# Introduction to Abstract Mathematics
# MA 103

## Exercises 1

- Relevant parts of the **Lecture notes**: Chapter 1 and Sections 2.1 – 2.8, 2.12 and 2.13.
- Relevant parts of the text books: **Biggs**: Chapter 1 and Sections 3.1 – 3.5;
  **Eccles**: Chapters 1 – 4.

- Try to do as many questions as you can, and hand in whatever you have.
- Hand in your homework by putting it in the **homework box of your class teacher** on the Ground Floor of Columbia House.
- Always write your name, the course number (MA 103), your class group number and your class teacher's name on your homework.
- Use standard A4 paper and leave room in the margins for remarks, corrections, etc. from your class teacher. Use paperclips or staples to keep the pages of your homework together.

Here are some general remarks about writing mathematics:

- **Always** justify your answers, whatever the wording of the question.
- Most answers to exercises in this course will require a careful explanation of why something is true or false. You should give your argument in enough detail for somebody else to follow what you write.
- Write your answers in **English**. That is, don't just use symbols, but use **words** to explain how you get from each line to the next.
- Avoid using the symbols "$\therefore$" or "$\because$" (if you don't know what they mean, that's just fine), or arrows like "$\longrightarrow$" or "$\Longrightarrow$" in mathematical arguments. Use words!

In the questions below, we assume that we are always dealing with natural numbers. So if a statement is made about certain numbers $n$, say, then the fuller version of this statement would start: *for all natural numbers n, ...* .

**1** Consider the following (false) statement:

*If $n$ is a multiple of 16, then $n$ is not a multiple of 6.*

What properties must the number $n$ have, for $n$ to be a *counterexample* to this statement?

Find a counterexample to the given statement.

**2** Show that the following statement is true, by giving a proof:

*If $n$ is a multiple of 4, then $9n + 30$ is a multiple of 6.*

**3** (a) Construct the truth table for the statement $p \Rightarrow (q \wedge r)$.

(b) Decide whether the statement in (a) is logically equivalent to $(p \Rightarrow q) \wedge (p \Rightarrow r)$ or not.

**4** (a) Construct the truth table for the statement $(p \Rightarrow q) \Rightarrow r$.

(b) Construct the truth table for the statement $p \Rightarrow (q \Rightarrow r)$.

(c) Construct the truth table for the statement $(p \Rightarrow q) \wedge (q \Rightarrow r)$.

(d) In view of your answers, why do you think we should never write $p \Rightarrow q \Rightarrow r$?

**5** (a) Write down the **converse** of the statements in Questions 1 and 2.

(b) For each of the converses, determine whether the converse statement is true or false. Justify your answer by giving a proof or a counterexample.

**6** (a) Write down the **contrapositive** of the statements in Questions 1 and 2.

(b) For each of the contrapositives, determine whether the statement is true or false.

**7** For which natural numbers $n$ is $3^n - 1$ a prime number? Justify your answer.

**8** (a) Explain what is wrong with the following proof of the statement:

*If $n$ and $m$ are natural numbers, then $4 = 3$.*

**Proof**: Denote the sum $n + m$ by $t$. Then we certainly have $n + m = t$.
This last statement can be rewritten as $(4n - 3n) + (4m - 3m) = 4t - 3t$.
Rearrangement leads to $4n + 4m - 4t = 3n + 3m - 3t$.
Taking out common factors on each side gives $4(n + m - t) = 3(n + m - t)$.
Removing the common term now leads to $4 = 3$. □

(b) Explain what is wrong with the following proof of the statement:

*If $n$ and $m$ are natural numbers, then $n = m$.*

**Proof**: Denote the sum $n + m$ by $t$. Then we certainly have $n + m = t$.
Multiplying both sides by $n - m$ yields $(n + m)(n - m) = t(n - m)$.
Multiplying out gives $n^2 - m^2 = tn - tm$.
This can be rewritten as $n^2 - tn = m^2 - tm$.
Adding $\frac{1}{4}t^2$ to both sides gives $n^2 - tn + \frac{1}{4}t^2 = m^2 - tm + \frac{1}{4}, t^2$.
This is the same as $(n - \frac{1}{2}t)^2 = (m - \frac{1}{2}t)^2$.
So we get $n - \frac{1}{2}t = m - \frac{1}{2}t$.
Adding $\frac{1}{2}t$ to both sides gives $n = m$. □

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## Exercises 2

- Relevant parts of the **Lecture notes**: Sections 2.9 – 2.17.
- Relevant parts of the text books: **Biggs**: Chapter 2 and Sections 3.6 – 3.7;
  **Eccles**: Sections 6.1 – 6.2, 7.1 – 7.4 and 7.6.

- Always justify your answers.
- In the questions below you must take care to note whether the question deals with 'natural numbers' or with 'integers'.

**1** (a) Write down the mathematical notation for the following sets:

(i) the set $A$ of all natural numbers that are multiples of five;

(ii) the set $B$ of all integers whose fourth power is less than 1000.

(b) List the elements of the set $B$ in (a)(ii).

**2** For each of the following pairs of sets $A, B$, determine whether the statement "$A \subseteq B$" is true or false. Make sure you explain why!

(a) $A = \{-1, -3, -9\}$, $\quad B = \{\, m \mid m \text{ is an odd integer} \,\}$;

(b) $A = \{0, 1, 3, 7, 1023\}$, $\quad B = \{\, m \mid m = 2^n - 1 \text{ for some natural number } n \,\}$;

(c) $A = \varnothing$, $\quad B = \{0\}$;

(d) $A = \{n \in \mathbb{N} \mid n \text{ is even}\}$, $\quad B = \{n \in \mathbb{N} \mid n^2 \text{ is even}\}$;

(e) $A = \{1\}$, $\quad B = \{\, \mathbb{N} \,\}$.

**3** Consider the set $X = \{\, A \mid A \subseteq \{\, 0, 1 \,\} \,\}$. In words: $X$ is the set consisting of those $A$ which are subsets of $\{0, 1\}$.

(a) Exactly what does it mean to say that $A$ is an *element* of $X$?

Exactly what does it mean to say that $B$ is a *subset* of $X$?

(*If you are not sure, go back and look at the definitions of these terms.*)

(b) Is $\{0\}$ an *element* of $X$?

(c) Is $\{0\}$ a *subset* of $X$?

(d) Is there a set which is both an element of $X$ and a subset of $X$?

Justify all your answers carefully.

**4** (a) Write out the following statements using mathematical notation for the sets and quantifiers. (Ordinary phrases like 'such that' are acceptable.)

*For every integer $x$, there is an integer $z$ such that $x \leq z \leq 2x$.*

*There is a natural number $n$ such that, for every natural number $m$, we have $n \leq m$.*

(b) Prove that the first statement in (a) is false by giving a counterexample.

(c) Is the second statement in (a) true or false? (Justify your answer in detail.)

(d) Formulate the negation for each of the two statements, and express these as simply as you can.

Write these negations out in mathematical notation as well.

**5** Decide whether the following statement is True or False, and justify your assertion by means of a proof or a counterexample:

*For all sets $A$, $B$ and $C$, we have $A \cup (B \cap C) = (A \cup B) \cap C$.*

**6** (a) Give a proof that, for all sets $A, B, C$, we have

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

(b) Give a proof that, for all sets $A, B, C$, we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

**7** *This is an extra question, and it is not directly related to the material discussed this week. Try to see how far you can get; it's a good way to test your mathematical skills.*

The numbers 1 to 25 are arranged in a square array of five rows and five columns in an arbitrary way. The greatest number in each row is determined, and then the least number of these five is taken; call that number $s$. Next, the least number in each column is determined, and then the greatest number of these five is taken; this number is called $t$.

(a) Arrange the numbers 1 to 25 in a square array by writing $1, 2, 3, 4, 5$ in the first row, then $6, 7, 8, 9, 10$ in the second row, etc. Determine $s$ and $t$ for this arrangement. (You should have $s = t$.)

(b) Find a way to arrange the numbers 1 to 25 in a square array which leads to values $s, t$ with $s \neq t$.

(c) Write out carefully a proof of the following statement:

*For every possible arrangement in a square array of the numbers 1 to 25, if we obtain $s$ and $t$ as described above, then we have $s \geq t$.*

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## Exercises 3

- Relevant parts of the **Lecture notes**: Sections 3.1 – 3.8 and 3.10 – 3.12.
- Relevant parts of the text books: **Biggs**: Chapter 5;
  **Eccles**: Chapter 5.
- Always justify your answers.

**1** (a) Use the principle of induction to prove that, for all $n \in \mathbb{N}$, $n^3 + 5n$ is a multiple of 3.

(b) Use the principle of induction to prove that, for all $n \in \mathbb{N}$, $n^3 + 5n$ is a multiple of 6.

(*You may use the result that, for every $k \in \mathbb{N}$, $k^2 + k$ is an even number.*)

**2** Prove that $\sum_{k=1}^{n} (4k-3) \;=\; n(2n-1)$, for all $n \geq 1$.

**3** (a) Prove that $2m \leq 2^m$, for all natural numbers $m$.

Now, for $m$ a natural number, let $P(m)$ be the statement: $m(m-1) + 10 \leq 2^m$.

(b) Use (a) to show that, for any natural number $m$, if $P(m)$ is True, then $P(m+1)$ is True.

(c) For which natural numbers $m$ is $P(m)$ True? Justify your answer carefully.

**4** The natural numbers $u_n$ are defined recursively as follows.

$$u_0 \;=\; 2; \quad u_n \;=\; 2u_{n-1} + 1 \quad (n \geq 1).$$

(a) Write down the first few values of $u_n$.

(b) On the basis of these values, guess a formula for $u_n$ of the form $u_n = a2^n + b$, where you need to guess the value of the integers $a$ and $b$.

(c) Use induction to show that your formula is correct.

**5** Write down explicit formulae for the expressions $u_n$, $v_n$ and $w_n$ defined as follows:

$$\begin{aligned} u_1 &= 1; \quad u_n \;=\; u_{n-1} + 3 \quad (n \geq 2); \\ v_1 &= 1; \quad v_n \;=\; n^2 v_{n-1} \quad (n \geq 2); \\ w_0 &= 0; \quad w_n \;=\; w_{n-1} + 3n \quad (n \geq 1). \end{aligned}$$

(*Start by working out the first few values in each case – without doing the arithmetic, so you might write, for instance, $u_2 = 1 + 3$, $u_3 = (1+3) + 3, \ldots$ – and try to spot the pattern emerging. It is a good idea to check that the formula you give is correct for the first one or two values of $n$.*)

**6** (a) For each of the sets $X, Y, Z$ below, find the least element in that set.

$$\begin{aligned} X &= \{\, x \in \mathbb{N} \mid x^3 \leq 500 \,\}; \\ Y &= \{\, x \in \mathbb{N} \mid x = k^2 - 100 \text{ for some } k \in \mathbb{N} \,\}; \\ Z &= \{\, x \in \mathbb{N} \mid x^2 + 100 < 25x \,\}. \end{aligned}$$

(b) And, if it exists, find the greatest element in $X$, $Y$ and $Z$. (Make sure you justify your answers.)

*Note: make sure you read the definitions of the sets carefully. What exactly does it mean, for instance, to say that 11 is an element of $Y$?*

**7** *This is an extra question, and it is not directly related to the material discussed this week. Try to see how far you can get; it's a good way to test your mathematical skills.*

Eddie and Ayesha, a married couple, go to a party in which there are four other couples. Some people from the group know each other, others have never met.

People who meet for the first time are introduced and shake hands. No one shakes hands with her/his own partner, and no one shakes hands more than once with the same person.

At the end of the party, Eddie asks the nine others how many people they had shaken hands with. To his surprise, all answers are different.

(a) How many people shook hands with Eddie's wife, Ayesha?

(b) If you find part (a) hard, you can try the following easier variants first:

- (i) What would the answer be if there was only **one** other couple?
  (Use the same assumptions on whom can shake hands with whom. But now assume at the end Eddie asks the three others how many people they shook hands with, and he gets three different answers. And the question is to find out how many people shook hands with Ayesha.)
- (ii) And what is the answer if there are **two** other couples?
  (Again, use the same assumptions: Eddie asks five people and gets five different answers.)
- (iii) Finally, try to answer the original question.

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## Exercises 4

- The material for this week is not included in the Lecture notes. You will be give an additional set of notes.
- Relevant parts of the text books: **Eccles**: Section 5.4.
- Always justify your answers.

**1** (a) Consider female cats that reproduce by the following rule: A cat born at time $n$ has two daughters both born at time $n+1$, and otherwise no further female offspring.

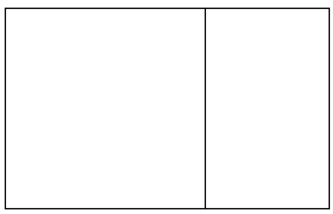
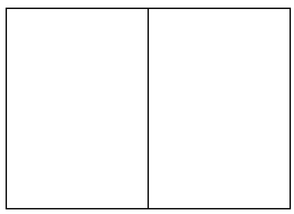
Starting with a single female cat born at time 1, what is the number of female cats born at time $n$, for any natural number $n$? Justify your answer.

(b) Suppose rabbit pairs reproduce by the following rule: A rabbit pair born at time $n$ gives birth to one further pair in every month starting from month $n+1$. Rabbits never die.

Starting with a single rabbit pair at time 1, what is the total number of rabbit pairs at time $n$, for any natural number $n$? Justify your answer.

**2** If a rectangle has sides of lengths $A$ and $B$ where $A \geq B$, then we call $A/B$ the *proportion* of the rectangle (so a square has proportion 1, and any non-square rectangle has proportion larger than 1).

The Golden Ratio $\phi$ is defined as the proportion of a rectangle with the property that, after removing a square, the remaining smaller rectangle has the same proportion $\phi$, as shown in the picture on the left:

Consider now the different proportion $\alpha$ of a rectangle that has the property that when dividing the longer side in half, the resulting two rectangles each have that same proportion $\alpha$, as shown in the above picture on the right.

(a) Determine $\alpha$.

The standard paper formats A0, A1, A2, A3, A4 etc. are defined by the rule that they have this proportion $\alpha$ and each is the previous format cut in half. So A4 is obtained by cutting A3 in half, for example. In addition, A0 is exactly one square metre in area.

(b) Determine the sizes (rounded to a millimetre) of A0, A1, A2, A3, A4 (the familiar A4 format allows you to check your result for plausibility).

(c) What is the weight of an A4 sheet of paper that weighs 80 grams per square metre?

**3** Analogous to the method for finding Binet's explicit formula for the Fibonacci numbers, show how to derive explicit formulas for the following recursively defined sequences (which, apart from (c), have been given, and proved to hold by induction, in an earlier lecture):

(a) $a_1 = 9,\ a_2 = 13,$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$;

(b) $b_1 = 7,\ b_2 = 23,$ and $b_n = 5b_{n-1} - 6b_{n-2}$ for $n \geq 3$;

(c) $c_1 = 1,\ c_2 = 2,$ and $c_n = 2c_{n-1} + 3c_{n-2}$ for $n \geq 3$.

**4** Recall that a *unit fraction* is of the form $1/a$ for a natural number $a$. An *Egyptian fraction* is a sum of distinct unit fractions.

(a) Show, using Egyptian fractions, how to divide 3 loaves of bread among 11 people so that everyone gets the same kinds of pieces.

The "greedy method" to present a fraction $p/q$, where $1 \leq p < q$, as an Egyptian fraction finds the smallest integer $a$ so that $p/q \geq 1/a$, which is $a = \lceil q/p \rceil$, and applies this recursively to the difference $p/q - 1/a$ until that difference is zero.

(b) Represent $7/15$ as an Egyptian fraction using the greedy method.

(c) Represent $3/7$ as an Egyptian fraction using the greedy method.

How does this help you to distribute 3 loaves of bread among 7 people? If this does not work, how would you modify the method? (Think of how to divide the bread.)

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## Exercises 5

- Relevant parts of the **Lecture notes**: Sections 4.1 – 4.5 and 4.8.
- Relevant parts of the text books: **Biggs**: Chapter 5 and Sections 6.1 – 6.3;
  **Eccles**: Sections 8.1 – 8.2, 8.4 – 8.5, 9.1 – 9.2 and 10.1.

- Always justify your answers.
- Before you start these exercises, make sure you have in front of you the *precise* definitions you need. In particular, you shoud know exactly what it means for a set to have $m$ elements.

**1** The functions $\alpha$ and $\beta$ from $\mathbb{N}$ to $\mathbb{N}$ are defined by

$$\alpha(n) = 2^n, \qquad \beta(n) = n^2 \qquad (n \in \mathbb{N}).$$

(a) Find formulae for the compositions $\alpha\beta$ and $\beta\alpha$.

(b) Determine whether $\alpha\beta$ and $\beta\alpha$ are different functions or not.

**2** Which of the following functions from $\mathbb{Z}$ to $\mathbb{Z}$ are injections, which are surjections, and which are bijections?

(a) $f(x) = x^3 \ (x \in \mathbb{Z})$;

(b) $g(x) = 3 - x \ (x \in \mathbb{Z})$;

(c) $h(x) = 2x + 3 \ (x \in \mathbb{Z})$;

(d) $j(x) = \begin{cases} x+1, & \text{if } x \text{ is even,} \\ x+3, & \text{if } x \text{ is odd,} \end{cases} \quad (x \in \mathbb{Z}).$

For each of the functions that are bijections, give the inverse function.

**3** The function $z$ from $\mathbb{N}$ to $\mathbb{N}$ is defined recursively by the rules

$$z(1) = 1 \qquad \text{and} \qquad z(n+1) = \begin{cases} \frac{1}{2}(z(n)+3), & \text{if } z(n) \text{ is odd,} \\ z(n)+5 & \text{if } z(n) \text{ is even,} \end{cases} \qquad \text{for all } n \geq 1.$$

Show that $z$ is neither an injection nor a surjection.

**4** Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$.

(a) How many functions are there from $X$ to $Y$?

(b) How many bijections are there from $X$ to $Y$?

(c) How many injections are there from $X$ to $Y$?

Explain your answers.

**5** In each of the following cases, find a value for $m$ such that a bijection $f : \mathbb{N}_m \longrightarrow X$ exists, and give a formula for such a bijection:

(a) $X = \{ 12, 15, 18, 21, 24, 27, 30 \}$;

(b) $X = \{ x \in \mathbb{Z} \mid -2 \leq x \leq 4 \}$;

(c) $X = \{ x \in \mathbb{Z} \mid x^2 \leq 16 \}$.

**6** Let $A$ be a finite set with $m$ elements, for some $m \in \mathbb{N}$. And suppose $x$ is an object that is not a member of $A$.

Prove, using the definitions, that $A \cup \{x\}$ has $m + 1$ elements.

(All you are told about $A$ is that it has $m$ elements. You need to show that there is a bijection from $\mathbb{N}_{m+1}$ to $A \cup \{x\}$.)

**7** Explain what is wrong with the following proof of the statement:

*If at least one person in the world has blue eyes, then everybody has blue eyes.*

**Proof**: Instead of the statement above, we prove the following:

**Theorem**: *For all $n \in \mathbb{N}$ we have: if in a group of $n$ persons there is at least one person with blue eyes, then everybody in that group has blue eyes.*

Once the theorem is proved, the original statement follows if we take for $n$ the number of people in the world.

**Proof of Theorem**: We prove the theorem using induction.

First, the result is true for $n = 1$. Since if we have a group of one person, and that person has blue eyes, then everybody in the group has blue eyes.

Now suppose the result is true for $n = k$, hence we suppose that: *if in a group of $k$ persons at least one person has blue eyes, then everybody in that group has blue eyes.*
We are going to prove that this gives the statement for $n = k + 1$: *if in a group of $k + 1$ persons there is at least one person with blue eyes, then everybody in that group has blue eyes.*
So take a group of $k + 1$ persons in which at least one person has blue eyes. Let's use the name A for one of the persons with blue eyes. Since $k + 1 \geq 2$, there is at least one person in the group who is not A. Call that person B. Now remove B from the group. What is left is a group of $k$ people in which at least one has blue eyes (since A is in that group). Hence everybody in that group has blue eyes (induction hypothesis).
Now put B back in, but remove another person. Again we have a group of $k$ people with at least one person with blue eyes. (In fact, all people except B are known to have blue eyes.) So also for this group we must conclude that everybody has blue eyes. In particular B must have blue eyes as well.
So we see that everybody in the original group of $k + 1$ persons has blue eyes.

By the principle of induction it follows that the result is true for all $n \in \mathbb{N}$. $\square$

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## Exercises 6

- Relevant parts of the **Lecture notes**: Sections 4.6 – 4.12. and 5.1 – 5.4.
- Relevant parts of the text books: **Biggs**: Section 6.3 – 6.4, 7.2 – 7.3 and 7.6;
  **Eccles**: Section 11.1 and Chapter 22.

  (The treatment of equivalence relations in **Eccles** is somewhat different of the way we do it, so read Chapter 22 with care.)
- Always justify your answers.

  (Hint: in the first three exercises, try using the Pigeonhole Principle.)

**1** Vladimir has 5 pairs of blue socks, 10 pairs of red socks, and 3 pairs of grey socks, all loose in a drawer. He picks a number of socks in the dark (i.e. he can't see the colours of the socks).

What is the minimum number of socks he has to pick to be guaranteed to have two of the same colour?

**2** Let $T$ be a set of 11 different natural numbers. Show that there are two elements $t_1, t_2 \in T$, $t_1 \neq t_2$, such that $t_2 - t_1$ is divisible by 10.

**3** In Question 7 of Exercises 3 we met Eddie and Ayesha. A couple of weeks after that question they attend another party. Again, some pairs of people shake hands, and some pairs don't. The total number of people at the party is $n$, for some $n \geq 2$.

Explain why, no matter what handshakes take place, there are certain to be two people at the party who shake the same number of hands.

**4** For each of the following relations on the set $\mathbb{N}$, decide whether the relation is reflexive, whether it is symmetric, whether it is transitive, and whether it is an equivalence relation. Justify your answers, giving short proofs or explicit counterexamples as appropriate.

(a) the relation $xRy$ given by "$x \geq y + 2$";

(b) the relation $xSy$ given by "$x \times y$ is even";

(c) the relation $xTy$ given by "$x$ is a multiple of $y$";

(d) the relation $xUy$ given by "$x = 1$ and $y = 1$".

**5** Show that the relation $V$ on $\mathbb{Z}$ defined by setting $xVy$ if $|x^2 - y^2| \leq 2$ is an equivalence relation.

Describe the equivalence classes of this relation.

**6** Explain what is wrong with the following proof of the statement:

> *If $S$ is a relation on a set $X$ that is both symmetric and transitive,*
> *then $S$ is an equivalence relation.*

**Proof**: Suppose $S$ is a symmetric and transitive relation on a set $X$, and let $a$ be any element of $X$. Now, if $aSb$, then $bSa$ (since $S$ is symmetric), and so $aSa$ (since $S$ is transitive). Therefore $S$ is reflexive, as well as being symmetric and transitive. So $S$ is an equivalence relation. □

*Hint: consider the relation $U$ in Question 4(d).*
*Metahint: why might we suggest you do that?*

**7** For a subset $S$ of $\mathbb{Z}$, we say that $m$ is a *lower bound of $S$* if for all $s \in S$ we have $m \leq S$. And $M$ is an *upper bound of $S$* if for all $s \in S$ we have $M \geq S$.

For each of the following (false) statements, give a counterexample (i.e. a set $S$ of integers with the first property, but not the second).

(a) If $S$ is a set of integers with an upper bound, then $S$ has a lower bound.

(b) If $S$ is a set of integers with an upper bound, then $\overline{S} = \{x \in \mathbb{Z} \mid x \notin S\}$ has a lower bound.

(c) If $S$ is a set of integers with no upper bound, then $\overline{S}$ has an upper bound.

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## Exercises 7

- Relevant parts of the **Lecture notes**: Chapter 6.
- Relevant parts of the text books: **Biggs**: Chapter 8;
  **Eccles**: Chapters 15 – 17 and Section 23.1.
- Always justify your answers.

**1** (a) Write down the definition of the statement "$a|b$" ($a$ divides $b$) for two integers $a, b$.

**Use this definition** to answer the following questions.

(b) For which integers $z$ is it true that $0|z$?

(c) For which integers $z$ is it true that $z|0$?

(d) For which integers $z$ is it true that $1|z$?

(e) For which integers $z$ is it true that $z|1$?

**2** Which of the following statements, for integers $a, b, c, d, p, q$, are true in general? Justify your answers, referring again to the definition of divisibility.

(a) If $d|a$ and $d|b$, then $d|(pa + qb)$.

(b) If $a|c$ and $b|c$, then $(ab)|c$.

(c) If $a|c$ and $b|d$, then $(a + b)|(c + d)$.

(d) If $a|c$ and $b|d$, then $(ab)|(cd)$.

**3** (a) Use Euclid's algorithm to find the greatest common divisor of 2009 and 820.

(b) Find integers $x, y$ so that $\gcd(2009, 820) = 2009x + 820y$.

**4** (a) Show that $\gcd(96, 42) = 6$, and find integers $x$ and $y$ satisfying $96x + 42y = 6$. Hence find integers $p$ and $q$ satisfying $96p + 42q = 60$.

(b) Let $a$ and $b$ be positive integers, and let $d = \gcd(a, b)$. Prove that, for $c \in \mathbb{Z}$, we have $d|c$ if and only if there are integers $x, y$ so that $c = xa + yb$.

**5** Let $a, b$ be integers, $b > 0$. In the lecture notes, it is proved that

**Theorem**: *if $a > 0$, then there exist $q, r \in \mathbb{Z}$ such that $a = qb + r$ with $0 \leq r < b$.*

Show how you can use this statement to deduce that

*if $a < 0$, then there exist $q, r \in \mathbb{Z}$ such that $a = qb + r$ with $0 \leq r < b$.*

**6** Let $n \geq 2$ be a number that is not a prime.

(a) Show that there is a divisor $q$ of $n$, with $q \geq 2$, so that $q^2 \leq n$.

(b) Show that there is a divisor $p$ of $n$ so that $p^2 \leq n$, $p \geq 2$ and $p$ is prime.

**7** Let $F_0, F_1, F_2, F_3, \ldots$ be the *Fibonacci numbers*, defined by

$$F_0 = 1; \qquad F_1 = 1; \qquad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2.$$

(a) Prove that for all $n \geq 0$ we have $\gcd(F_{n+1}, F_n) = 1$.

(b) Prove that for all $n \geq 0$ we also have $\gcd(F_{n+2}, F_n) = 1$.

(Hint: check what Euclid's algorithm would do if you started to compute $\gcd(F_{n+1}, F_n)$ or $\gcd(F_{n+2}, F_n)$.)

(c) Is it true that, for all $n \geq 0$, we have $\gcd(F_{n+3}, F_n) = 1$?

(d) Show that, for $n \geq 3$, $F_{n+3} = 4F_n + F_{n-3}$. What does this tell us about $\gcd(F_{n+3}, F_n)$?

## Maple questions

Since all students following this course are also supposed to follow (or have followed) MA100 *Mathematical Methods*, everybody should have some experience with using the computer package "Maple". This program is available on all PCs in the school.

Future exercises sheets will contain some questions in which you are asked to use Maple to explore aspects of the theory. In your answers, you should say what commands you used and what the output is. You may attach a printout of your work as well if you wish.

**8** For all $k \in \mathbb{N}$, let $p_k$ be the $k$-th prime, so $p_1 = 2$, $p_2 = 3$, etc.

(a) The maple command to find the $k$-th prime is "`ithprime(k)`".

Use this command to find $p_{15}$, $p_{150}$ and $p_{1,500}$.

(b) The maple command to check if a number $n$ is prime is "`isprime(n)`".

Use this command to find all prime numbers between 1,000,000 and 1,000,010.

We prove in lectures that the set of primes is infinite, using a proof by contradiction. This proof looks at the numbers

$$N_m = p_1 \cdot p_2 \cdot \cdots \cdot p_m + 1, \qquad \text{for } m \geq 1.$$

So $N_1 = p_1 + 1 = 2 + 1 = 3$, $N_2 = p_1 p_2 + 1 = 2 \cdot 3 + 1 = 7$, etc.

We prove that $N_m$ is not divisible by any of $p_1, p_2, \ldots, p_m$, so that $N_m$ is either a prime or it is divisible by a prime larger than $p_m$.

(c) Use Maple to find out which of these numbers $N_m$, for $m = 1, 2, \ldots, 15$, is actually prime.

The Maple command to find the prime factors of $n$ is "`numtheory[factorset](n)`". (It is easier to first load the "number theory" package by typing "`with(numtheory)`", and then use the command "`factorset(n)`".)

(d) Use Maple to compare $p_m$ with the smallest prime number that divides $N_m$, for $m = 1, 2, \ldots, 15$.

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## Exercises 8

- Relevant parts of the **Lecture notes**: Chapter 7.
- Relevant parts of the text books: **Biggs**: Sections 13.1 – 13.3;
  **Eccles**: Chapters 19 – 21.
- Always justify your answers.

**1** (a) Find the representations of $(2008)_{10}$ in base 2, base 3 and base 11.

(b) What is the normal decimal notation for $(100011)_2$ and $(100011)_3$?

(c) For any integer $m \geq 2$, what is 1 in base $m$? And what is $m$ in base $m$?

**2** Let $m$ be an integer, $m \geq 2$.

(a) Prove that, for all elements $r, s, t \in \mathbb{Z}_m$, we have $(r \oplus s) \oplus t = r \oplus (s \oplus t)$.
(Remember that an element of $\mathbb{Z}_m$ is an equivalence class $[x]_m$ for some $x \in \mathbb{Z}$.)

(b) Prove that for all elements $r, s \in \mathbb{Z}_m$ we have $r \otimes s = s \otimes r$.

(c) Show that in $\mathbb{Z}_m$ we have $(m-1)^{-1} = m - 1$.

**3** Solve the following systems of equations; i.e., in each case, find **all** pairs $(x, y)$ satisfying both equations. Make sure to justify each step carefully.

(a) $\begin{cases} 2x + \phantom{2}y = 1, \\ \phantom{2}x + 2y = 3, \end{cases}$ in $\mathbb{Z}_7$; (b) $\begin{cases} 2x + \phantom{2}y = 1, \\ \phantom{2}x + 2y = 3, \end{cases}$ in $\mathbb{Z}_6$;

(c) $\begin{cases} 2x + \phantom{2}y = 2, \\ \phantom{2}x + 2y = 4, \end{cases}$ in $\mathbb{Z}_6$.

**4** Find the inverses (if they exist) of

(a) 3 in $\mathbb{Z}_{11}$; (b) 11 in $\mathbb{Z}_{16}$; (c) 14 in $\mathbb{Z}_{16}$. (d) 100 in $\mathbb{Z}_{2007}$.

**5** For part (a) and (b) of this question, we work in $\mathbb{Z}_p$, where $p$ is a prime. (*So what result from lectures is likely to be useful?*)

(a) Suppose $a, b \in \mathbb{Z}_p$ have the property that $ab = 0$ in $\mathbb{Z}_p$. Show that $a = 0$ or $b = 0$ in $\mathbb{Z}_p$.

(b) Using the result in (a), show that the equation $x^2 = 1$ has only the solutions $x = 1$ and $x = -1$ in $\mathbb{Z}_p$ (which is the same as $x = 1$ and $x = p - 1$ in $\mathbb{Z}_p$).

(c) How many solutions are there for the equation $x^2 = 1$ in $\mathbb{Z}_2$?

(d) Find an integer $m \geq 2$ so that the equation $x^2 = 1$ in $\mathbb{Z}_m$ has more than two solutions.

**6** We want to solve the quadratic equation $y^2 + y - 11 = 0$ in $\mathbb{Z}_{41}$.

(a) Solve the equation $2a = 1$ in $\mathbb{Z}_{41}$.

(b) Find elements $a, b \in \mathbb{Z}_{41}$ such that $y^2 + y - 11 = (y + a)^2 - b$ in $\mathbb{Z}_{41}$.

(c) Use (b) and Question 5(b) to write down the solution(s) of $y^2 + y - 11 = 0$ in $\mathbb{Z}_{41}$.

## Maple questions

**7** (a) Find all primes $p$ with $41 \leq p \leq 107$. (Remember that the Maple command to find the $i$-th prime number is "`ithprime(i)`".)

The Maple command to solve equations in modular arithmetic $\mathbb{Z}_m$ is "`msolve`". Use the Maple help (or type "`?msolve`") to find out how to use this command for different values of $m$.

(b) Solve the equation $y^2 + y - 11 = 0$ in $\mathbb{Z}_p$ for all primes $p$ with $41 \leq p \leq 107$.

You will notice that there are no solutions (or at least, Maple doesn't give solutions) for certain values of $p$.

(c) Use your results (and if necessary check for more prime values) to make a conjecture describing the primes $p$ for which there are solutions to $y^2 + y - 11 = 0$ in $\mathbb{Z}_p$.

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## Exercises 9

- Relevant parts of the **Lecture notes**: Section 8.1 – 8.4.
- Relevant parts of the text books: **Biggs**: Sections 6.6, 9.1 – 9.5 and 9.7;
  **Eccles**: Chapters 13 – 14.

  (The treatment of this material in **Eccles** is somewhat different of the way we do it, so read those chapters with care.)
- Always justify your answers.

**1** In the lectures, we gave a construction for the rational numbers. This started by looking at the set $S$ of all pairs of the form $(a, b)$, with $a, b \in \mathbb{Z}$ and $b \neq 0$, and then considering the relation $R$ on $R$ defined by:

$$(a, b) R (c, d) \qquad \text{if and only if} \qquad ad = bc.$$

(a) Explain why things would go badly wrong if we allow $S$ to include pairs $(a, b)$ with $b = 0$.

(The answer is **not** that if $b = 0$ the division $a/b$ doesn't make sense. We define "division" only **after** we've established certain properties of the relation $R$, and defined $\mathbb{Q}$ and its arithmetic operations.)

We define the set $\mathbb{Q}$ to be the set of equivalence classes of the relation $R$, and defined an "addition" operation on $\mathbb{Q}$ by setting $[a, b] + [e, f] = [af + be, bf]$, for all $(a, b), (e, f) \in S$. (We're using the shorthand $[a, b]$ here for $[(a, b)]$.)

(b) Suppose that $(a, b)$, $(c, d)$ and $(e, f)$ are in $S$, and that $(a, b) R (c, d)$. Show that this means that $(af + be, bf) R (cf + de, df)$.

Use this to show that if $(r, s) R (t, u)$ and $(v, w) R (x, y)$, then

$$[r, s] + [v, w] = [t, u] + [x, y].$$

(This means that the addition operation defined on $\mathbb{Q}$ is *well-defined*.)

**2** (a) Write down 5 rational numbers lying strictly between $4/3$ and $41/30$.

(b) Explain how you would extend the method you used in (a) to find 5,000 rational numbers lying strictly between $4/3$ and $41/30$.

**3** (a) Express the following rational numbers in decimal representation:

(i) $\dfrac{14}{13}$; (ii) $\dfrac{7}{-25}$.

(b) Express the recurring decimal $-0.8\overline{765}$ as a rational number $p/q$, where $p$ and $q$ are integers.

**4** In the lectures and text books (Example 1 of Section 9.4 in Biggs; Theorem 13.3.2 in Eccles), it is proved that $\sqrt{2}$ is an irrational number. We also observe that $1 < \sqrt{2} < 2$.

(a) Prove that for any rationals $x, y$, with $y \neq 0$, also $x + y\sqrt{2}$ is an irrational number.

(b) Explain how you can use these facts about $\sqrt{2}$ to find an irrational number lying strictly between any two rational numbers $a/b$ and $c/d$, where $a/b < c/d$.

**5** If $z$ is a real number, then $z^n$ is defined recursively for non-negative integers $n$: $z^0 = 1$, and, for $n \geq 0$, $z^{n+1} = z \times z^n$.

If $z$ is a positive real number, and $b$ a natural number, then we define the positive $b^{th}$ root $y = z^{1/b}$ to be the positive real number $y$ such that $y^b = z$. You may assume that a positive real number always has a positive $b^{th}$ root.

(a) Given a positive real number $z$, is it possible for there to be two different positive $b^{th}$ roots of $z$? Justify your answer briefly.

(b) Suppose that $a, b, c, d$ are natural numbers with $ad = bc$, and $x$ is a positive real number. Show, from the definitions above, that $(x^a)^{1/b} = (x^c)^{1/d}$.

Explain why this result allows you to define $x^q$ when $x$ is a positive real number and $q$ is a positive rational number.

(c) Show that $2^{3/4}$ is irrational.

## Maple questions

**6** The numbers $x_n$ for $n \in \mathbb{N}$ are defined by

$$\begin{aligned} x_1 &= 1; \\ x_n &= \frac{1}{2}\left(x_{n-1} + \frac{3}{x_{n-1}}\right), \qquad \text{for } n \geq 2. \end{aligned}$$

(a) Prove that $x_n$ is a rational number for all $n \geq 1$.

(b) Use Maple to calculate $x_1, x_2, \ldots, x_8$ as rational numbers.

The Maple command to find approximate decimal representations of numbers is "`evalf(.)`". By default it gives 10 digits of the results. Check the Maple help of "`evalf`" to find out how to obtain results with a different (prescribed) number of digits.

(c) Approximate the decimal representation of $x_1, x_2, \ldots, x_8$ with 20 digits precision. Calculate $\sqrt{3}$ with 20 digits precision as well.

(d) For what $x_n$ does it appear that $x_n$ gives an approximation of $\sqrt{3}$ with 20 or more digits precision?

(e) By further increasing the number of digits, try to find out in how many digits $\sqrt{3}$ and $x_8$ coincide.

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## Exercises 10

- Relevant parts of the **Lecture notes**: Section 8.5.
- The two text books **Biggs** and **Eccles** don't do complex numbers.

  You can find some further reading on complex numbers in one of the MA100 text books: M. Anthony and M. Harvey, *Linear Algebra: Concepts and Methods*, Chapter 13.
- Always justify your answers.

**1** Write the following sum, product and quotient of complex numbers as complex numbers:

(a) $(2+3i)+(-5+4i)$; (b) $(2+3i)(-5+4i)$; (c) $\dfrac{2+3i}{-5+4i}$.

**2** Find the square roots of the following numbers. (That is, for each of the numbers, find the complex numbers $z$ such that $z^2$ equals the number.)

(a) $-9$; (b) $i$; (c) $4-3i$.

**3** Solve the quadratic equation

$$x^2 - 2ix - 5 + 3i \;=\; 0.$$

**4** DeMoivre's formula tells us that

$$(\cos\theta + i\sin\theta)^3 \;=\; \cos(3\theta) + i\sin(3\theta).$$

By expanding the left hand-side of this equation, and by using $\sin^2\theta + \cos^2\theta = 1$, prove the following trigonometrical identities:

$$\cos(3\theta) \;=\; 4\cos^3\theta - 3\cos\theta \qquad \text{and} \qquad \sin(3\theta) \;=\; 3\sin\theta - 4\sin^3\theta.$$

**5** Write the following complex numbers in the standard form $x+yi$:

(a) $\left(\cos\left(\dfrac{3\pi}{14}\right) + i\sin\left(\dfrac{3\pi}{14}\right)\right)^7$; (b) $(\sqrt{3}+i)^{57}$.

**6** Find the modulus and principal argument of the following complex numbers:

(a) $1+i$, (b) $-1-\sqrt{3}i$.

**7** (a) Consider the equation $z^5 = a$, where $a$ is a positive real number.

What can you say about the modulus of any solution $z$? What can you say about the principal argument?

Draw all the five solutions of $z^5 - 32 = 0$ on the Argand diagram.

(b) Write down a polynomial $P(z)$ such that

$$(z-2)P(z) = z^5 - 32.$$

What can you say about the solutions of $P(z) = 0$?

**8** Consider the polynomial $P(z) = z^4 + z^2 - 2z + 6$.

(a) Show that $z^* = 1 + i$ is a root of $P(z)$, i.e. a solution of $P(z) = 0$.

(b) Hence find all the roots of $P(z)$.