

Learning about Bond Prices

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Learning and Bond price volatility:

- Many studies on stock price volatility, fewer on bond price volatility
- We learnt a lot about stock prices using models where agents learn about prices (internal rationality).
- Recent literature studies the role of expectations on many aspects of the economy.
- How much can we learn about bond yields using models of internally rational learning?

Roadmap

- 1 Yields Data, Some Puzzles
- 2 Literature
- 3 A Model
- 4 Internal Rationality
- 5 A Simple Model of Learning about the Slope

- Sample 1983:12 to 2019:12.
- Monthly data, US government bonds
- Nominal zero-coupon yields
- Surveys: Reuters' Blue Chip Financial Forecasters survey on expected bond yields.
- One-year maturities, leads, forecasting horizons.

Flat Term structure of Yield Volatilities

Table 1: Yields 1st and 2d moments

	1y	2y	3y	5y	7y	10y
$\hat{Y}^n$	388.83	417.23	441.39	481.19	510.89	541.69
$\hat{\sigma}_{Y^n}$	289.15	294.34	291.41	281.62	272.50	258.74
$\widehat{\text{corr}}(Y^n)_{-1y}$	0.89	0.90	0.91	0.93	0.93	0.93

Hard to reconcile with RE and reasonable serial correlation of expected inflation.

Failure of Expectations Hypothesis

$$Y_{t+1y}^{n-1} - Y_t^n = \alpha^n + \beta^n \frac{Y_t^n - Y_t^1}{n-1} + U_t^n \quad (1)$$

EH says β 's ≈ 1

Table 2: Failure of Expectations Hypothesis

	2y	3y	5y	7y	10y
$\hat{\beta}^n$	-0.31	-0.50	-0.99	-1.28	-1.67
t-stat	-0.43	-0.60	-1.21	-1.56	-2.16

horizon= 1y here and in all tables below
 n means " n years"

Failure of EH: "Modern" version

Let Q_t^n be the price of a zero-coupon bond of maturity n .

Define excess bond returns

$$xr_{t+1y}^n = \log \left(\frac{Q_{t+1y}^{n-1}}{Q_t^n} \right) - \log \left(\frac{1}{Q_t^1} \right)$$

Or, in terms of yields $Y_t^n \equiv -\frac{\log(Q_t^n)}{n}$

$$xr_{t+1y}^n = -(n-1)Y_{t+1y}^{n-1} + nY_t^n - Y_t^1$$

Failure of EH: "Modern" version

EH can also be expressed in terms of xr forecasts:

$$xr_{t+1y}^n = \alpha_n^{xr} + b_n^{xr} \frac{Y_t^n - Y_t^1}{n-1} + U_t^n$$

EH implies b^{xr} 's ≈ 0 .

LHS are marginal profits from investing in long bonds.

Table 3: Failure of EH "modern" version

	2y	3y	6y	8y	11y
$\hat{b}_n^{xr}$	0.61	2.54	10.69	15.87	24.17
$t - stat$	0.52	1.14	2.40	2.87	3.54
R^2	0.32	1.59	6.00	7.88	10.41

High volatility of excess bond returns

$$xr_{t+1y}^n = \log \left(\frac{Q_{t+1y}^{n-1}}{Q_t^n} \right) - \log \left(\frac{1}{Q_t^1} \right)$$

Table 4: Excess Returns volatility

maturity n	2y	3y	5y	7y	10y
$\hat{\sigma}_{xr^n}$	136.60	256.20	455.74	636.45	880.10

Excess volatility of yields, Intuition

$$\begin{aligned} Q_t^n &= Q_t^1 E_t^{\mathbb{Q}}(Q_{t+1}^{n-1}) \\ &= E_t^{\mathbb{Q}}\left(\prod_{i=0}^{n-1} Q_{t+i}^1\right) \end{aligned}$$

where $\mathbb{Q}$ refers to "risk-neutral" distribution.

Taking log-linear approximation

$$Y_t^n \approx E_t^{\mathbb{Q}}\left(\frac{\sum_{i=0}^{n-1} Y_{t+i}^1}{n}\right)$$

therefore we expect $\sigma_{Y_t^n} < \sigma_{Y_t^1}$

Excess volatility of excess bond returns, Intuition

Furthermore

$$\begin{aligned}xr_{t+1}^n &= nY_t^n - (n-1)Y_{t+1}^{n-1} - Y_t^1 \\ &\approx (E_t^{\mathbb{Q}} - E_{t+1}^{\mathbb{Q}}) \left(\sum_{i=1}^{n-1} Y_{t+i}^1 \right)\end{aligned}$$

so $\sigma_{xr_t^n}^2$ variance of a change in expectations, so it should be small.

Bond survey expectations

- Much recent work on surveys about asset prices
- Giacometti, Laursen and Singleton (2020) and Singleton (2021) AFA Presidential Address (GLS) :
 - Describe investor bond yield expectations
 - Investors (consensus) bond yield expectations are on average similar to those arising from an affine model with principal components

Adam, Marcet, Beutel (2017)

Let $\mathcal{E}$ denote survey expectations.

Consider a regression like EH but with expectations in left hand side

$$\mathcal{E}_t(xr_{t+1y}^n) = \alpha_n^{\mathcal{E}} + b_n^{\mathcal{E}} \frac{Y_t^n - Y_t^1}{n-1} + U_t^{n,\mathcal{E}}$$

Under RE $b_n^{xr} = b_n^{\mathcal{E}}$.

Related to Coibion and Gorodnichenko approach, two differences:

- CG work with differenced regression
- CG introduce forecast errors or forecast updates on the RHS

RE Regressions

$$xr_{t+1y}^n = \alpha_n^{xr} + b_n^{xr} \frac{Y_t^n - Y_t^1}{n-1} + U_t^{n,xr}$$
$$\mathcal{E}_t(xr_{t+1y}^n) = \alpha_n^{\mathcal{E}} + b_n^{\mathcal{E}} \frac{Y_t^n - Y_t^1}{n-1} + U_t^{n,\mathcal{E}}$$

Table 5: RE Regressions

	2y	3y	6y	8y	11y
$\widehat{b}_n^{xr}$	0.61	2.54	10.69	15.87	24.17
$\widehat{b}_n^{\mathcal{E}}$	0.16	0.80	2.50	1.27	3.69
t-stat $\widehat{b}_n^{xr} = \widehat{b}_n^{\mathcal{E}}$	0.37	0.76	1.75	2.47	2.70

Surveys underestimate the role of the slope in predicting xr_{t+1y}^n

t – stat rejects RE, more strongly for long maturities

Test would be valid for any regressor in the information set at t

RE Regressions, additional versions

Alternative regressors:

- PC_t^2 : 2d Estimated Principal Component

- $\hat{s}_t$: Investor's estimate of underlying slope according to our model

Table 6: RE Regressions

regressor		2y	3y	6y	8y	11y
PC_t^2	$\hat{b}_n^{xr}$	0.06	0.13	0.43	0.61	0.89
	$\hat{b}_n^{\mathcal{E}}$	0.02	0.05	0.14	0.13	0.19
	t-stat $\hat{b}_n^{xr} = \hat{b}_n^{\mathcal{E}}$	0.85	1.11	1.98	2.57	2.87
$\hat{s}_{t-3}$	$\hat{b}_n^{xr}$	1.04	2.16	9.14	13.90	22.67
	$\hat{b}_n^{\mathcal{E}}$	0.41	0.57	0.28	-2.28	-2.86
	t-stat $\hat{b}_n^{xr} = \hat{b}_n^{\mathcal{E}}$	0.77	0.95	2.12	2.79	3.08
$\hat{s}_t, \hat{s}_{t-3}$	p-value $\hat{b}_n^{xr} = \hat{b}_n^{\mathcal{E}}$	0.70	0.59	0.09	0.01	0.00

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- Difficult to match EH-failure and volatilities with standard equilibrium asset pricing models.
High volatility of xr documented since Shiller (1979) but very rarely addressed.
- Much progress has been made:
 - Time-varying reward for risk:
Dai and Singleton (2002), Wachter (2006), Gabaix (2008), Bansal and Shaliastovich (2013)
 - Bayesian/RE: agents disagree about forecast of inflation but they know the pricing function:
Xiong and Yan (2010), Buraschi and Whelan (2022), Buraschi and Jiltsov (2007), Ehling Gallmeyer, Heyerdahl-Larsen and Illreditsch (2018).

- Tables 1-4 remain hard to explain. Giglio et al (2018) argue standard models can't explain observations.
- Tables 5-6 are a new puzzle
- It matters whether the model is RE with time-varying risk or learning!!!

Literature, Adaptive Learning, partial information and bond yields

- Sinha (2009), Piazzesi and Schneider (2006), Laubach, Tetlow, and Williams (2007) models of adaptive learning about how macro variables or inflation influence bond prices.
- Ellison Tischbirekz (2017) beauty contest.

This Project

Improve match of the data under *Internal Rationality*.

- investors have a *consistent system of beliefs* about bond prices
- investors are *fully rational* given their beliefs.

A "minor" deviation from standard RE paradigm

Beliefs about bond prices are not RE, but they are quite good in equilibrium:

- close to RE beliefs
- model converges to RE
- "close" to the true distribution, (hard to reject using observed data)
- investors' beliefs match some features of expectations surveys

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A model of bond prices

N zero-coupon nominal bonds of maturity $n = 1, \dots, N$

Bond prices $Q_t = [Q_t^1, \dots, Q_t^N]'$.

Homogeneous investors with $E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t u(c_t)$

Investor beliefs represented by $\mathcal{P}$.

All bonds with $n > 1$ are resold one period after being purchased.

Budget constraint:

$$P_t c_t + \sum_{n=1}^N Q_t^n B_t^n = w_t P_t + \sum_{n=1}^N Q_t^{n-1} B_{t-1}^n$$

$$Q_t^0 = 1$$

A model of Bond Prices

Assume bond limits:

$$\overline{B} \geq B_t^n \geq \underline{B} \quad (2)$$

Inflation process:

$$\log P_t - \log P_{t-1} \equiv \pi_t = \mu(1 - \rho) + \rho\pi_{t-1} + \eta_t^\pi$$

$$\eta_t^\pi \sim \mathcal{N}(0, \sigma_{\eta^\pi}^2), \text{ iid.}$$

The Model

Equilibrium conditions:

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2$$

The short rate Y_t^1 is exogenous, given by

$$Y_t^1 = -\log(\delta) + \pi_t^e + \xi_t$$

ξ_t a mean-zero exogenous process (policy shock).

So bond limit is binding for $n = 1$.

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Investors' probability space under IR

$$\Omega \equiv \Omega_Q \times \Omega_{\pi, \xi}$$

where Ω_Q contains possible histories of $Q = [Q^n]_{n=1}^N$, and $\Omega_{\pi, \xi}$ contains possible histories of inflation π, ξ .

Investors have a belief system $\mathcal{P}$ with (consistent) probabilities on Ω .

RE is a special case when agents' beliefs about π, ξ are correct and they know the true pricing function determining Q .

Bayesian/RE a special case when agents' don't know everything about π, ξ but they know the true pricing function determining Q .

Model with learning about Q

However if agents don't know the pricing function prices are given by

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2 \quad (3)$$

Therefore we need to state investors' beliefs about joint distribution of Q, π .

Simple case

Assume for the rest of the talk

- ξ_t iid mean zero
- no risk aversion: $u(c) = c$
- Investors know the inflation process
- $\rho = 1$

RE prices

Under RE

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2$$

becomes

$$\begin{aligned} Q_t^{RE,n} &= \delta^n E_t \left(\frac{P_t}{P_{t+n}} \right) \\ Y_t^{RE,n} &= -\log \delta + \pi_t \end{aligned}$$

Model with learning about Q

Agents' beliefs about Q 's are given by a perceived yield curve
(in blue we show investors beliefs equations)

$$Y_t^n = -\log \delta + \pi_t^e + \mathbf{s}_t n + u_t^n \quad (4)$$

$\mathbf{s}$ a perceived, unobserved, "underlying slope".

In other words:

- Investors know the level of the yield curve and how inflation determines yields
- but (different from RE) agents believe there is a time-varying slope $\mathbf{s}_t$

A popular myth among academics:

"Investors with beliefs about bond prices Q are irrational."

The justification for this myth:

If investors are rational they can apply LIE:

$$Q_t^n = \delta^n E_t^{\mathcal{P}} \left(\frac{u'(c_{t+n})}{u'(c_t)} \frac{P_t}{P_{t+n}} \right)$$

And if they know inflation process then

$$Q_t^n = \delta^n E_t \left(\frac{u'(c_{t+n})}{u'(c_t)} \frac{P_t}{P_{t+n}} \right) = Q_t^{n,RE}$$

But:

Two reasons why LIE does not apply

- 1 LIE can not be applied with individual knowledge, it needs market knowledge.
(Adam Marcet 2011)
- 2 Under IR $u'(c_{t+n})$ depends on future prices thus on price beliefs, so $E_t^{\mathcal{P}}$ can not become E_t
(Adam, Marcet, Beutel, 2017)

Therefore investors with a consistent system of beliefs $\mathcal{P}$ do not see anything that contradicts their price beliefs.

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A Simple Model

Recall investors' perceived yield curve

$$Y_t^n = -\log \delta + \mathbf{s}_t n + \pi_t^e + u_t^n$$

To complete investors' beliefs assume investors believe

$$\mathbf{s}_t = \mathbf{s}_{t-1} + w_t$$

Optimal filtering implies

$$E_t^{\mathcal{P}}(\mathbf{s}_t) \equiv \hat{\mathbf{s}}_t = \hat{\mathbf{s}}_{t-1} + g(s_{t-1} - \hat{\mathbf{s}}_{t-1})$$

where s_t is the best estimate of period- t slope:

$$s_t = E_t^{\mathcal{P}}(s_t \mid Y_t^N, \dots, Y_t^1)$$

to be specified later. Then

$$E_t^{\mathcal{P}}(Y_{t+1}^n) = -\log \delta + \hat{\mathbf{s}}_t n + \pi_t^e$$

Learning Equilibrium

Then, equilibrium under learning with a log-linear approximation, actual yield curve (equations that hold in equilibrium in brown)

$$Y_t^n = -\log \delta + \hat{\mathbf{s}}_t \cdot n \left(\frac{n-1}{n} \right)^2 + \pi_t^e$$

Beliefs "near-rational"

Recall perceived yield curve:

$$Y_t^n = -\log \delta + \hat{\mathbf{s}}_t n + \pi_t^e + \hat{u}_t^n$$

Conceptually not a large deviation from RE:

- beliefs close to RE beliefs if $\sigma_w \approx 0$.
- just because agents believe there is a time-varying slope there is a time-varying slope (see below)
- model converges to RE (see below)
- perceived yield curve similar to actual yield curve, beliefs hard to reject

A simple estimate s_t (the perceived slope at t)

In principle investors should use all maturities Y_t^n to estimate s_t .

A simple case:

assume $\text{var}^{\mathcal{P}}(u_t^m) \ll \text{var}^{\mathcal{P}}(u_t^n)$ for a reference bond of maturity m .

GLS means use only $n = m, 1$ to filter s with t -information:

$$E_t^{\mathcal{P}}(s_t \mid Y_t^N, \dots, Y_t^1) = s_t = \frac{Y_t^m - Y_t^1}{m - 1}$$

Learning Equilibrium with "simple" estimate s_t

Plugging in equilibrium yields

$$\begin{aligned} Y_t^n &= -\log(\delta) + \hat{\mathbf{s}}_t \, n \left(\frac{n-1}{n} \right)^2 + \pi_t^e \\ s_t &= \frac{Y_t^m - Y_t^1}{m-1} = \hat{\mathbf{s}}_t \frac{m-1}{m} - \frac{\xi_{t-1}}{m-1} \\ \hat{\mathbf{s}}_t &= \hat{\mathbf{s}}_{t-1} + g(s_{t-1} - \hat{\mathbf{s}}_{t-1}) \end{aligned}$$

So, under self-referential learning:

$$\uparrow s_t = \frac{Y_t^m - Y_t^1}{m-1} \implies \uparrow \hat{\mathbf{s}}_{t+1} \implies \uparrow Y_{t+1}^n \implies \uparrow s_{t+1} = \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} \dots$$

The T-map

When agents' perceived expectation (PLM) is

$$E_t^{\mathcal{P}} \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} = \hat{\mathbf{s}}_t$$

then the true expectation (ALM) is

$$E_t \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} \equiv \frac{m-1}{m} \hat{\mathbf{s}}_t \equiv T(\hat{\mathbf{s}}_t)$$

Convergence to RE

We have $T' < 1$.

Standard theorems in the learning literature imply that model converges to RE:

$$Y_t \rightarrow Y_t^{RE}$$

if either

- OLS instead of constant gain ($g = 1/t$) or
- $g \rightarrow 0$

(Marcet and Sargent (1989), Williams (2017), Evans Honkapohja (2002))

Fluctuations around RE

Furthermore, for $g > 0$, plugging in equilibrium value of s_t

$$\hat{s}_t = \left(1 - \frac{g}{m}\right) \hat{s}_{t-1} - g \frac{\xi_{t-1}}{m-1}$$

So $\hat{s}_t$ very persistent and $E(\hat{s}_t) = 0$.

The learning model fluctuates around RE.

Collecting equilibrium relationships

$$\widehat{\mathbf{s}}_t = \left(1 - \frac{g}{m}\right) \widehat{\mathbf{s}}_{t-1} - g \frac{\xi_{t-1}}{m-1}$$

$$Y_t^n = -\log(\delta) + \widehat{\mathbf{s}}_t \frac{(n-1)^2}{n} + \pi_t^e$$

$$\begin{aligned} xr_{t+1}^n = & \left[2n - 3 + (n-2)^2 \frac{g}{m}\right] \widehat{\mathbf{s}}_t \\ & + (n-1) \eta_{t+1}^\pi + \left[\frac{(n-2)^2 g}{m-1} - 1\right] \xi_t \end{aligned}$$

$$E_t^{\mathcal{P}}(xr_{t+1}^n) = -\log \delta - \xi_t$$

With this we can easily compute all the variances and correlations.

Generic Properties of the Learning Model

In this simple case with unit root inflation we see analytically:

- $\text{var}(\hat{\mathbf{s}}_t)$ contributes to increase $\text{var}(Y^n)$ and $\text{var}(xr^n)$.
More so for large n .
- Consistent with RE regression:
 - $\hat{b}_n^{xr} > 0$, increases with n
 - $\hat{b}_n^{xr} > \hat{b}_n^{\mathcal{E}}$
 - $\hat{b}_n^{xr} - \hat{b}_n^{\mathcal{E}}$, increases with n

$$\hat{b}_n^{xr} > \hat{b}_n^{\mathcal{E}}$$

Recall PLM and ALM:

$$Y_t^n = -\log \delta + \left(\frac{n-1}{n} \right)^2 \hat{\mathbf{s}}_t n + \pi_t^e$$

$$Y_t^n = -\log \delta + \mathbf{s}_t n + \pi_t^e + u_t^n$$

So forecasts of future yields

$$E_t(Y_{t+1}^n) = -\log \delta + \left(\frac{n-1}{n} \right)^2 \left(1 - \frac{g}{m} \right) \hat{\mathbf{s}}_t n + \pi_t^e$$

$$E_t^{\mathcal{P}}(Y_{t+1}^n) = -\log \delta + \hat{\mathbf{s}}_t n + \pi_t^e + u_t^n$$

So agents *overpredict* the influence of $\hat{\mathbf{s}}_t$ on future yields.

Therefore investors *underpredict* the influence of $\hat{\mathbf{s}}_t$ on future *xr*

It can explain that in the data $\hat{b}_n^{xr} > \hat{b}_n^{\mathcal{E}}$.

Starting to fit a Calibrated model

- inflation is AR(1) with $\rho < 1$.
- use a yearly model
- Calibrate to sample values σ_{π^e} , ρ , σ_{η^π} , σ_ξ for $\xi = Y^1 - \pi^e$.

Only one free parameter g .

Fiddled with it to select $g = 0.3$

Comparing with the data: Calibration

Table 7: Model fit

	3y			10y		
	<i>data</i>	<i>RE</i>	<i>Learn</i>	<i>Data</i>	<i>RE</i>	<i>Learn</i>
σ_{Y^n}	291	85	96	258	58	175
$\rho_{Y^n, -1y}$	0.91	0.94	0.94	0.93	0.94	0.97
σ_{xr^n}	254	56	185	871	201	642
$\widehat{b}_n^{xr}$	3.22	0	2.02	24.6	0	7.01
$\widehat{b}_n^{\mathcal{E}}$	1.02	0	1.93	2.5	0	4.00

To be done

- policy shock ξ serially correlated as in the data
- monthly model
- learning about the intercept in the yield curve
- risk aversion
- disagreement
- learning when investors observe factors

Conclusion

A very simple model suggests learning about the slope helps to explain the data.

It is consistent with RE regressions

Moves second moments in the right direction.

But room for improvement!