CHORDS IN LONGEST CYCLES

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(*based on joint work with Alexey Pokrovskiy*)

Let G be a graph on n vertices, and let C be a cycle in G. A *chord* in C is an edge in G between two vertices of C which does not already form part of the cycle. The following 1976 conjecture of Thomassen has garnered a lot of attention over the years:

Conjecture 1. *Let G be 3-connected. Every cycle of maximum order in G has a chord.*

Thomassen himself proved that the conjecture holds in the case that G is cubic (see [1]), and several other authors have looked at variations of the problem in various other classes of graphs; and in general these graphs are usually quite sparse. Indeed, for very dense graphs—even without any connectivity assumptions—the result is obvious, e.g., if $\delta(G) \geqslant \frac{n}{2}$ then by Dirac's theorem G is Hamiltonian and thus (provided $n \geqslant 5$), any cycle of maximum order has a chord. On the other

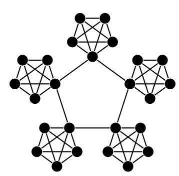

FIGURE 1. Illustration of Harvey's construction with $t = 5$.

hand, the following construction of Harvey shows that graphs having $\delta(G) < \sqrt{n}$ can avoid chords: take t copies of K_t, and join them around an t-cycle, as illustrated in figure 1. This graph has $\delta(G) = \sqrt{n}-1$, yet the central cycle is both of maximum length and chordless. In [2], he conjectures that this bound is tight:

Conjecture 2. *Let G be a graph on n vertices with $\delta(G) > \sqrt{n} - 1$. Every cycle of maximum order in G has a chord.*

In the same paper, he shows that this is true with the slightly stronger assumption that $\delta(G) \geqslant \frac{3+\sqrt{17}}{2\sqrt{2}}\sqrt{n} \approx 2.52\sqrt{n}$. We prove the following asymptotic form of the conjecture:

Theorem 3. *Fix $\epsilon > 0$, and let G be an graph on n vertices satisfying $\delta(G) \geqslant (1+\epsilon)\sqrt{n}$. Then if n is large enough, every cycle of maximum order in G has a chord.*

References

[1] C. Thomassen, *Chords in longest cycles*, Journal of Combinatorial Theory, Series B, 71(2):211–214, 1997.

[2] D. J. Harvey, *A cycle of maximum order in a graph of high minimum degree has a chord*, The Electronic Journal of Combinatorics 24(4), 2017.