MA203 Exam-2016
Solutions

Question 1

Solution to Question 1 (a) [2+2=4 marks]

(i) [2 marks] Let $s_n = \sum_{k=1}^{n} x_k$ for each $n \in \mathbb{N}$. We say that the series $\sum_{n=1}^{\infty} x_n$ converges, if the sequence (s_n) of partial sums converges. In such case we write $\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$ and we call this limit the sum or the value of the series. □

(ii) [2 marks] We say that the series $\sum_{n=1}^{\infty} x_n$ converges absolutely if $\sum_{n=1}^{\infty} |x_n|$ is convergent.

Solution to Question 1 (b) [5 marks]

Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Since the sequence $(\frac{1}{n})$ has positive terms, is decreasing and converges to zero, it follows by the Leibniz Alternating Series Test that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. However the series is not absolutely convergent as $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right|$ is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent. □

Solution to Question 1 (c) [5 marks]

The statement is false. Consider for example the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. Then

$$\lim_{n\to\infty} |s_{n+1} - s_n| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

However the sequence (s_n) is not Cauchy. Observe that for any n in $\mathbb{N}$ we have that

$$s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{n}{2n} = \frac{1}{2}.$$

This shows that (s_n) is not Cauchy and so the series $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges. □

Solution to Question 1 (d) [4 marks]

The proof is wrong. The fact $(|x_n|)$ converges to zero does not imply that $\sum_{n=1}^{\infty} |x_n|$ converges. As a counterexample consider the harmonic series. □

Solution to Question 1 (e) [7 marks]

Let $x_n = \frac{n}{2^n(3n-1)}$. Then we see that

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \frac{3n^2 + 2n - 1}{6n^2 + 4n} = \frac{1}{2}.$$

It follows that the power series converges for $|x - 1| < 2$, that is for $-1 < x < 3$, and diverges for $x > 3$ and $x < -1$.

If $x = -1$ then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n-1}.$$

Let $a_n = \frac{(-1)^n n}{3n-1}$. The sequence (a_n) does not converge to zero (it is actually divergent since $a_{2n} \to \frac{1}{3}$ and $a_{2n-1} \to -\frac{1}{3}$), and so the series diverges.
If $x = 3$ then the series becomes $\sum_{n=1}^{\infty} \frac{n}{3n-1}$. The sequence $(\frac{n}{3n-1})$ does not converge to zero and so the series diverges.
We conclude that the power series diverges for $x \in (-\infty, -1] \cup [3, \infty)$. $\square$

Question 2
Solution to Question 2 (a) [**6 marks**]
(i) A set F is closed if and only if F^c is open.
(ii) A set F is closed if and only if $F = \{x \in X |\ V_\epsilon(x) \cap F \neq \emptyset \text{ for all } \epsilon > 0\}$.
(iii) A set F is closed if only if for every sequence (x_n) in F which converges, we have that $\lim x_n \in F$.

Solution to Question 2 (b) [**2+4+3+2=11 marks**]
(i) The illustration is given in a separate file.
(ii) The set E is neither open nor closed.
E is not open. Consider the point $(2, 3)$. Then no ϵ-nhood, $V_\epsilon(2, 3)$, of $(2, 3)$ is contained in E. For example $(2 + \frac{\epsilon}{2}, 3) \in V_\epsilon(2, 3)$ but $(2 + \frac{\epsilon}{2}, 3) \notin E$.
E is not closed since $E \neq cl(E)$. For example consider the point $x = (1, 1)$. Then every ϵ-nhood of x intersects E, but x does not belong to E.
(iii) The interior of the set E is given by

$$int(E) = \{(x, y) \in \mathbb{R}^2 |\ 1 < x < 2,\ 1 < y < 4\}.$$

The closure of the set E is given by

$$cl(E) = \{(x, y) \in \mathbb{R}^2 |\ 1 \leq x \leq 2,\ 1 \leq y \leq 4\} \cup \{(2, y) \in \mathbb{R}^2 | -\infty < y < \infty\}.$$

(iv) The boundary of E^c is given by,

$$bd(E^c) = \{(x, 1)|1 \leq x \leq 2\} \cup \{(x, 4)|1 \leq x \leq 2\} \cup \{(1, y)|1 \leq y \leq 4\} \cup \{(2, y)|y \in \mathbb{R}\}.$$

A graphical illustration of the boundary is also acceptable. One point will be given if the set is not given but it is mentioned that the boundary of E^c is equal to the boundary of E.

Solution to Question 2 (c) [**4+4=8 marks**]
(i) First we show that $cl(V_r(x)) \subseteq C_r(x)$. Since $V_r(x) \subseteq C_r(x)$ and $C_r(x)$ is a closed set and $cl(V_r(x))$ is the smallest closed set containing $V_r(x)$, it follows that $cl(V_r(x)) \subseteq C_r(x)$.
For the converse inclusion let $y \in C_r(x)$. We claim that there is a sequence (y_n) in $V_r(x)$ such that $y_n \to y$. It follows that $y \in cl(V_r(x))$ and so $C_r(x) \subseteq cl(V_r(x))$.
Proof of claim: For every $n \in \mathbb{N}$ set $y_n = \frac{1}{n}x + (1 - \frac{1}{n})y$. Then $y_n \in V_r(x)$ for each $n \in \mathbb{N}$. Therefore (y_n) is a sequence in $V_r(x)$. Also for any $n \in \mathbb{N}$

$$||y - y_n||_2 = \frac{1}{n}||x - y||_2 < \frac{1}{n}r,$$

and thus $y_n \to y$. □
(ii) The metrics induced by the norms $|| \cdot ||_2$ and $|| \cdot ||_\infty$ are strongly equivalent. Specifically we have that

$$d_\infty(x, y) \leq d_2(x, y) \leq 2d_\infty(x, y).$$

It follows that the d_2 and d_∞ metrics are topologically equivalent and thus give rise to the same collection of open sets. So since $C_r(x)$ is closed in $(\mathbb{R}^2, || \cdot ||_2)$, it must also be closed in $(\mathbb{R}^2, || \cdot ||_\infty)$. □

Question 3
Solution to Question 3 (a) [**2+4=6 marks**]
(i) A sequence (x_n) in a metric space (X, d) is said to converge to $x \in X$ if for any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > N$.
(ii) For $n \in \mathbb{N}$ let $\epsilon = \frac{1}{n} > 0$. Then, since $V_{\frac{1}{n}}(x) \cap E \neq \emptyset$ there is $x_n \in V_{\frac{1}{n}}(x) \cap E$. The sequence (x_n) is in E and $d(x_n, x) < \frac{1}{n}$ for all $n \in \mathbb{N}$. The latter implies that $\lim_{n\to\infty} d(x_n, x) = 0$ and thus $\lim_{n\to\infty} x_n = x$. □

Solution to Question 3 (b) [**2+3+5=10 marks**]
(i) A subset C of X is said to be compact in X if every open cover of C has a finite subcover.
(ii) Either $\mathcal{U} = \{V_n(x)|n \in \mathbb{N}\}$ for some $x \in E$ or $\mathcal{U} = \{V_1(x)|x \in E\}$ for $\epsilon = 1$. Graphical illustrations are provided in a separate file.
(iii) Let (X, d) be a discrete metric space and $C \subseteq X$ be compact. The collection $\mathcal{U} = \{V_1(x)|x \in C\}$, is an open cover for C and since C is compact it has a finite subcover. This finite subcover is of the form $\mathcal{U}' = \{V_1(x_1), \ldots, V_1(x_k)\}$, where $x_1, \ldots, x_k \in C$. Since $V_1(x) = \{x\}$ for every $x \in X$ and since $\mathcal{U}'$ covers C we have that

$$C \subseteq \bigcup_{i=1}^{k} V_1(x_i) = \{x_1, \ldots, x_k\}.$$

It follows that C is finite. $\square$

Solution to Question 3 (c) [**4+5=9 marks**]

(i) Let (X, d) be a compact metric space and suppose (x_n) is a Cauchy sequence in X. Then by the compactness of X we have that (x_n) has a convergent subsequence, say (x_{n_k}), with $\lim x_{n_k} = x \in X$. Since (x_n) is Cauchy we have that (x_n) converges to x and so X is complete. $\square$

(ii) The converse statement is: Every complete metric space is compact. The statement is false. The metric space $(\mathbb{R}, |\cdot|)$ is complete but not compact.

To show that $\mathbb{R}$ is not compact we either show that the sequence (n) does not have a convergent subsequence or that the open cover $\{(-n, n)|\ n \in \mathbb{N}\}$ does not have a finite subcover.

Method 1: Let (x_{n_k}) be any subsequence of (n). The distance between any two distinct terms of (x_{n_k}) is at least 1 and thus (x_{n_k}) cannot be Cauchy and hence not convergent. It follows that $\mathbb{R}$ is not compact.

Method 2: Consider the family of open sets given below:

$$\mathcal{U} = \{V_n(0) \mid n \in \mathbb{N}\} = \{(-n, n) \mid n \in \mathbb{N}\}.$$

First we show that $\mathbb{R} \subseteq \bigcup_{n\in\mathbb{N}}(-n, n)$. Indeed (see Archimedean property) for any $x \in \mathbb{R}$ there is $m \in \mathbb{N}$ such that $x < m$. Hence $x \in (-m, m)$ and so $x \in \bigcup_{n\in\mathbb{N}}(-n, n)$.

The open cover $\mathcal{U}$ does not have a finite subcover for $\mathbb{R}$. Any finite subcover will be of the form

$$\{(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)\},$$

for some $k \in \mathbb{N}$ and does not cover $\mathbb{R}$. Indeed if $N = \max\{n_1, n_2, \ldots, n_k\}$ then clearly $N + 1 \in \mathbb{R}$ but not in $\bigcup_{i=1}^{k}(-n_i, n_i)$. So $\mathbb{R}$ is not compact. $\square$

Question 4

Solution to Question 4 (a) [**4+5=9 marks**]

Let $c \in X$. By continuity, given $\epsilon = 1 > 0$ there is $\delta_c = \delta(1, c) > 0$ such that $|f(x) - f(c)| < 1$ for all $x \in V_\delta(c)$. It follows that if $x \in V_\delta(c)$, then $|f(x)| < M_c$ where $M_c = 1 + |f(c)| > 0$.

(ii) From part (i) we have that around each $c \in X$ there is a nhood $V_{\delta_c}(c)$ where the function f is bounded by a constant $M_c > 0$.

X not compact. A global bound for the function could be $M = \sup\limits_{c\in X} M_c$. However, unless the set X is finite, M may not exist.

Compactness means that the set X has some sort of finite structure. The finite structure of the compact set X allows us to construct a global bound as follows. The family

$$\{V_{\delta_c}(c) \mid c \in X\}$$

is an open cover for X and thus it has a finite subcover, say $\{V_{\delta_{c_1}}(c_1), \ldots, V_{\delta_{c_k}}(c_k)\}$. In each of the nhoods $V_{\delta_{c_i}}(c_i)$ the function f is bounded by a constant $M_{c_i} > 0$. The maximum of these finitely many local bounds is a global bound for f on X. Also acceptable if a proof and a counterexample are given.

Solution to Question 4 (b) [**6 marks**]
To show that $f^{-1} : Y \to X$ is continuous it is enough to show that for every closed subset C of X the set $(f^{-1})^{-1}(C) = f(C)$ is a closed subset of Y. Let C be a closed subset of X. Then since every closed subset of a compact metric space is compact, we have that C is compact. By continuity of f we have that $f(C)$ is a compact subset of Y. Since every compact set is closed, it follows that $f(C)$ is closed and thus f^{-1} is continuous. □

Solution to Question 4 (c) [**3+5+2=10 marks**]
(i) $f : X \to Y$ is not uniformly continuous, if there is $\epsilon_0 > 0$ such that for all $\delta > 0$ there are $x, z \in X$ (x, z depend on δ) with $d_X(x, z) < \delta$ and $d_Y(f(x), f(z)) \geq \epsilon_0$.

(ii) To show that $x \mapsto x^2$ is not uniformly continuous on $\mathbb{R}$ we will use the nonuniform continuity criterion that is derived from part (i). If we take $\delta = \frac{1}{n}$ then we have that $f : X \to Y$ is not uniformly continuous, if there is $\epsilon_0 > 0$ and two sequences (x_n) and (z_n) in X, such that, $\lim d_X(x_n, z_n) = 0$ and $d_Y(f(x_n), f(z_n)) \geq \epsilon_0$.

Let $\epsilon_0 = 1 > 0$. For every $n \in \mathbb{N}$, let $x_n = 2n$ and $y_n = 2n + \frac{1}{2n}$. Then for all $n \in \mathbb{N}$

$$|x_n - y_n| = \frac{1}{2n} < \frac{1}{n}$$

and so $\lim |x_n - y_n| = 0$. Also

$$|x_n^2 - y_n^2| = |(2n)^2 - (2n + \frac{1}{2n})^2| = 2 + \frac{1}{4n^2} > 1.$$

It follows that $x \mapsto x^2$ is not uniformly continuous on $\mathbb{R}$. □

(iii) Let C be a compact subset of $\mathbb{R}$ then the following is true: If $f : C \to \mathbb{R}$ is continuous then f is uniformly continuous. Also any closed and bounded interval $[a, b]$ is compact. So since $x \mapsto x^2$ is continuous on $[a, b]$ is it will also be uniformly continuous.

Question 5

Solution to Question 5 (a) [2+4=6 marks]

(i) Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

(ii) Let $x > 0$. Then the function $g - f$ is continuous on $[0,x]$ and differentiable on $(0,x)$. It follows by the MVT that there is $c \in (0,x)$ such that

$$(g-f)(x) - (g-f)(0) = (g-f)'(c)(x-0).$$

So, since $g'(c) \geq f'(c)$, we have that for all $x > 0$,

$$g(x) - f(x) = x((g'(c) - f'(c)) \geq 0.$$

This, together with the fact that $f(0) = g(0)$ imply that $f(x) \leq g(x)$ for all $x \geq 0$. □

Solution to Question 5 (b) [3+6=9 marks] (i) A bounded function $f : [a,b] \to \mathbb{R}$ is called Riemann integrable if its lower and upper integrals are equal,

$$\sup_{\mathrm{P} \in \mathcal{P}} L(f, \mathrm{P}) = L(f) = U(f) = \inf_{\mathrm{P} \in \mathcal{P}} U(f, \mathrm{P}).$$

The common value of the upper and lower integrals is denoted by $\int_a^b f(x)dx$ and is called the Riemann integral of f.
Where

$$U(f, \mathrm{P}) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \text{ and } L(f, \mathrm{P}) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}),$$

and

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) \text{ and } m_k = \inf_{x \in [x_{k-1}, x_k]} f(x).$$

(ii) For any $n \geq 2$ let

$$\mathrm{P}_n = \{0, 4 - \frac{1}{n}, 4 + \frac{1}{n}, 5\},$$

be a partition of $[0,5]$. Then $I_1 = [0, 4 - \frac{1}{n}]$, $I_2 = [4 - \frac{1}{n}, 4 + \frac{1}{n}]$ and $I_3 = [4 + \frac{1}{n}, 5]$. Thus

$$U(f, \mathrm{P}_n) = 3(4 - \frac{1}{n}) + 3\frac{2}{n} + 3(1 - \frac{1}{n}),$$

and

$$L(f, \mathrm{P}_n) = 3(4 - \frac{1}{n}) + 1\frac{2}{n} + 3(1 - \frac{1}{n}).$$

Hence

$$U(f,\mathrm{P}_n) - L(f,\mathrm{P}_n) = 0 + \frac{4}{n} + 0 = \frac{4}{n}.$$

Hence $\lim\limits_{n\to\infty}(U(f,\mathrm{P}_n) - L(f,\mathrm{P}_n)) = 0$ and so f is integrable with

$$\int_0^5 f = \lim_{n\to\infty} U(f,\mathrm{P}_n) = \lim_{n\to\infty}(12 - \frac{3}{n} + \frac{6}{n} + 3 - \frac{3}{n}) = 15. \qquad \square$$

Solution to Question 5 (c) [3+7=10 marks]

(i) Let (f_n) be a sequence of real valued functions defined on E. The sequence (f_n) is said to **converge uniformly** to a function $f : E \to \mathbb{R}$ if

$$\forall \epsilon > 0,\ \exists N = N(\epsilon) \in \mathbb{N} \text{ such that } \ \forall x \in E,\ |f_n(x) - f(x)| < \epsilon,\ \forall n > N.$$

(ii) We will use the following criterion.

Let (f_n) be a sequence of real-valued functions defined on a set E, that converges pointwise to $f : E \to \mathbb{R}$. Set

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

Then $f_n \rightrightarrows f$ if and only if $M_n \to 0$ as $n \to \infty$.

Observe that for all $x \in \mathbb{R}$ we have that

$$-\frac{1}{n} \le \frac{\sin n^2 x}{n} \le \frac{1}{n}.$$

It follows that $f_n(x) \to 0$ for all $x \in \mathbb{R}$. Hence the sequence (f_n) converges pointwise to the zero function, f, on $\mathbb{R}$.

We also have that

$$M_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |\frac{\sin n^2 x}{n}| = \frac{1}{n}.$$

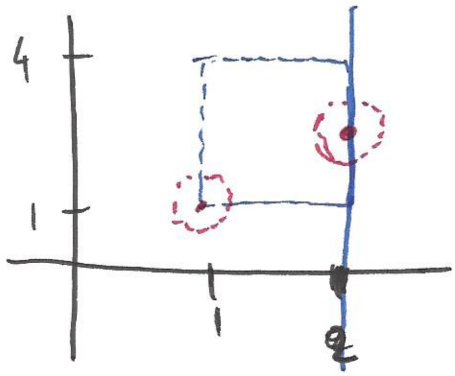
Since $\lim\limits_{n\to\infty} M_n = 0$ we conclude that the sequence (f_n) converges uniformly to the zero function, f, on $\mathbb{R}$. $\square$

MA203: Real Analysis exam (2016)

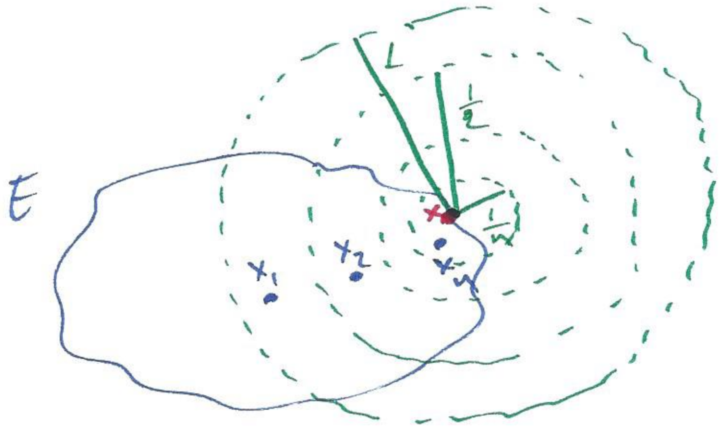
Graphical illustrations

The graphical illustrations guide is hand written to clearly indicate what is expected by the students.

Question 2(b)(i)

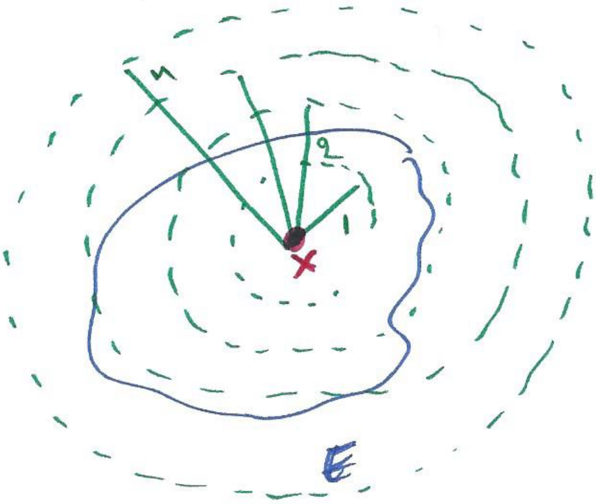


Question 3(a)(ii)



Question 3(b)(ii)

Example 1



Example 2