THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

Summer 2016 examination

MA203

Real Analysis

2015/2016 syllabus only — not for resit candidates

Instructions to candidates

This paper contains **5** questions. You may attempt as many questions as you wish, but only your **best 4** answers will count towards the final mark. All questions carry equal numbers of marks.

Answers should be justified by showing work.

Please write your answers in dark ink (black or blue) only.

Time Allowed	**Reading Time:**	*None*
	Writing Time:	*2 hours*
You are supplied with:		*Answer booklets*
You may also use:		*No additional materials*
Calculators:		*Calculators are not allowed in this examination*

Question 1

(a) Let (x_n) be a sequence of real numbers. State carefully what does it mean to say that the series $\sum_{n=1}^{\infty} x_n$

(i) converges

(ii) converges absolutely.

(b) Give an example of a series $\sum_{n=1}^{\infty} x_n$ which converges but does not converge absolutely. You must justify your claims.

(c) Let (s_n) be the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$ and consider the following statement:

> If $\lim\limits_{n\to\infty} |s_{n+1} - s_n| = 0$ then the sequence (s_n) is Cauchy and thus the series $\sum_{n=1}^{\infty} x_n$ converges.

Determine whether the statement is true or false and explain.

(d) Let (x_n) and (y_n) be sequences in $\mathbb{R}$. We want to show that if $|x_n| \le y_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} y_n$ converges then $\sum_{n=1}^{\infty} x_n$ converges as well. For this we suggest the following proof:

> We have that $\underline{\lim\limits_{n\to\infty} y_n = 0}$ and that $|x_n| \le y_n$ for all $n \in \mathbb{N}$.
>
> $\underline{\text{Therefore the sequence } (|x_n|) \text{ converges to zero}}$. $\underline{\text{The latter implies}}$
>
> $\underline{\text{that } \sum_{n=1}^{\infty} |x_n| \text{ converges}}$ and $\underline{\text{thus the series } \sum_{n=1}^{\infty} x_n \text{ converges}}$.

Determine whether the above proof is correct or wrong. If it is correct, explain the underlined statements. If it is wrong, identify the incorrect claim and explain why it is wrong.

(e) Consider the power series

$$\sum_{n=1}^{\infty} \frac{n}{2^n(3n-1)}(x-1)^n.$$

Find all the values of $x \in \mathbb{R}$ for which the power series diverges.

Question 2

(a) Let F be a subset of a metric space (X, d). State carefully what does it mean to say that F is a closed set. You need to give your answer in the following three equivalent ways.

(i) In terms of open sets.

(ii) In terms of ϵ-neighborhoods.

(iii) In terms of sequences.

(b) Let E be the subset of the normed space $(\mathbb{R}^2, ||\cdot||_2)$ given by

$$E = \{(x, y) \in \mathbb{R}^2 |\ 1 < x < 2,\ 1 < y < 4\} \cup \{(2, y) \in \mathbb{R}^2 | -\infty < y < \infty\}.$$

(i) Illustrate the set E graphically.

(ii) Is the set E, closed, open or neither? Explain.

(iii) Determine the interior, $int(E)$, and closure, $cl(E)$, of the set E. You only need to give the sets.

(iv) What is the boundary of the set E^c? You only need to give the set.

(c) Let $x \in \mathbb{R}^2$ and $r > 0$. Consider the closed subset of $\mathbb{R}^2$ defined by

$$C_r(x) = \{y \in \mathbb{R}^2 |\ ||x - y||_2 \leq r\}.$$

(i) Show that $cl(V_r(x)) = C_r(x)$ in $(\mathbb{R}^2, ||\cdot||_2)$.
Hint: You may use without proof any known characterization of the closure and that if $y \in C_r(x)$ and $n \in \mathbb{N}$ then $y_n = \frac{1}{n}x + (1 - \frac{1}{n})y \in V_r(x)$.

(ii) Is the set $C_r(x)$ closed in $(\mathbb{R}^2, ||\cdot||_\infty)$? Explain in terms of equivalent metrics.

Question 3

(a) In what follows, (X, d) is a metric space.

(i) Let (x_n) be a sequence in X. What does it mean to say that (x_n) converges to $x \in X$?

(ii) Let $x \in X$ and $E \subseteq X$. Show that if $V_\epsilon(x) \cap E \neq \emptyset$ for all $\epsilon > 0$, then there is a sequence (x_n) in E such that $\lim_{n\to\infty} x_n = x$. You must also provide a graphical illustration of your proof.

(b) In what follows, (X, d) is a metric space.

(i) Let $C \subseteq X$. Use open covers to state what does it mean to say that C is a compact set.

(ii) Let $E \subseteq X$, $x \in E$ and $\epsilon > 0$. Give an example of an open cover, $\mathcal{U}$, for E, that is made of open neighbourhoods and either has x fixed and ϵ varying, or has x varying and ϵ fixed. Graphically illustrate your example.

(iii) Show that a compact subset of a discrete metric space must be finite.

(c) (i) Show that every compact metric space is complete.

(ii) State the converse of the statement in part (c)(i) and determine whether it is true or false. If it is true, explain briefly. If it is false you need to give a counterexample and explain why the space in your example is not compact.

In both parts of (c) you may use either the open cover or the sequential characterization of compactness.

Question 4

(a) Let (X, d) be a metric space.

(i) Assume that $f : X \to \mathbb{R}$ is continuous on X and let $c \in X$. Show that there are $\delta_c > 0$ and $M_c > 0$ such that $|f(x)| \leq M_c$ for all $x \in V_{\delta_c}(c)$.

(ii) In part (i) of this question we showed that a continuous function, $f : X \to \mathbb{R}$, is locally bounded. We know that if in addition X is compact then a continuous function $f : X \to \mathbb{R}$ is (globally) bounded on X. Explain why the local bounds of f cannot be used to construct a global bound when X is not compact, and how the compactness of X turns the local bounds of f to a global bound. Your answer needs to be descriptive (uses few "formulas" and mathematical symbols and provides intuition).

In what follows (X, d_X) and (Y, d_Y) are two metric spaces.

(b) *Let $f : X \to Y$ be a one-to-one, continuous function from X onto Y. If X is compact then f^{-1} is continuous on Y.*
A sketchy proof of this statement is given below. Read this proof carefully and write a detailed one explaining the points indicated by (*why?*).
Sketchy proof: Let C be a closed subset of X. Then C is compact (why?) and so $f(C)$ is a compact (why?) subset of Y. It follows that $f(C)$ is closed (why?) and thus f^{-1} is continuous (why?). $\square$

(c) (i) Explain carefully what does it mean to say that $f : X \to Y$ is not uniformly continuous.

(ii) Show that the function $x \mapsto x^2$ is not uniformly continuous on $\mathbb{R}$.

(iii) Determine an interval where the function $x \mapsto x^2$ is uniformly continuous. Explain briefly.

Question 5

(a) (i) State the Mean Value Theorem.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two differentiable functions on $\mathbb{R}$, with $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.

(b) (i) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. State carefully what does it mean to say that f is Riemann integrable.

(ii) Let $f : [0, 5] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3 & \text{if } x \neq 4 \\ 1 & \text{if } x = 4. \end{cases}$$

Determine whether the function f is Riemann integrable. If the function is not Riemann integrable, explain. If f is Riemann integrable then use either the definition or an appropriate integrability criterion and calculate its integral. You may <u>not</u> use any integration methods known from calculus.

(c) (i) Let (f_n) be a sequence of real valued functions defined on the set E. State carefully what does it mean to say that (f_n) converges uniformly to a function $f : E \to \mathbb{R}$.

(ii) Let $f_n(x) = \frac{1}{n}\sin(n^2 x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Determine whether the sequence (f_n) converges uniformly on $\mathbb{R}$. If it does, find its limit. If you use a uniform convergence criterion, other than the definition, you need to clearly state it.

END OF PAPER