Diagonal Ramsey via effective quasirandomness

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$$R(k+1, \ell+1) \le k^{-c_\varepsilon \log k/\log\log k}\binom{k+\ell}{k},$$

if $\ell/k \in [\varepsilon, 1]$, $\ell > C_\varepsilon$.

New Ramsey bounds

Theorem (S. '20+)

For each $\varepsilon \in (0, 1/2)$ *there is* $c_\varepsilon > 0$ *such that*

$$R(k+1, \ell+1) \le e^{-c_\varepsilon (\log k)^2} \binom{k+\ell}{k}$$

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Corollary (S. '20+)

There is an absolute constant $c > 0$ *such that for* $k \ge 3$,

$$R(k+1, k+1) \le e^{-c(\log k)^2} \binom{2k}{k}.$$

Discussion

- Current Ramsey bounds:

$$(1+o(1))\frac{k}{e}2^{\frac{k+1}{2}} \le R(k,k) \le e^{-c(\log k)^2}4^k.$$

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- Our improvement in the exponent originates in **optimal** "effective quasirandomness" results (deriving from global structure of signed graph densities).
- The optimality demonstrates this is a natural barrier.

Thomason framework

- Erdős–Szekeres inductive proof:

$$R(k+1,\ell+1) \le R(k,\ell+1) + R(k+1,\ell)$$
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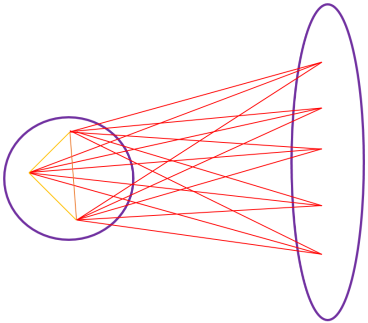

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- Thus we control $\#K_3 + \#\overline{K}_3$ relatively well (Goodman's formula):

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- This gives an upper bound on the left side.

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- This is **optimal**, replacing Ramsey graphs by graphs with degrees within $pn \pm \mu n$ and with codegrees bounded by $p^2 n + \nu n$, where $\mu, \nu \approx r/k$.

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where $\mathbf{x} = (x_v)_{v\in V(H)}$ and $d\mathbf{x}$ is the product measure on $\Omega^{V(H)}$.

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$$W_{x_1,\ldots,x_r} = \mathbb{E}_y \prod_{i=1}^{r} W(x_i, y).$$

Graphons (visual)

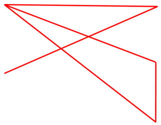

Figure 1: $t_H(W)$

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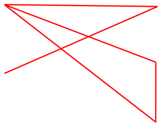

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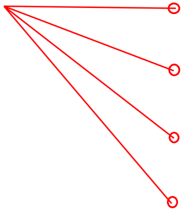

Figure 2: W_{x_1,x_2,x_3,x_4}

Measuring quasirandomness

Definition

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- The key point is that

$$t_{K_{2,a}}(f_{p,G}) = \mathbb{E}_{x,y}(f_{p,G})_{x,y}^a = O(\nu_{p,G}^a + n^{-1})$$

almost immediately follows (sign issue when $2|a$).

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  where $\sum^*$ is taken over isomorphism classes of graphs without isolated vertices and $c_{J,H}$ is the number of subgraphs of H isomorphic to J.
- The sum of $c_{J,H}$ over J with s vertices is at most $\binom{r}{s} 2^{\binom{s}{2}}$.
- So, we really want bounds on $t_J(f)$ of the form $2^{-\Omega(s^2)}$ or so for $s = v(J) \le r$, given $\mu_{p,G}, \nu_{p,G} \approx r/k$.

A local bound

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- $\nu_{p,G}^{s/4} \approx k^{-s/4} = \exp(-\Omega(s^2))$ as long as $s = O(\log k)$.

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- Check: $\mu_{1/2,G}, \nu_{1/2,G} = O(1/k)$ with high probability (if n is large).
- Check: $2^{-\binom{r}{2}} t_{K_r}(W_G) = 2^{\Omega(r^2)}$ (n large) if $r = \Omega(\log k)$. (Since *global bound* essentially optimal for this G.)

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- dense regularity:sparse regularity::effective quasirandomness:**?**