MA100 Mathematical Methods
Background You Should Know and Exercises

This is background material for MA100 and is intended to be a review of your A-level Mathematics course. Please work through it before term or in your spare time. (You do not need and should not use a calculator.) If you have difficulties with anything, don't panic – but do make a note of any background you are missing and work on it, or see one of the lecturers.

It is also recommended that you look through the review booklets, <u>*An Algebra Refresher*</u> and <u>*A Calculus Refresher*</u>. These can be found at: `www.maths.lse.ac.uk/Refreshers/`. References are made to these booklets (*AlgR* and *CalcR*) for corresponding material in the exercises that follow.

Solutions to the exercises can be found on the MA100 Moodle page at the start of term.

1. The set of real numbers, denoted $\mathbb{R}$, includes the following subsets:
$\mathbb{N}$, the set of natural numbers: $1, 2, 3, 4, \ldots$
$\mathbb{Z}$, the set of integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbb{Q}$, the set of rational numbers: p/q with $p, q \in \mathbb{Z}$, $q \neq 0$; such as, $\frac{2}{5}$, $-\frac{9}{2}$, $\frac{4}{1} = 4$.
the set of irrational numbers: real numbers which are not rational; for example, $\sqrt{2}$, π.

The absolute value (modulus) of a real number: $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$.

The absolute value satisfies $|a| = \sqrt{a^2}$, and the Triangle Inequality: $|a+b| \leq |a| + |b|$.

Intervals of real numbers, such as: $(a, b) = \{x \mid a < x < b\}$, $[a, b] = \{x \mid a \leq x \leq b\}$,
$(-\infty, b) = \{x \mid x < b\}$ $\quad [a, \infty) = \{x \mid a \leq x\}$.

2. Polynomials of degree n: $P_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$,
where the a_i are real constants, $a_n \neq 0$, and x is a real variable.

How to graph: straight lines, $P_1(x) = a_0 + a_1x$
quadratics, $P_2(x) = a_0 + a_1x + a_2x^2$
cubics, $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

The laws of indices: $x^r x^s = x^{r+s}$ $\quad (x^r)^s = x^{rs}$

How to expand expressions like $(1+x)^n$, $(x+y)^n$

How to factorize quadratics (such as $x^2 - 5x + 6$ and $3x^2 + 14x - 5$)
and simple cubics, (such as $x^3 - 1$ and $x^3 - 2x^2 - 5x + 6$).

The Quadratic Formula.

How to sum: arithmetic series, $a + (a+d) + ... + (a + (n-1)d)$
geometric series, $a + ar + ar^2 + ... + ar^{n-1}$

Exercises. *(AlgR, sections 2–3, 8–10)*

(2.1) Sketch $f(x) = 4 - x^2$ and $g(x) = 2x + 1$ and find their points of intersection.

(2.2) Expand $(2 + 5x)^4$ and $\left(x^2 - 4/x\right)^3$.

(2.3) Find an expression for the profit function $\Pi(q)$ as a function of q given that

$$\Pi(q) = pq - C(q), \qquad \text{where} \quad 2p + 0.4q = 155 \quad \text{and} \quad C(q) = q^2 - 10q + 300 \ .$$

(2.4) Find the complete solution of each of the following systems of equations, both algebraically (by solving the equations simultaneously) and graphically (by illustrating them as lines on a graph).

$$\text{(a)} \begin{cases} x + 2y = 4 \\ 2x - y = 3 \end{cases} \qquad \text{(b)} \begin{cases} x + 2y = 4 \\ 2x + 4y = 4 \end{cases} \qquad \text{(c)} \begin{cases} x + 2y = 4 \\ 2x + 4y = 8 \end{cases}$$

NOTE. (x, y) is a solution of a system of equations if it satisfies all equations simultaneously.

(2.5) Factorise $P(x) = x^3 - 7x + 6$ and sketch the graph of $P(x)$.

3. Be able to manipulate algebraic expressions accurately and efficiently.
Be able to manipulate indices.

Exercises. *(AlgR, sections 1–7)*

(3.1) Simplify $\quad (a)\ 6ab - \dfrac{a}{b}(b^2 - 4bc) \qquad (b)\ \dfrac{49x^{-2}}{35y} - \dfrac{4xy^2}{(2xy)^3}$

(3.2) Solve for s: $\quad \dfrac{5}{3s+1} - \dfrac{2}{s+1} = 0$.

(3.3) Rewrite the simultaneous equations

$$x = \frac{a - c - by}{2b} \qquad\qquad y = \frac{a - c - f - bx}{2b}$$

as a system of linear equations in x and y. Solve the system for x and y (in terms of the constants a, b, c, f), and then show that

$$x + y = \frac{1}{3b}(2a - 2c - f) \ .$$

4. Properties of the exponential (e^x) and logarithmic ($\ln x = \log_e x$) functions and their graphs:

$$\ln e^x = x \qquad (e^r)^s = e^{rs} \qquad \ln x^r = r \ln x$$

$$e^{\ln y} = y \qquad e^{r+s} = e^r e^s \qquad \ln(xy) = \ln x + \ln y$$

Exercises. (*AlgR, sections 2–3, 8–10*)

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Exercises.

(4.1) Sketch the graphs of e^x, e^{-x}, $\ln x$.

(4.2) What is the value of e^0, $\ln 0$, $\ln 1$?

(4.3) Show that: $\ln\left(\frac{a}{b}\right) = \ln a - \ln b \quad (a > 0,\ b > 0)$.

(4.4) Simplify: $\ln x^5 - 2\ln x + \ln y^3 \quad (x > 0,\ y > 0)$.
For which values of x and y is this expression positive?

5. Properties of the trigonometric functions (sin, tan, *etc.*) using radian measure of an angle:
Be able to sketch their graphs.
Know their values at multiples of π, $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{6}$.
(Know the triangles associated with these last values.)

Know the basic trigonometric identities, *e.g.*

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,; \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b\,;$$

$$\sin x = -\sin(-x) \quad \text{is an odd function;} \qquad \cos x = \cos(-x) \quad \text{is an even function.}$$

Exercises.

(5.1) Sketch the graphs of $\sin x$, $\cos x$, $2\cos 3x$, $\tan x$.

(5.2) Deduce the formulas for $\sin 2x$, $\cos 2x$ from the identities above, and deduce that

$$\sec^2 x = \tan^2 x + 1\,; \qquad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}\,;$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]\ .$$

(5.3) Evaluate (**without using a calculator):** $\sec\frac{\pi}{3}$, $\cot(-\frac{\pi}{6})$, $\sin(\frac{5\pi}{4})$, $\cos(\frac{11\pi}{4})$.

6. Rules of differentiation: Product, quotient and chain rules
Be able to differentiate polynomials.
Know the derivatives of the exponential, logarithmic and trigonometric functions.

Exercises. (*CalcR, sections 1–9*)

(6.1) Calculate

(a) $\frac{d}{dx}\left(e^{-3x}\right)$ (b) $\frac{d}{dx}\ln(5-2x)$ (c) $\frac{d}{dx}\left(e^{-3x}\ln(5-2x)\right)$ (d) $\frac{d}{dx}\ln\left(\frac{1}{x^2}\right)$

(e) $\frac{d}{dx}\left(\frac{xe^x}{2x^2+1}\right)$ (f) $\frac{d}{dx}\sqrt{3x^2-1}$ (g) $\frac{d}{dx}\tan 3x$ (h) $\frac{d}{dx}\sin(x^2-5)$

7. Know that the equation of the tangent line to the curve $y = f(x)$ at the point where $x = a$ is given by:

$$y - f(a) = f'(x)(x - a)$$

Exercise.

(7.1) Find the equation of the tangent line to the curve $y = 2x^3 - 9x^2 - 38x + 21$ at the point where $x = 1$.

8. Find the stationary (turning) points of graphs of functions and determine which point is a maximum and which is a minimum.

Exercises.

(8.1) Do this for the function $y = 2x^3 - 9x^2 - 38x + 21$.

(8.2) Sketch the graph of $f(x) = x^3 - 3x^2$.
Find the maximum and the minimum value of f on the interval $[0, 3]$, and on the interval $[0, 4]$.

9. Techniques of integration: partial fractions, integration by parts, substitution.

Exercise. (*CalcR, sections 10-17; AlgR, section 11*)

(9.1) Calculate

$$\text{(a)} \int \sin 3x \, dx \qquad \text{(b)} \int_0^{\frac{\pi}{4}} \sec^2 x \, dx \qquad \text{(c)} \int \frac{1}{x^2 - 5x + 6} \, dx$$

$$\text{(d)} \int \frac{x}{x-1} \, dx \qquad \text{(e)} \int x \cos 2x \, dx \qquad \text{(f)} \int x^2 e^x \, dx$$

$$\text{(g)} \int x(x^2 - 2)^4 \, dx \qquad \text{(h)} \int (2x + 2) e^{x^2 + 2x + 3} \, dx$$

10. The technique of *Completing the Square* in a quadratic expression.

Exercise. (*AlgR, section 9*)

(10.1) Complete the squares in the expressions: $x^2 + 6x + 11$ and $2x^2 - 4x + 7$.

(10.2) Complete the square in $ax^2 + bx + c$.
Use this to derive the quadratic formula to solve $ax^2 + bx + c = 0$ for x.