties (axioms) one wishes that this point should satisfy.

Nash's solution to the bargaining problem, $\mathrm{F}(S, (a_0, u_0)) = (a^*, u^*)$, is the unique point for which $(a^* - a_0)(u^* - u_0) \geq (a - a_0)(u - u_0)$ for all $(a, u) \in S$. It satisfies the following axioms:

1. *Individual rationality*: $a^* \geq a_0$ *and* $u^* \geq u_0$.
2. *Feasibility*: $(a^*, u^*) \in S$.
3. *Pareto optimality*: If $(a, u) \in S$ and $(a, u) \geq (a^*, u^*)$, then $(a, u) = (a^*, u^*)$.
4. *Independence of irrelevant alternatives*: If S' is a subset of S containing both (a_0, u_0) and (a^*, u^*), then $\mathrm{F}(S', (a_0, u_0)) = \mathrm{F}(S, (a_0, u_0))$.
5. *Invariance with respect to linear utility transformations*: If S' is obtained from S by a linear transformation: $\grave{a} = c_1 a + b_1$ and $\grave{u} = c_2 u + b_2$ where $c_1, c_2 > 0$, then $\mathrm{F}(S', (\grave{a}_0, \grave{u}_0)) = (c_1 a^* + b_1, c_2 u^* + b_2)$.
6. *Symmetry*: If $a_0 = u_0$ and $(a, u) \in S$ if $(u, a) \in S$, then $a^* = u^*$.

In some presentations the first three axioms are combined into a single axiom entitled "Pareto optimality" (see e.g., Luce and Raiffa 1957: 127).[1]

The meaning of the first axiom is that a player should not be considered 'individually rational' if he accepts an agreement that provides him a utility lower than the one he can obtain by unilateral action. We argue that, *in addition,* a player should not be considered 'individually rational' if he accepts an agreement that provides him with a utility lower than the minimal utility he can derive in case the parties reach a Pareto optimal agreement. We believe that both these utility points would be used as anchors by an arbitrator asked by the parties to propose a reasonable agreement.

Following this logic, we define in the next section a Minimal Utility Point (MUP), as well as a New Reference Point (NRP), which are based on Roth's (1977) as well as on our requirements from 'individual rationality'. The definition of NRP results in modifications of the first and fourth axioms, and in a new proposed solution of the bargaining problem.

2 Definitions, modified axioms, and a new proposed solution

Let (a', u'') be a feasible Pareto optimal point where A obtains his highest possible utility in S, and let (a'', u') be a feasible Pareto optimal point where U obtains his highest possible utility in S. It therefore follows that the point depicting the minimum utilities of both players in case they reach a Pareto optimal agreement is $\mathrm{MUP} = (a'', u'')$. This *minimal utility point* need not belong to S.

The MUP point is important in determining what should be the relevant 'bargaining universe' within S. Following our argument that one should use as reference points both (a_0, u_0) and (a'', u''), we define a *new reference point* $\mathrm{NRP} = (a^\wedge, u^\wedge)$, where:

(i) $a^\wedge = \max(a_0, a'')$.
(ii) $u^\wedge = \max(u_0, u'')$.

By employing this new reference point we now modify Nash's axiom 1 (individual rationality) and axiom 4 (independence of irrelevant alternatives) as follows:

[1]In fact, the concept of 'individual rationality' should be attributed to Roth (1977). In his original 1950 paper, Nash only introduces Weak Pareto Optimality—and no Individual Rationality—among his axioms. Roth observed that the Nash solution could also be characterized by replacing Weak Pareto Optimality with Strict Pareto Optimality and Strict Individual Rationality.

Modified axiom 1: $a^* \geq a^\wedge$ and $u^* \geq u^\wedge$.

Modified axiom 4: If S' is a subspace of S such that (a^*, u^*) is in S' and $(a^\wedge, u^\wedge)_S = (a^\wedge, u^\wedge)_{S'}$, then $\mathrm{F}(S, (a_{0,} u_0)) = \mathrm{F}(S', (a_0, u_0))$.

By employing Nash's original proof it is easy to prove that based on these modified axioms and the original axioms 2, 3, 5, and 6, Nash's modified bargaining solution that we propose is the unique point (a^*, u^*) for which $(a^* - a^\wedge)(u^* - u^\wedge) \geq (a - a^\wedge)(u - u^\wedge)$ for all $(a, u) \in S$.

3 Examples

We demonstrate the possible differences between Nash's original solution and our proposed modification of this solution by means of the following examples.

Example 1 Assume that:

(1) Two players, A and U, must divide between them 100 utiles;
(2) $(a_0, u_0) = (0, 0)$ (i.e., if A and U fail to reach an agreement they get nothing);
(3) $(a'', u'') = (0, 0)$ (i.e., it is possible for the players to reach an agreement where one of them gets all the 100 utiles while the other gets nothing).

In this case both the original Nash solution as well as our modification of this solution awards each player 50 utiles.

Now suppose that, *ceteris paribus*, assumption (3) above is changed such that:

(3) $(a'', u'') = (0, 20)$ (i.e., that if A and U reach a Pareto optimal agreement, then U must get, for some reason, at least 20 utiles and, therefore, A cannot get more than 80 utiles).

In this case the original Nash solution would still be $(50, 50)$, while our modified Nash solution would be $(40, 60)$.

Example 2 Assume that:

(1) Two players, A and U, must divide between them 100 utiles;
(2) $(a_0, u_0) = (40, 0)$.
(3) $(a'', u'') = (0, 20)$.

Hence,

(4) $(a^\wedge, u^\wedge) = (40, 20)$.

We argue that in this case where $a_0 > a''$ but $u_0 < u''$, individual rationality should reflect both the logic associated with the status quo (maximin) point as well as that associated with MUP. In other words, we have here a situation where A can get 40 utiles by unilateral action but no more than 80 utiles if he reaches a Pareto optimal agreement with U, while U gets nothing if an agreement is not reached but can get at least 20 utiles if he reaches a Pareto optimal agreement with A. Hence in this case the original Nash solution is $(70, 30)$, while our modified Nash solution would be $(60, 40)$.

As is demonstrated in the following example, the logic of reference to MUP in solving a two-person bargaining problem was invoked already a long time ago.

Example 3 The Babylonian Talmud discusses a case where two persons, A and U, argue over a division of a *talit*.[2] A claims that the entire *talit* belongs to him while U claims only half of the *talit*. Both A and U know that if they fail to reach an agreement the *talit* will be confiscated by the authorities and they will get nothing.

So if we apply our notation we have here a situation where $(a_0, u_0) = (0, 0)$ while $(a'', u'') = (1/2, 0)$. Consequently, here, too, the original Nash solution would be $(1/2, 1/2)$, while our modified Nash solution would be $(3/4, 1/4)$—which happens to be also the solution adopted by the Talmud. Rabbi Shlomo Yitzhaki[3] explained the logic of the Talmudic ruling as if though it invoked the axiom of Individual Rationality coupled with the MUP:

> The one who claims 'half of it is mine' concedes that half of the *talit* belongs to the other claimant [who concedes nothing and should therefore get half the *talit* right away]. Since only half the *talit* is contested, each of the claimants should swear that he owns at least one-half of the contested part, and thereafter each takes his half [i.e., they divide the contested half equally].

4 Conclusion

It has been demonstrated in two independent laboratory experiments (Felsenthal and Diskin 1982; Schellenberg 1988) that the majority of actual players confronted with similar two-person bargaining problems where $(a'', u'') > (a_0, u_0)$, adopted agreements which were much closer to our modified Nash solution than to other investigated solutions, including the original Nash solution. We hope that this note supplies a normative justification to their behavior.

Employment of reference points similar to MUP was later suggested by others. Thus, Conley et al. (1997) used a similar idea to prove that impossibilities of social choice theory are often derived from lack of a proper reference point, and that when a point similar to MUP is employed these impossibilities are 'solved'. Another quite well-known reference to MUP is that of Herrero (1998). To the best of our knowledge the employment of both (a'', u'') and (a_0, u_0) has not yet been suggested.

Dagan et al. (2002) proved that Nash's axiom of Independence of Irrelevant Alternatives may be substituted by three other axioms such that his unique solution of the bargaining problem does not change. It therefore follows that when this substitution is made coupled with the employment of NRP and modifying the axiom of Individual Rationality as we have, one obtains a solution to the bargaining problem which is identical to our proposed modified Nash solution.

Moreover, even if one views the set of axioms underlying an alternative solution to the two-person bargaining problem—e.g., that proposed by Kalai and Smorodinsky (1975)—as more reasonable than the set of axioms proposed by Nash, we would still hold that it ought to be modified by employing a modified Individual Rationality axiom.[4]

It also seems that a reference point similar to NRP should be used in situations where more than two bargainers are involved.

---

[2]In Talmudic times a *talit* was a top garment in the form of a wide cloak, similar to the Roman toga. This particular example is also discussed by Aumann and Maschler (1985) and by Young (1994, Chap. 4).

[3]Rabbi Shlomo Yitzhaki (1040–1105 AD) is probably the most well known Jewish commentator on the bible and the Talmud who is better known by his acronym RASHY. The quoted Talmudic ruling and his comments on it appear in Chap. 2, Article 2, of *Baba Metzia* (Middle Book) of the Babylonian Talmud.

[4]In our Example 1 above the original Kalai–Smorodinsky (KS) solution is $(a, u) = (\frac{400}{9}, \frac{500}{9})$; in our Example 2 the KS solution is $(a, u) = (64, 36)$, while in our Example 3 the KS solution is $(a, u) = (2/3, 1/3)$. In all

Acknowledgements The authors are grateful to Robert J. Aumann, Moshé Koppel and an anonymous referee for their helpful comments.

References

Aumann, R. J., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, *36*, 195–213.

Butler, C. K. (2004). Modeling compromise at the international table. *Conflict Management and Peace Science*, *21*, 159–177.

Conley, J. P., McLean, R. P., & Wilkie, S. (1997). Reference functions and possibility theorems for cardinal social choice problems. *Social Choice and Welfare*, *14*, 65–78.

Dagan, N., Volij, O., & Winter, E. (2002). A characterization of the Nash bargaining solution. *Social Choice and Welfare*, *19*, 811–823.

Felsenthal, D. S., & Diskin, A. (1982). The bargaining problem revisited: minimum utility point, restricted monotonicity axiom, and the mean as an estimate of expected utility. *Journal of Conflict Resolution*, *26*, 664–691.

Herrero, C. (1998). Endogenous reference points and the adjusted proportional solution for bargaining problems with claims. *Social Choice and Welfare*, *15*, 113–119.

Kalai, E., & Smorodinsky, M. (1975). Other solutions to Nash's bargaining problem. *Econometrica*, *45*, 513–518.

Luce, D. R., & Raiffa, H. (1957). *Games and decisions: Introduction and critical survey*. New York: Wiley.

Nash, J. F. Jr. (1950). The bargaining problem. *Econometrica*, *18*, 155–162.

Roth, A. E. (1977). Individual rationality and Nash's solution to the bargaining problem. *Mathematics of Operations Research*, *2*(1), 64–65.

Schellenberg, J. A. (1988). A comparative test of three models for solving the bargaining problem. *Behavioral Science*, *33*, 81–96.

Young, H. P. (1994). *Equity in theory and practice*. Princeton: Princeton University Press.

these three examples if one modifies the KS solution by replacing (a_0, u_0) by $(a^\wedge, u^\wedge)$, one obtains a modified KS solution, which is identical to our modified Nash solution. However, one can easily demonstrate that our modified Nash solution may be different than our modified KS solution. It should also be noted that Butler (2004) compared between Nash, KS, and Felsenthal–Diskin (1982) solutions, and concluded that "the most compromise is proposed by the Felsenthal–Diskin bargaining solution, followed by the Kalai–Smorodinsky, and then the Nash bargaining solutions" (Butler, 2004: 159).