Firm Cyclicality and Financial Frictions*

Alex Clymo† Filip Rozsypal‡

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Abstract

Using administrative micro data we document how firms' sensitivities to business cycles differ by size and age. Among the youngest firms, small firms are more cyclical than large, but the reverse is true among older firms. The differences in cyclicality are large: "young and small firms" are nearly twice as cyclical as large firms, who respond one-and-half to one to the aggregate business cycle. In contrast, "old and small" firms are almost acyclical on average. High leverage firms are more cyclical than low leverage firms which—when combined with the age-profiles and cyclicalities of financial variables—suggests that financial frictions are likely to explain the excess cyclicality of "young and small" firms, but not of large firms. Augmenting a dynamic heterogeneous-firm model with heterogeneous returns-to-scale and entrant wealth allows it to replicate these findings, and implies that financial policies targeted at young firms become less effective in stimulating aggregate output while the opposite is true for direct labor subsidies.

Keywords: firm age, firm size, cyclicality, financial frictions

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†Department of Economics, University of Essex. Wivenhoe Park, Colchester CO4 3SQ, United Kingdom. Email: a.clymo@essex.ac.uk

‡Research Unit, Danmarks Nationalbank, Langelinie Allé 47, 2100 København Ø, Denmark and Center for Macroeconomics (CFM). Email: firo@nationalbanken.dk

1 Introduction

Why does aggregate GDP fall in recessions? Behind aggregate GDP are the many thousands of firms which make up a modern economy, and so to understand the answer to this question, we must understand which firms are suffering most during recessions. That is, which firms are the most *cyclical*. The financial accelerator literature (Bernanke and Gertler, 1989; Gertler and Gilchrist, 1994; Bernanke et al., 1999) emphasizes financial frictions as important drivers of recessions, with the implication that younger or smaller firms should be the most cyclical, since they are the most financially vulnerable. More recently, Crouzet and Mehrotra (2020) argued that while smaller firms are indeed more cyclical, this is not due to financial frictions. In this paper propose to reconcile these two views.

This paper's contributions are to empirically document the cyclicality of firms not just by age *or* size, but also the joint relationship between age *and* size, and to build a quantitative model that matches the observed facts. Using firm-level administrative and balance sheet data from the universe of Danish firms, we make two empirical contributions. First, measured using sales, assets, or employment, we document that young firms are more cyclical than old firms, but that the relationship between size and cyclicality is non-monotone: among old firms, large firms are more cyclical than small firms, while the opposite is true among young firms. Second, we use firm-level financial data to provide suggestive evidence of a role for financial frictions, which we argue are more severe for younger firms. We then build a quantitative heterogeneous firm model and show that matching these new stylized facts on cyclicality has implications for the potency of different cyclical policies.

These new stylized facts create a challenge for financial frictions models. A basic collateral constraint introduces a direct link between size and age: young firms typically enter smaller, and lack accumulated net worth, making them financially constrained. Therefore, the cyclicality of young firms and small firms is linked through financial constraints, and making cyclicality negatively correlated with size. However, this is at adds with our evidence that, among older firms, cyclicality increases with size. We propose extensions to this class of model by introducing heterogeneous returns to scale and starting net worth, which break the mechanical link between cyclicality by age and size. This allows financial frictions to primarily affect young firms, for whom cyclicality is indeed decreasing with size, while coexisting with the general tendency of larger firms to be more cyclical.

A long literature dating back to Gertler and Gilchrist (1994) emphasizes that firm size may act as a proxy for financial frictions, and thus argues that small firms being more cyclical than large firms could be interpreted as evidence in favor of financial frictions. More recently, it has been shown that firm age is a more important predictor of both the average level and cyclicality of firm growth than firm size (see Fort et al. (2013); Haltiwanger et al. (2013), for evidence from the US). In the US data, small firms are only more cyclical to the extent that they tend to be young, while older small firms display no excess cyclicality. Thus, the relationship between age, size, and cyclicality, including any underlying role of financial frictions, is complicated, and all elements must be studied at once in order to create a full picture. It is this challenge that we tackle in this paper.

Our administrative data allows us to study the behavior of younger and smaller firms, who are typically unlisted and hence not available in databases like Compustat. Additionally, our data includes measures of employment, turnover, firm age as well as a broad range of financial data from firms' balance sheets. Accordingly, we can directly investigate the role of financial frictions across the whole firm age/size distribution using balance sheet data, which are typically hard to find for young, unlisted firms.

Our first set of empirical results show that the cyclicality of firms depends not just on their age or size but, crucially, on the joint relationship between age and size. In particular, we measure cyclicality by regressing firm-level growth rates on aggregate GDP growth, with interactions for joint age-size bin. This extends the methodology of Crouzet and Mehrotra (2020) to additionally study firm age, and we take sales, assets, and employment as our measures of firm-level performance. We find that young firms are more cyclical than old firms, even conditioning on firm size. The relationship between firm size and cyclicality is more complicated, and flips sign between young and old firms. As a consequence, we find that "young and small" firms are the most cyclical in the economy, followed by large firms, while "old and small" firms are the least cyclical. This finding highlights the importance of studying firm age and size together: over the business cycle, the relationship between age and size is confounded by forces which move firms of different sizes differently depending on their age.

Our second set of empirical results investigate the role of finance in driving cyclicality by firm age and size. We look at measures of total firm debt, as well as their leverage, measured as the debt to asset ratio and find suggestive evidence for financial frictions, especially for younger firms. Starting with the firm lifecycle, we find that leverage is higher at young firms, even conditioning for firm size. This is consistent with models of financial accelerator where younger, smaller firms are more financially constrained. Moreover, young firms also have the highest growth rates of debt and leverage, suggesting that they are actively trying to increase their debt, and hence more likely to be affected if the access to debt is restricted in a recession. Older firms, on the other hand, all have shrinking leverage ratios, showing that they are reducing their reliance on debt (presumably either by accumulating retaining earnings or switching to equity financing). Finally, we find that, after controlling for age, firms of all sizes have very similar leverage ratios, as well as similar growth rates of their leverage. This suggests that financial constraints are unlikely to bind more at larger firms than small firms. We then move on to directly studying the relationship between leverage and cyclicality, and find that high leverage firms are more cyclical than low leverage firms. Overall the results paint a suggestive picture of the role of finance in driving cyclicality over the business cycle. We observe that young firms are more cyclical than old, which is plausibly linked to financial frictions since young firms have higher leverage. On the other hand, we observe that large firms are more cyclical than small (among older firms) which is unlikely to be driven by finance, since these firms all have similar leverage levels.[1] We also document an inverse-U relationship between leverage and the growth rate of sales, employment, and assets, suggesting that both very low or very high leverage might hinder firms' growth.

Our theoretical contribution is to demonstrate the importance of capturing the heterogeneity of cyclicality along the full joint size and age distribution using a quantitative heterogeneous firm model. Our model framework builds on the seminal work of Khan and Thomas (2013), who, building on insights of Bernanke and Gertler (1989); Kiyotaki and Moore (1997); Bernanke et al. (1999) and Jermann and Quadrini (2012), set up a heterogeneous firm model with financial frictions, which they use for business cycle analysis. In

[1] Given the nature of our dataset, our empirical investigation of financial frictions does not have an ambition to achieve causal identification. However, our results are consistent with evidence with direct identification of financial friction via firm-bank matches, such as Chodorow-Reich (2014), that finds that firms borrowing from financially distressed banks contracted more during the Great Recession, and that this effect is largest at smaller firms.

this framework, firms are born poor, and hence financially constrained, and grow out of these financial constraints as they age. We extend the model to match the cross-sectional age and size distributions of firms, making it a natural laboratory to study the cyclicality of firms by age and size over the cycle.

First, we set up a basic calibration of the model to cross-sectional moments. We find that this model is unable to generate our facts on the cyclicality of firms by joint age-size bin, when confronted with a range of standard shocks. Second, we propose two extensions to the model in order to allow it to match our empirical evidence. The first extension is permanent differences in returns to scale across firms. We assign four firm size groups, with smaller firms having more decreasing returns to scale. We show that differences in returns to scale provide a simple and under-explored channel for driving cyclicality by firm size. In particular, making some firms have more decreasing returns to scale naturally makes them smaller, and also makes them less cyclical, as in the data. Put differently, small firms are fundamentally different from large firms (think a local shop versus Carlsberg Group), and these fundamental differences in firm scope help explain why small firms are less cyclical than large.[2] Our second extension is differences in the initial net worth that firms enter with, depending on their size group. We allow larger entrants to enter with more net worth, and hence to be less financially constrained. This matches both our lifecycle data (larger entrants grow less fast than smaller entrants) and cyclical data. Specifically, the calibration finds that large entrants start less financially constrained than small entrants. This makes the excess cyclicality of young firms smaller for larger firms, as in our data. With these two twists, we show that our new model is able to replicate the cyclicality of firms by joint age-size bin from our data.

Finally, we show that our results have important bite for the aggregate economy by turning to their implications for business cycle policies. Among large firms, we found empirically that there is little difference in cyclicality between young and old firms. This implies that large young firms are less financially constrained than a standard model would suggest. This dramatically reduces the power of an "age based" policy, which targets young firms by offering debt relief, since this policy now mostly only affects small

[2] Gavazza et al. (2018) also use permanent differences in returns to scale to drive differences in firm size, in a model of recruiting intensity calibrated to the US economy. They find that this helps match why recruiting intensity (measured as the vacancy yield) is more cyclical at small firms than large firms, by creating financially unconstrained small firms. We show how, among financially unconstrained (older) firms, differences in returns to scale can explain why large firms are more cyclical than small firms.

entrants who have limited impact on the aggregate economy. In contrast, a policy of simply offering a wage bill subsidy to all firms becomes more powerful, since large firms now respond more strongly to the policy. These results show that it is crucial to understand and match the responsiveness of the full age-size distribution of firms in order to perform robust policy evaluation through the lens of structural models.

Related literature: There is a broad literature studying the effect of age and size on firm decisions and outcomes. Different papers find different, sometimes conflicting, results, which is partly driven by different samples of firms available (by age and size) in a given dataset. Gertler and Gilchrist (1994) investigate the cyclicality of small versus large firms and find that small firms are more sensitive to periods of credit market tightening than large firms. Khan and Thomas (2013) show that small firms contracted more than large firms during the financial crisis, and Gavazza et al. (2018) show that the vacancy yield was more cyclical at small than large firms during this same period. On the other hand, Moscarini and Postel-Vinay (2012) find that larger firms (in terms of number of employees) are more cyclical, when aggregate conditions are measured using the (HP-filtered) level of the unemployment rate. Similarly, Mian and Sufi (2014) show that larger establishments contracted more in areas with larger declines in house prices. Fort et al. (2013) discuss the conflicting results by firm size, and add age to this analysis. They find that young firms are more cyclical than old firms, and that this difference is much more important than the differential between small and large firms. While they do not have direct financial data at the firm level, they use state-level house price data to argue that financial frictions may drive this result.[3] Compared to the latter two papers, we do not use any local identification for financial shocks, but instead directly measure how firm-level financial variables vary over age-size bins.

Due to data limitations, much of the knowledge about cyclicality and firm finance is based on large publicly traded firms. Sharpe (1994) uses Compustat data to document that high leverage firms are more cyclical than low leverage firms. Giroud and Mueller

[3] Another strand of literature examines the cyclicality of firm financing, both in terms of empirics and also model building. For examples, see Jermann and Quadrini (2012) (investigate the cyclicality debt and equity issuance), Covas and Haan (2011) (the cyclicality of financing is different across firms of different sizes, with the procyclicality of equity issuance decreasing monotonically with firm size), Crouzet (2017) (the choice of bank and bond financing), Begenau and Salomao (2018) (firm size and debt/equity cyclicality) or Nikolov et al. (2018) (size and source of financial constraints).

(2017) combine Compustat data with establishment-level employment data to show that the decline in house prices during the Great Recession, as investigated by Mian and Sufi (2014), was transmitted to declines in employment through high leverage firms. Conversely, **?** use Compustat data find that firms with low default risk, including those with low debt burdens, are the most responsive to monetary shocks. Relative to their paper, our sample includes non-listed firms, and thus younger firms who may behave differently to financial frictions than older, listed firms. Jeenas (2019) investigates the role of liquidity and leverage in driving heterogeneous investment dynamics, and finds that leverage ceases to be important once liquidity is controlled for. However, publicly traded firms are only a small subset[4] and as such are not representative of the whole firm population. Cloyne et al. (2019) use data for the US and UK to show that younger, non-dividend paying firms exhibit the largest and most significant changes in investment following monetary policy shocks. Beyond focusing on public firms, they also measure age as time since incorporation due to data availability, rather than foundation. In contrast, we are able to measure age since foundation. We also focus on overall cyclicality rather than the response to identified monetary policy shocks.

The two papers closest to our work are Crouzet and Mehrotra (2020) and Dinlersoz et al. (2018). Both study the US, and go beyond Compustat to achieve wider firm coverage, so their results are not based only on public firms. Crouzet and Mehrotra (2020) find that only the largest firms (99th percentile and above measured by assets) are less cyclical than the rest, which goes in the opposite direction to our results where cyclicality increases in size. However, there is a large difference in the definition of size groups, as they focus on the top 10% of firms (measured by assets) while we investigate firms across the whole size distribution (measured by employees). We extend their empirical specification by including firm age, and more importantly, the interactions of size and age. We also study the cyclicality of employment, thereby bridging to the literature focusing on employment fluctuations and financial frictions (Chodorow-Reich (2014); Duygan-Bump et al. (2015)). Dinlersoz et al. (2018) merge balance sheet data from Compustat and Orbis into the US Longitudinal Business Database (LBD). Similarly to our analysis, they are able to analyse both private and public firms and can measure firms employment and

[4]In the US there are around 4000 publicly traded firms (Gupta et al., 2021) in the population of over 5 million firms.

age since foundation. They argue that small private firms are plausibly financially constrained both before and after the financial crisis, while larger private firms may have only become constrained during the crisis, and large public firms appear to never be financially constrained.

Overall, our key contribution is to study the joint relationship between firm age and size over the entire population of firms, measuring both firm-level averages and cyclicality. We stress the interaction between age and size in driving financial frictions, and present the novel result that the answer to the question "are large or small firms more cyclical?" depends on whether one looks at young or old firms. Finally, we complement our empirical work with policy implications from a quantitative heterogeneous firm model.

The rest of this paper is organised as follows. In Section 2 we discuss the data, construction of our key variables and the estimation specification we use. In Section 3 we present our empirical results, and in Section 4 we present our quantitative model findings. Finally, in Section 5 we conclude.

2 Data

Our dataset covers the universe of firms in Denmark between 2001 and 2019 at annual frequency. In order to analyse firm outcomes and financial balance sheet data together, we merge two datasets ("data registers") provided by Statistics Denmark (DST): the FIRE dataset ("Regnskabsstatistikken"), which broadly contains data on accounting variables, is merged with the FIRM dataset ("Firmastatistik"), containing data regarding "economic, employment and accounting information at company level.

The quality of this data is generally believed to be very high, as Statistics Denmark is a government agency, and most of the variables we use are originally collected by Denmark's tax authority, SKAT.[5] Additionally, DST also runs independent checks on the datasets. Individual firms are identified by a unique number that is generated at the time of registration. The merging of the datasets is done using this identifier, and thus provides exact matches. More information on data itself and the cleaning process is provided in appendix A.1.

[5]Sales, assets, liabilities, investment and information about employment based on payroll.

Our cleaned dataset is an unbalanced panel capturing employer firms in Denmark. It contains roughly 2 million firm-year observations, with approximately 100,000 firms per year (in the early 2000's the number of firms is around 80,000 but it grows to 120,000 at the end of the sample). Crucially, our dataset therefore contains firms which are both publicly listed on the stock exchange and privately owned, and which span the entire distribution of firm age and size. Combined with the fact that we have access to firm balance sheet financial data, this makes our dataset uniquely suited to studying the role of financial frictions across the whole distribution of firms, especially at younger and smaller firms that are not featured in datasets like COMPUSTAT.

Given the start and end date of the underlying registers one might reasonably worry whether our results might be overweighting the role of finance due to the financial crisis. While it would be theoretically possible to extend our sample by using alternative datasets that cover different time periods, we believe that there was enough other variation in Danish business cycle that other shocks are also well represented. According to OECD[6], our sample covers the following business cycle turning points: troughs in 2003M7, 2009M7, and 2014M4 and peaks in 2006M7, 2011M4 and 2019. This means that while certainly dominant, the financial crisis is not the only recession in our dataset.

2.1 Key variables

Since or focus is on firm cyclicality and financial frictions, we will mostly use firm production and balance sheet variables from our merged dataset. On the production side, we use data on sales, employment (both headcount, which we use for definition of size groups and the number of full time equivalent workers, which we use in the regressions), profits and investment. On the financial side, we use data on total assets, total liabilities, and the stock of debt of all maturities. We use the ratio of debt to assets as a measure of leverage. We additionally use data on a firm's sector of operation.

We define firm size by its lagged employment (headcount).[7] The firm size measure thus changes as the firm grows or shrinks as it ages and is hit by shocks. We put firms into bins based on four quantile thresholds (0-30th, 30-60th, 60-90th and 90+) of size across the

[6] See OECD turning points.

[7] For robustness, we also follow Crouzet and Mehrotra (2020), and measure firm size by the value of its assets (we report the alternative results in the appendix A.3).

population of firms active that year. In Figure 9 in the appendix we plot the employment thresholds defining these size bins, and how they evolve over time. The thresholds are relatively stable over time, with a minor expansion of the largest firms at the end of the sample.[8] The firm size distribution is heavily skewed: the top size bin (containing only the 10% largest firms) represents over 70% of aggregate employment.

We sort firms into four age groups: 0-3, 4-8, 9-19, and 20+ years old. The number of firms within each age group changes over time, and as Figure 9 shows, cyclical fluctuations in entry create swings in cohort size that propagate over the age distribution. Firm age is measured from the moment the firm is registered.[9] This notion of age is thus the true age since foundation of the firm, which distinguishes us from other datasets which can only measure age since, for example, the firm was publicly listed on stock markets. As with our size measure, in our empirical work we do not work with age directly, but put firms into our four age bins.

Firm-level outcomes are defined the normalised growth rates suggested by Haltiwanger et al. (2013): for any firm-level variable $x_{i,t}$, we measure growth from $t-1$ to t as

$$\hat{g}_{x_{i,t}} \equiv \frac{x_{i,t} - x_{i,t-1}}{\frac{1}{2}(x_{i,t} + x_{i,t-1})},$$

where i indexes firms and t years. As discussed by Haltiwanger et al. (2013), this growth rate, which uses the average of the current and past value as the denominator, rather than just the past value, is more robust and typically has better properties in firm-level data. For our cyclical measure, we use the standard growth rate of aggregate GDP, which we denote as $y_t \equiv \frac{GDP_t - GDP_{t-1}}{GDP_{t-1}}$, collected from the DST National accounts.

2.2 Estimation framework

To study the intricate interplay between firm size and age we allow for interactions between size and age bins. Therefore, the effect of being old, for example, is allowed to be different for small and large firms. Using the definition of groups from the previous section, we run a regression with a set of dummies controlling for the interaction of size

[8] For assets, the pattern is similar, see the figures in A.3.

[9] Given that it takes very little time to start a new firm in Denmark, we believe there is not a large need to formally register the firm long before the firm becomes economically active.

and age. Formally, we run two types of regressions to get at the differences in levels and differences in cyclicality:

$$x_{i,t} = \sum_j \sum_k \alpha_{j,k} \mathbb{1}_{i \in I_t^j} \mathbb{1}_{i \in A(k)} + \sum_l \gamma_l \mathbb{1}_{i \in S(l)}, \tag{1}$$

$$\hat{g}_{x_{i,t}} = \sum_j \sum_k (\alpha_{j,k} + \beta_{j,k} y_t) \mathbb{1}_{i \in I_t^j} \mathbb{1}_{i \in A(k)} + \sum_l (\gamma_l + \delta_l y_t) \mathbb{1}_{i \in S(l)}, \tag{2}$$

where $\hat{g}_{x_{i,t}}$ denotes the firm-level normalised growth-rate of the variable of interest, such as turnover or employment at firm i. The indices j, k, and l index firm size bins, firm age bins, and firm sectors respectively.[10] $\mathbb{1}_{i \in I_t^j}$ is an indicator variable for firm i being in size group j at time t (and similarly $\mathbb{1}_{i \in A(k)}$ for age and $\mathbb{1}_{i \in S(l)}$ sector). Our baseline results are obtained by OLS estimation.[11] Primarily, we present the results graphically, combining the coefficients and plotting against the size, grouping the age bins by a line of age-specific colour.

The regression equation (1) is used to gain insight about the basic age-size distribution of variables of interest. We do so by grouping the coefficients α by age group and and plotting the values over firm size bin. The results should not strictly be interpreted as life-cycle profiles as the dataset is not a balanced panel due to firm exit.[12] We include sectoral controls so the results are to be interpreted as the within sector levels differences driven by age and size.

Equation (2) is used to study firm cyclicality. We regress firm level growth rates, $\hat{g}_{x_{i,t}}$, on aggregate growth, y_t, using dummy variables to separately estimate the cyclicality of different groups of firms. For each size bin, α_{jk} captures the marginal effect on the average growth rate of firms of being in that size bin. For these regressions, we are more interested in the β_{jk} parameters, which capture how the firm-level growth rates, $\hat{g}_{x_{i,t}}$, are differently related to the aggregate growth rate, y_t. The interpretation of β_{jk} is that a 1pp increase in aggregate growth is on average associated with a "β_{jk}"pp increase in firm-level growth for firms in size group j and age group k, on top of any additional effects captured by sector specific cyclicality. Thus, the β_{jk} capture the cyclicalities of each firm age-size group. Similarly, δ_l coefficients control the cyclicalities of the different sectors,

[10] We use Danish 36 sector industrial classification DB07, based on NACE rev.2

[11] For the sake of robustness, we also estimate the effects a corresponding specification with firm fixed effects, but the results are very similar (see appendix A.5).

[12] For more discussion of the estimation of the effect of age using Danish firm micro data, see Andersen and Rozsypal (2021).

to strip out the potentially differing average cyclicality of different industries. Thus, the effects when comparing coefficients from different, for example, age groups should be interpreted as within-industry effects.

This specification is an extension of Crouzet and Mehrotra (2020)'s regression (their equation (1)) to include firm age categories interacted with the size bins. The equation is essentially a regression of firm-level growth rates on a constant term and the aggregate growth rates, with interaction terms allowing for group-specific means and loadings on the aggregate growth rate. Dinlersoz et al. (2018) also estimate the effect of age and size. To capture the effect, they employ quadratic terms in age or (log of) size. The contribution of the present paper is to measure the effect of size *as it changes with age* (or vice versa). Technically the difference is in inclusion of the interaction term and using size and age bins rather than having a parametric specification for the effect. They also study how coefficients change during the Great recession, which we do not do. We also discuss the results specification where we do not allow for interaction of size and age in Appendix A.4.

3 How does firm age and size determine firm outcomes?

In this section we investigate how firm age and size affect firm averages, and firm cyclicality. In Section 3.1 we set the stage by examining the distributions of firms along the size dimension and how the size distribution varies across age groups. The size distribution is wide, with the smallest (largest) bins having average employment of 1.6 (115.9) employees respectively. However, even conditioning on firm size, firm age is a strong predictor of the *growth rate* of a firm, with young firms often growing by over 10% per year, while older firms grow very little on average. "Old small" firms are thus fundamentally different from "young small" firms; they have reached they optimal size and are not growing anymore. In the remaining subsections, we then build on this foundation to study firm cyclicality and its relationship to firm finance.

Table 1: Averages of Variables of Interest by Age and Size

	Age groups				Size groups			
	0-3	4-8	9-19	20+	0-30	30-60	60-90	90+
Employment	7.7	12.2	17.5	26.2	1.6	4.0	11.2	115.9
sales	17435	28038	45910	93294	4219	9333	25012	320649
Assets	18705	32859	56870	128151	10115	20451	24029	360309
Debt	10595	17615	30204	69040	5237	10809	12760	192787
Equity	4627	11155	21058	49123	4197	7620	9894	144007
DA (w)	0.85	0.79	0.69	0.62	0.75	0.75	0.73	0.68

Note: Sales, assets, debt and equity in thousands of DKK (during the sample period 1000 DKK represented value of 134 EUR (stable due to fixed exchange rate) or 150-200 USD). Reported numbers are the average values within in bin. Debt/assets (DA). Only continuing firms.

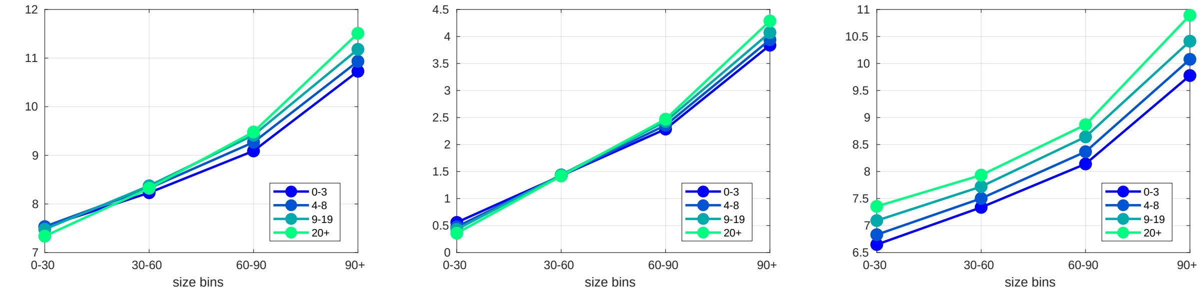
3.1 Levels and growth rates of real variables

To give an overview of the distribution of various variables in the data, we provide a summary table with basic moments of the variables of interest over firm size or age. However, in the present paper we argue that the interaction between size and age is important, so in addition to the standard moments that we document in Table 1, we also show series of plots where we group firms by age groups and plot how various characteristics change along the size dimension (after taking away the sector contributions). Here we focus on "real" variables, by which we mean non-financial variables, and we turn to financial variables in later sections.

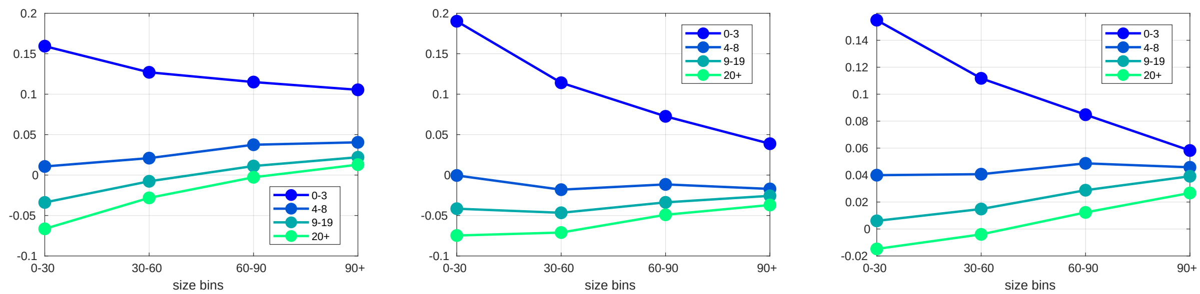
As Figure 1 shows, for sales and employment there there is only little separation by age in each size group. For employment, this is natural, as the size bins are defined by employment. In contrast, within each size bin the assets are uniformly increasing with age, to the extent that entrants (the first age bin) in the second size bins have roughly the same assets as firms in the first size bins that have been around for 20+ years (the last age bin). If we interpret assets as firms' capital, this disconnect in proportionality between employment and capital is at odds with optimality conditions implied by common production functions.

The panels in the second row of Figure 1 show the growth rates of the same variables by size $\times$ age groups. Not surprisingly, the growth rate for all variables is the highest for the entrants, suggesting that on average firms start below their optimal size. The

Figure 1: Average levels and growth rates by size and age



(a) (log) Sales (b) (log) Employment (c) (log) Assets



(d) Growth rate of sales (e) Growth rate of employment (f) Growth rate of assets

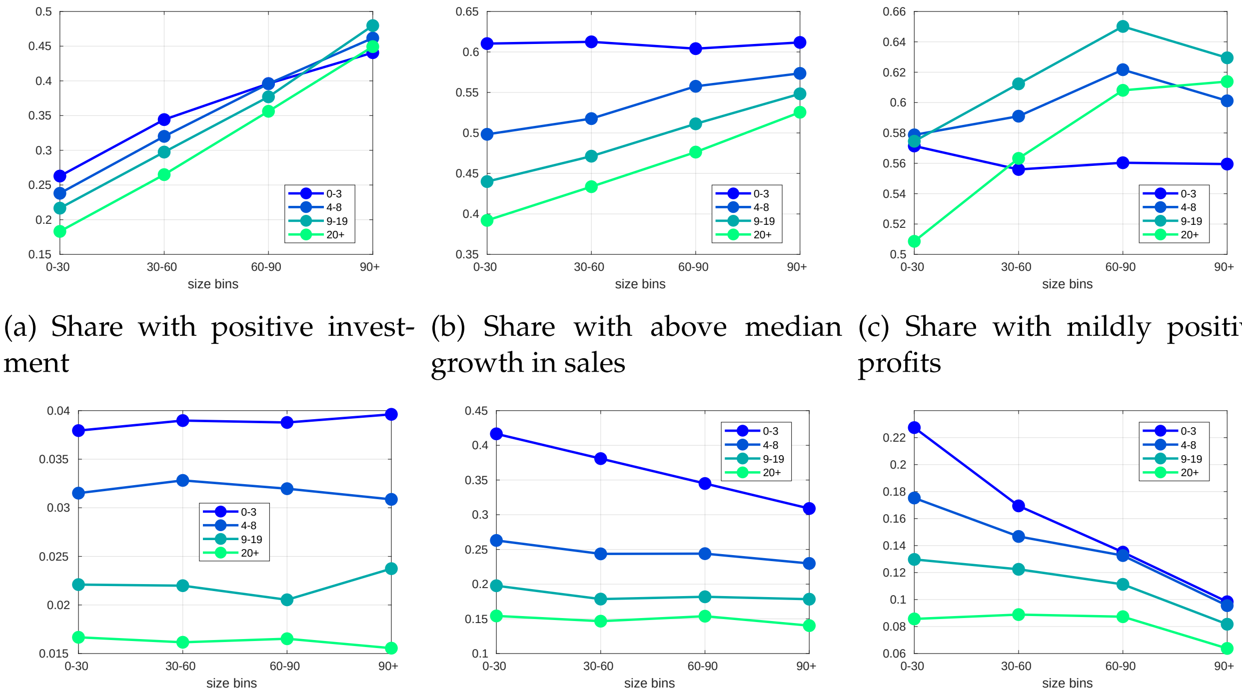
Note: Average level is computed as coefficient α_{lk} from regression (1)*. Different lines correspond to firms of different size bins whereas firm age bin is on x axis.*

entrants that start in the smallest bin also grow the fastest, both the inputs (employment and assets) and also output (sales), suggesting that the dispersion of the starting size is actually larger than the dispersion in long run optimal size. Compared to entrants, older firms on average grow slower and the size gradient is heavily muted or even reversed: larger firms grow faster than smaller firms of any age group beyond the entrants. This is again consistent with a world where different firms have different optimal sizes and most growth comes from the young firms that enter below their optimal size.

Figure 1 thus shows that Gibrat's law—the idea that growth rates should be independent of size—fails. Separating the different age groups also allows us to notice that the directionality of the failure is different for entrants, for whom it fails the most, than for firms of all other ages.

To get further insights we look at the distribution of investment, growth rate of sales and profits in Figure 2. In panel (a) we see that larger firms are more likely to have positive investment with qualitatively little separation by age. On the other hand, panel (b) shows that the odds of very large investment do not depend on size and they are clearly

Figure 2: Investment, Growth of Sales and Profitability by Size and Age



(a) Share with positive investment (b) Share with above median growth in sales (c) Share with mildly positive profits

(d) Share with high investment (e) Share with high sales growth (f) Share with high profits

Note: Results using regression (1). *High investment defined as investment/assets>0.25. Mild profits are defined as profits/assets>0.03, high profits are defined as profits/assets>0.25. High sales growth is defined as sales growth larger than 75th percentile of sales growth distribution.*

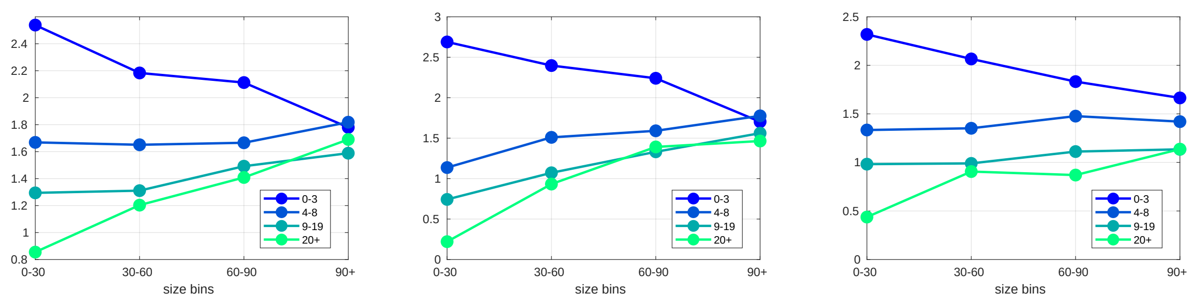
separated by age. This suggest that investment is lumpy which makes it difficult for small firms to engage in small projects (among small firms, it is still the younger than are more likely to invest). For sales growth, in panels (b) and (e) we look at how many firms within each bin perform better than the economy-wide median and 75th percentile firm respectively. Again, the patterns with respect to size differ for these two measures: the share of firms is increasing (decreasing) with size for above median (above 75th percentile) growth, with the oldest (youngest) firms showing the largest gradient. In panels (c) and (f) we look at the share of firms with positive and high profits respectively. The share of firms with large profits looks qualitatively similar to the share of firms with large growth of sales, but for the shares with mildly positive profits we do not observe any clear pattern between size and age.

To summarize, often, albeit not always, the effect of size on firm outcomes is different as firms get older. Sometimes the gradients even have opposite sign, meaning that it could be the case that the average gradient is zero when the whole population is taken into account, hiding size patterns that only hold within age groups. In our view, this highlights the importance of taking both age and size into account when understanding firm dynamics, and we show that this insight remains true when considering business cycle cyclicality in the following sections.

3.2 Cyclicality of real variables

In this section we investigate the cyclical sensitivity of firms by size and age. We do so without any reference to financial frictions, or other underlying causes of the differing levels of cyclicality. Thus, the results in this section are meant to be interpreted as theory free, and provide us with our basic stylised facts about firm cyclicality in the Danish economy. Following Crouzet and Mehrotra (2020) "cyclical sensitivity" refers to the extent that a worsening in aggregate conditions is systematically associated with declines in outcomes at firms of various groups.

Figure 3: Cyclicality by Size and Age



(a) Sales (b) Employment (c) Assets

Note: Cyclicality is computed as a sum of coefficients β_{lk} corresponding to given size and age bin from regression (2). Apart from the youngest firms, cyclicality increases with age and the level is shifted down by as firms get older. For the youngest firms, cyclicality falls with firm size. Combined with young firms being more cyclical than old, we observe that large firms are much more homogeneous group in terms of cyclicality than small firms.

We are primarily interested in the effect of firm age and size on cyclicality, and so will not display the sectoral coefficients, which we treat as control variables. We present our results from regression specification given by equation (2) for firm-level sales, employment, and asset growth. In the main text, we present our results by plotting the relevant coefficients graphically in Figure 3. We have a separate cyclicality coefficient, $\beta_{j,k}$, for every age-size pair. In the appendix, we give the regression results in Table 4.

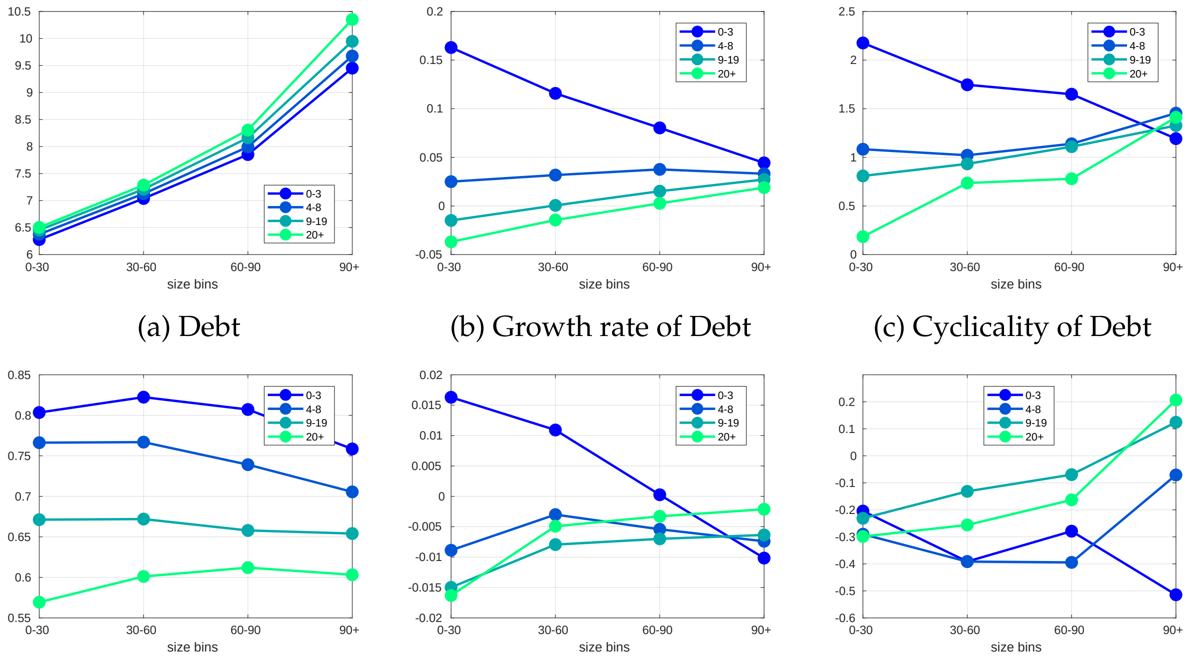
The results in Figure 3 show two general patterns which hold regardless of whether we measure cyclicality using sales, employment, or assets. First, younger firms are more cyclical across most of size distribution. Second, the effect of size is different for entrants (the age 0-3 group) and for firms of all other sizes. For the entrants, the larger firms are on average less cyclical. In contrast, among all the older firms, larger firms tend to be more cyclical and gradient is the largest for the firms in the oldest age bin. The corollary is that the difference in cyclicality between young vs. old firms is much larger among the smallest firms than among the largest firms. Indeed we detect a distinct non-linearity in the age-size relationship. In particular, for smaller firms, those in the 0-30th size percentile, the difference in cyclicality between young-small firms and old-small firms is dramatic. However, once we get the the largest (90%+) size bin, cyclicality is very similar for young and olds firms (especially when measured by sales or employment). In other words, all large firms are alike, but small firms can be very different, and small young firms and small old firms are not alike.

It is interesting to compare our full joint age-size cyclicality results to simpler specifications which only investigate the role of size or age on cyclicality independently. We run these specifications, and present the results in Figure 13 and Table 5 in the appendix. When studied alone, we find that younger firms are more cyclical than old firms. But when studied alone, the regression does not find consistently find economically or statistically significant differences in cyclicality by firm size.[13] Our more nuanced joint age-size regression can explain these results. Size alone does not predict cyclicality since the relationship between size and cyclicality has opposite signs for young and old firms, which roughly cancel out on average. Consequently, policies that target all small firms, might have different effects on small firms that are old and those that are young.[14]

[13]The exception is by employment, where smaller firms are less cyclical than larger, but the same does not hold for sales or assets.

[14]One reason why firms grow is that they accumulate consumers. A larger customer base might provide insurance against fluctuations in demand, as long as the demand coming from individual customers is not perfectly correlated. However, this channel would manifest itself in differences in unconditional volatility rather than cyclicality, because aggregate business cycles by definition involve some degree of commonality.

Figure 4: Average Levels, Growth Rates and Cyclicality of Debt and Leverage



(a) Debt (b) Growth rate of Debt (c) Cyclicality of Debt

(d) Leverage (D/A) (e) Growth rate of leverage (f) Cyclicality of leverage

Note: Cyclicality of debt follows the same pattern by age and size as cyclicality of employment and sales. In contrast, most firms have counter-cyclical leverage. Leverage (Debt over assets (DA)) is winsorized at 99.5%.

3.3 Levels, growth rates, and cyclicality of financial variables

So far, we have presented results about the cyclicality of firms by joint age-size, without reference to any underlying theory which could explain these results. This places the results in a similar approach as the cyclicality results in Fort et al. (2013). In this section, we aim to go further, and use our firm-level financial data to provide evidence about the role of financial frictions in driving cyclicality by age and size.

To gain more insight about which firms are more likely to be more severely credit constrained, we examine firm debt and leverage using the same approach as we used on for sales, employment and assets. We study the differences in levels, growth rates and cyclicality of financial variables for different sizes and ages, with the results presented in Figure 4.

Starting with the levels of leverage (panel d), we find that the younger firms are on average more leveraged than older firms. These firms are the likely candidates for being the most constrained, both because they already have the most (in relative terms) debt and the shortest track-record with lenders. Note that the gradient with respect to size is greatly dominated by the effect of age, and to the extent that there is a size gradient it has inconsistent signs across groups. For most firms the larger size seems to decrease the leverage (the exemption being the the smallest entrants and top 10% of firms in terms of size). Turning to the levels of debt (panel a), we see the reverse; older firms having more debt across all size groups. However, given what we already know about leverage (Debt/Assets), it must be the case that older firm just have even more assets than debt and hence one can hypothesize that they are probably less credit constrained than the young firms, despite having higher debt levels.

This conjecture is further supported by the growth rate of debt and leverage (panels b and e), which shows the young and small firms experiencing much high growth rate of both. For large entrants (age 0-3 and size 90%+) the growth rate of both debt and leverage is in line with the other large firms, which suggests entrants of this size are not being treated by banks worse than older firms and are likely not more (or less) financially constrained. This distinction between the financial position of small versus large entrants will be important in our theoretical work.

Finally, turning to the cyclicality results, we see that debt (panel c) follows the same pattern as sales and other variables from Figure 3. That is, entrants have more cyclical debt and its cyclicality decreases with size and for all older firms it increases with size. Leverage seems to be countercyclical for almost all age-size groups, possibly because asset values decline more than debt in recessions. It is also noteworthy that the cyclicality of leverage is the only variable where the dispersion among the small firms is much larger than than dispersion among the large firms.

3.4 How does leverage affect firm growth and cyclicality?

To explore the relationship between leverage and firm outcomes further, consider the following two regression specifications:

$$\hat{g}_{x_{i,t}} = \sum_m (\omega_m + \psi_m y_t) \mathbb{1}_{i \in DA(m)} + \sum_l (\gamma_l + \delta_l y_t) \mathbb{1}_{i \in S(l)}, \tag{3}$$

$$\hat{g}_{x_{i,t}} = \sum_m (\omega_m + \psi_m y_t) \mathbb{1}_{i \in DA(m)} + \sum_j \sum_k (\alpha_{j,k} + \beta_{j,k} y_t) \mathbb{1}_{i \in I_t^j} \mathbb{1}_{i \in A(k)} + \sum_l (\gamma_l + \delta_l y_t) \mathbb{1}_{i \in S(l)}, \tag{4}$$

Here we use five bins—the lagged leverage quintile—to control the the leverage $\mathbb{1}_{i \in DA(m)}$. These regression equations are a variation on (2). In (3) we regress firm outcomes on the leverage bin and it's interaction with aggregate growth. In (4) we additionally include the original age-size interactions from our previous regression. We report the coefficients ω and ψ in Table 2. We take sales, employment, and assets as the variables to be explained, and plot specification (3) in odd columns, and (4) in even columns.

The cyclicality of firms in each leverage bin is given by the coefficients in rows 7 to 10, with quintile 40%-60% being the omitted quintile. Starting with specification without age-size controls, columns 1, 3, and 5 show that across all three left-hand-side variables, more highly leveraged firms are more cyclical. The results are strongly statistically significant for employment and assets, and less so for sales, while maintaining the same coefficient pattern. This finding meshes very natural with our previous results, where we found that young firms are more cyclical than old, and that young firms are more highly leveraged than old. Here, we directly find that highly leveraged firms are more cyclical.

What do we learn from comparing specifications (3) and (4)? Comparing the odd and even columns in Table 2), we see that the effect of leverage bin on cyclicality gets compressed when the joint age-size dummies are added to the regression. For example, for employment in columns 3 and 4, the difference in cyclicality between the highest and lowest leverage bin is $0.335 - (-0.395) = 0.73$ without age-size controls, and $0.309 - (-0.180) = 0.489$ with age-size controls. This means that some of the difference in cyclicality between low and high leverage firms is related to the differences in cyclicality between different firm age-size groups that we documented in Section 3.2.

Equally interesting are the average growth rates of firms in each leverage bin, given by the coefficients in rows 3 to 6. For employment, we find there is an *inverse-U* relationship between leverage and average growth rate, so firms with both very low and very high leverage tend to grow slower (even after accounting for size and age effects). On the

Table 2: Effects of Leverage on Average Growth Rate and Sensitivity to Business Cycle

	(1) Sales	(2) Sales	(3) Employment	(4) Employment	(5) Assets	(6) Assets
Constant	0.012*	0.007	-0.009	-0.043***	0.026***	0.014*
	(1.99)	(1.16)	(-1.44)	(-6.69)	(4.40)	(2.31)
y	1.528***	1.717***	1.301***	1.473***	1.289***	1.224***
	(5.25)	(5.70)	(4.30)	(4.80)	(4.54)	(4.18)
0-20	-0.045***	-0.034***	-0.022***	-0.017***	0.010***	0.017***
	(-29.33)	(-22.21)	(-16.30)	(-12.45)	(6.43)	(10.90)
20-40	-0.018***	-0.015***	-0.007***	-0.005***	0.013***	0.015***
	(-12.00)	(-9.99)	(-5.55)	(-4.05)	(8.25)	(9.67)
60-80	0.005***	0.001	-0.004**	-0.009***	-0.019***	-0.022***
	(3.45)	(0.96)	(-3.15)	(-6.56)	(-11.94)	(-13.73)
80+	0.005**	-0.004*	-0.042***	-0.059***	-0.023***	-0.031***
	(2.84)	(-2.26)	(-26.81)	(-37.76)	(-11.75)	(-15.97)
0-20 × y	-0.102	0.018	-0.395***	-0.180**	-0.503***	-0.407***
	(-1.43)	(0.25)	(-6.37)	(-2.88)	(-7.46)	(-5.91)
20-40 × y	-0.145*	-0.109	-0.145*	-0.080	-0.182**	-0.148*
	(-2.13)	(-1.59)	(-2.47)	(-1.38)	(-2.71)	(-2.22)
60-80 × y	-0.021	-0.028	0.193**	0.172**	0.111	0.090
	(-0.30)	(-0.39)	(3.11)	(2.80)	(1.56)	(1.27)
80+ × y	0.061	0.023	0.335***	0.309***	0.437***	0.354***
	(0.77)	(0.28)	(4.62)	(4.24)	(5.01)	(4.02)
Observations	595547	595547	674084	674084	594646	594646
adj-r2	0.017	0.035	0.015	0.053	0.010	0.021
Sectors	yes	yes	yes	yes	yes	yes
Clustering level	firm	firm	firm	firm	firm	firm
Size and Age controls	no	yes	no	yes	no	yes

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Note: The degree of leverage captured by Debt/Assets quintile. Firms with median leverage (between 40th and 60th percentile) are treated as the base group. Odd columns correspond to regression without size X age controls (equation (3)), even to to specification with age x size controls (equation (4)).

other hand, we find that firms with higher leverage have higher sales growth, but lower asset growth.

Additional results: To dig deeper into the relationship between cyclicality by age-size and leverage, we also explore what happens to coefficients on by age and size when adding leverage to the regression. To reduce the number of coefficients, we do not estimate cyclicality coefficients for each age-size bin separately as we did in (2). Instead, we treat the size bin as a continuous variable and compute an age-bin-specific size slope.

These results for cyclicality of employment growth are presented in Table 7 in the appendix.[15] In these tables, the results without leverage controls are effectively simplified

[15] We repeat these exercises for sales and asset growth cyclicality in Table 8. There is some commonality,

versions of (2), where the age-size interaction is represented by a single size-slope for each age bin. Comparing the results with and without leverage controls shows that there is not much of a change in the coefficients capturing the cyclicality of firms by age and size when we add leverage. In this regression, we estimate one slope coefficient for size (both treating size bin as continuous variable or alternatively including a log of firm headcount in the previous year). This result is in line with the findings of Crouzet and Mehrotra (2020). Since we do find direct evidence that high leverage firms are more cyclical, we interpret this evidence that *both* leverage *and* other features related to age and size matter for cyclicality, and a full model needs to account for both. As an example, Casiraghi et al. (2021) build a model where some young firms are not allowed to borrow. Our model will feature differences in returns to scale which drive differences in cyclicality unrelated to financial constraints.

As a robustness check for the effect of leverage, we compare our baseline leverage specification where the leverage quintile is economy-wide, to a specification where it is computed within each age x size bin. The results are very similar (for details see Table 6). This results is driven by the fact that while there is a difference *the average* leverage across size and age, there is large dispersion of leverage within each bin.

3.5 Summary of empirical evidence

We have presented results on the cyclicality of firms by age and size, and their interaction. The central finding is that cyclicality differs in non-monotone ways across the age-size distribution: "young and small" firms are the most cyclical, large firms are the second most cyclical, and "old and small" firms are the least cyclical. We then turn to understanding what drives this pattern.

The patterns that we observe for financial variables are in line with financial frictions affecting *young and small* firms more than the rest, and explaining why they are the most cyclical. In particular, young firms have higher leverage than old firms, while leverage does not vary by firm size (when controlling for age). Among young firms, the smallest firms have the highest leverage growth, while all large firms have low leverage growth.

but also some difference. Sales behave like employment with respect to the effect of leverage on the average growth rate, but leverage's effect of cyclicality not significantly different from zero. The cyclicality of assets follows the same pattern as employment, but the average growth rate is negatively related to the leverage quintile.

Directly looking at leverage, we find that the highest leverage firms are the most cyclical, and that this partially—but not totally—explains the differences in cyclicality by age and size.

4 Quantitative model

In this section we build a dynamic, quantitative heterogeneous-firm model. The model builds on classic heterogeneous-firm financial frictions models, such as Khan and Thomas (2013),[16] which we extend in several ways in order to match our new stylised facts.

4.1 Description of the model

The model features a continuum of heterogeneous firms, with both ex-ante and ex-post heterogeneity. There is a representative consumer, who owns firms and supplies labor. The model also features a final goods aggregating firm. The key features that we use to connect the model to our empirical work are financial frictions at the firm level, and differences in returns to scale across different classes of firm.

The model is set in continuous time with an infinite horizon. Let $t \in [0, \infty)$ denote time. We focus on the case without aggregate uncertainty, and conduct business-cycle experiments using unanticipated one-time shocks. The model is presented below in steady state, for expositional simplicity, and we therefore drop the time subscript, t, in most of what follows.

4.1.1 Final goods producer

A continuum of heterogeneous firms are our firms of interest. These are technically intermediate goods producing firms, and we will refer to them simply as "firms" where it does not cause confusion. These firms produce a firm-specific good using capital and labor. Their goods are all sold to a representative final goods producer ("Final Goods Producer") who combines them to produce a composite final good, which is used for

[16]For examples of other work building on this framework, see Jo and Senga (2019), Ottonello and Winberry (2020), and the references therein.

consumption and investment. The final good is the numeraire, with price normalised to one in all periods.

The representative final goods producer purchases the output of the heterogeneous intermediate goods firms to produce the composite final good. Let q_i be production (gross output) of firm i and p_i the price of its good in terms of the numeraire. The production function is $Q = \left(\int_0^G q_i^\theta \, \mathrm{d}i\right)^{\frac{1}{\theta}}$ where Q is units of production of the final good. G is the mass of active intermediate firms, which can be endogenous or exogenous. Define $\epsilon = 1/(1-\theta)$ as the elasticity of substitution between varieties and restrict to gross substitutability ($\epsilon \geq 1 \Leftrightarrow 0 \leq \theta \leq 1$). This ensures that intermediate goods firms have decreasing returns to scale in revenue, even if they have constant returns to scale in production.

The final-goods firm is a price taker in both the final and intermediates markets. Their profit is given by $\pi = \left(\int_0^G q_i^\theta \, \mathrm{d}i\right)^{\frac{1}{\theta}} - \int_0^G p_i q_i \, \mathrm{d}i$. Profit for the final goods producer, and indeed all firms, is denoted in units of the numeraire final good. The final good producer's first order condition for good purchase from firm i gives

$$q_i = p_i^{-\epsilon} Q \tag{5}$$

This is the demand curve for the intermediate goods firms. For each firm let $y_i \equiv p_i q_i$ denote value added, equal to sales revenue. Aggregate GDP is the sum over firms $Y \equiv \int_0^G y_i di$, and goods market clearing gives $Y = C + I$.

4.1.2 Intermediate goods firms (a.k.a. "Firms")

There is a mass G of firms which arises via firm entry and exit. Firms have both ex-ante and ex-post heterogeneity. Firms face downwards facing demand curves (i.e. have well defined optimal size) and financial frictions. Firms are owned by the representative household, and discount the future at the interest rate, r.

At birth, firms draw a permanent "size type" $s = \{1, 2, ..., S\}$, which determines features which we wish to relate to firm size. Specifically, their returns to scale, η_s, depends on this size type, as well as a permanent component of their physical productivity, which we label z_s^S. To capture features of the firm lifecycle unrelated to financial frictions, we additionally introduce a "lifecycle shock", which we denote $g = \{1, 2\}$. All firms are born "young", with $g = 1$. At an exogenous rate α_G they transition to $g = 2$ and become "old". This shock controls a lifecycle component of their productivity, z_g^G, as well as firm exit rates, ζ_g, which we discuss further below. We normalize $z_2^G = 1$, so that z_1^G gives the productivity disadvantage of young firms. We abstract from any further idiosyncratic shocks to firms, in order to focus on cross-sectional size and age heterogeneity.

All firms share the common production function

$$q = z \min \left\{ k, \frac{l}{\alpha} \right\}^{\eta_s} \tag{6}$$

where $z \equiv z_s^S z_g^G$ denotes overall productivity, which combines the size and lifecycle components. Firms have Leontief production functions in capital and labor, with labor share determined by α.[17] If all firms had $\eta_s = 1$ then all firms would have constant returns to scale in production, and $\eta_s < 0$ denotes decreasing returns to scale. The demand curve is (5), and a firm's revenue is therefore $pq = z^\theta \min \left\{ k, \frac{l}{\alpha} \right\}^{\eta_s \theta} Q^{1-\theta}$. Value added is equal to revenue: $y = pq$.

At the firm level, all factors of production can be adjusted freely without cost. We are in continuous time, and there is no time to build for capital. It is convenient to first optimise labor for a given level of capital. Static profit is

$$\pi(k, s, g) = \max_{l \geq 0} \left\{ (z_s^S z_g^G)^\theta \min \left\{ k, \frac{l}{\alpha} \right\}^{\eta_s \theta} Q^{1-\theta} - wl \right\} \tag{7}$$

The Leontief production function gives the solution simply as $l(k) = \alpha k$, and $\pi(k, s, g) = (z_s^S z_g^G)^\theta k^{\eta_s \theta} Q^{1-\theta} - \alpha w k$.

A firm's capital stock evolves through a standard accumulation equation. Given investment i per unit of time and depreciation rate δ we have: $\dot{k} = i - \delta k$. One unit of investment costs p_K units of final good. Old capital and investment are perfect substitutes for firms, so capital also trades at the price p_K.

Firms can borrow using a risk-free short-term bond b with interest rate r. They face a borrowing constraint which limits the amount they can borrow according to the amount they can post as collateral. Specifically, we assume that borrowing is limited by the constraint $b \leq \lambda p_K k$, where recall that k is a firm's physical capital. The parameter λ controls

[17] The use of a Leontief production function is helpful in matching the wide size distribution in the data, when combined with financial frictions which directly affect the purchase of capital only, and not labor. With a Cobb Douglas production function, a financially constrained firm heavily substitutes from capital to labor while young. By ruling this out, the Leontief production function forces firms to maintain a fixed capital-labor ratio, so that financial frictions directly affect both capital and labor equally. This helps keep firms of size type s in the percentile group (0-30% and so on) that they are designed to match.

the tightness of the borrowing limit, with smaller λ making the constraint tighter and restricting the amount a firm can borrow for a given quantity of collateral. In the business cycle experiments, we allow λ to evolve as an aggregate financial shock.

A firm's net worth, n is defined as its assets less its liabilities: $n = p_K k - b$. Combining this with the borrowing limit gives $k \leq \frac{n}{p_K(1-\lambda)}$. Define a firm's leverage, ϕ, as $\phi \equiv p_K k/n$. Combining this with the borrowing limit gives the constraint as a constraint on leverage instead: $\phi \leq \bar{\phi}$, where $\bar{\phi} \equiv \frac{1}{1-\lambda}$ is the exogenous leverage limit.

A firm's net worth evolves according to

$$\dot{n} = \left(\frac{\pi(k,s,g)}{k} - (\delta + r) p_K \right) k + rn - d \tag{8}$$

where the first term is the net return on leveraged investment, and d denotes the dividend payout flow. We assume that firms cannot raise equity at all after the moment of birth, and so impose $d \geq 0$. We simplify the dividend payout policy, and impose that firms payout dividends only when net worth exceeds an exogenous level $\bar{n}$, and payout such that net worth remains at $\bar{n}$. Firms therefore pay no dividends while they are young, but then start paying out dividends when they are older and have achieved sufficient scale.

Firm exit is exogenous, and occurs at rate ζ_g. This is allowed to depend on the current lifecycle state, g, in order to match the data that young firms exit at a higher rate than old firms. When firms exit, they pay out their remaining net worth, n, as a final dividend.

As part of our calibration procedure, we allow a small number of firms to become "superstar" firms to match the importance of a few very large older firms in the data. We assign all firms a very small probability of becoming a superstar, which happens at rate $\alpha_\star$, so that only 0.5% of firms are superstars in steady state. When a firm becomes a superstar, it switches to a special superstar state with productivity $z_\star$ and returns to scale $\eta_\star = \eta_S$. Given the enormous change in optimal size that happens at this point, we allow firms to raise equity at the moment they become superstars, and allow them continuous access to equity from then on. They therefore become "Modigliani Miller" firms and their financial structure becomes undefined and they follow the efficient investment and production policies. We assume superstars hold a constant leverage rate, calibrated to that of the largest firms in the economy, and that they exit at a low rate of 1% per year, in line with the low exit rates of very large, old firms in the data.

The firm's problem can be stated recursively using a Hamilton Jacobi Bellman (HJB) equation. For any firm which has not transitioned to superstar status, denote optimized firm value as $v(n,s,g)$. This can be expressed as

$$rv(n,s,g) = \max_{0\le p_K k \le \bar{\phi} n} d(n) + v_n(n,s,g)\left(\left(\frac{\pi(k,s,g)}{k} - (\delta+r)p_K\right)k + rn - d(n)\right) +$$
$$+ \zeta_g\,(n - v(n,s,g)) + \alpha_\star\,(v^\star + n - v(n,s,g)) + \mathbf{1}_{g=1}\alpha_G\,(v(n,s,2) - v(n,s,1)) \quad (9)$$

Here, $d(n)$ is the exogenous dividend payout policy for the current level of net worth. The v_n term is the drift in net worth, which depends on the capital choice and dividend payout. The $\alpha_\star$ term captures the transition to superstar status. Since superstars face no financial frictions, superstar value can be expressed as $v^\star + n$ for some constant $v^\star$. The ζ_g term captures firm exit, and the final term captures the transition from lifecycle state $g=1$ to $g=2$.

The firm investment policy in this setting can be expressed as an unconstrained optimal capital stock, which firms will achieve only if they are financially unconstrained. The first order condition with respect to capital is $v_n(n,s,g)\,(\pi_k(k,s,g) - (\delta+r)p_K) = \mu_k$, where $\mu_k \ge 0$ is the multiplier on the borrowing constraint. If a firm hits its borrowing constraint then we know that $k = \bar{\phi}n/p_K$. If a firm is rich enough to be unconstrained, then $\mu_k = 0$ and the capital FOC gives us $\frac{\pi_k(k,s,g)}{p_K} = \delta + r$. This gives the unconstrained investment policy if unconstrained, $k^{unc}(s,g)$, which has an analytic solution. The overall investment policy can then be simply expressed as $k(n,s,g) = \min\{\bar{\phi}n/p_K, k^{unc}(s,g)\}$.

Finally, denote by μ_0 the flow rate at which new firms enter, which is assumed to be constant. After entry, new firms draw their permanent size type, with γ_s^S denoting the probability of drawing type s. New entrants are endowed with some initial amount of net worth, n, from an initial equity injection by their owners. We suppose that firms start life with net worth equal to the fraction n_s^e of the net worth required to become financially unconstrained.[18] This is allowed to differ by size type, which will be important for matching the data on cyclicality by joint age-size bin.

[18]Let $n^{unc}(s)$ denote the amount of net worth required to become financially unconstrained for a firm of size type s. This is easy to calculate using the leverage constraint, as $k^{unc}(s,2) = \bar{\phi}n^{unc}(s)/p_K \rightarrow n^{unc}(s) = k^{unc}(s,2)p_K/\bar{\phi}$, where we define $n^{unc}(s)$ as the level of net worth to be able to afford the unconstrained level of capital, given the borrowing constraint, when they reach lifecycle maturity ($g = 2$). Entrants therefore start life with net worth equal to $n_s^e \times n^{unc}(s)$.

4.1.3 Closing the model

Given the solution to the firm problem, we can simulate the endogenous firm distribution in steady state or in transitions. We can then calculate aggregates such as output and employment, and moments of the firm size and age distribution. We close the model by specifying how the prices that firms face (real wage, interest rate, and capital prices) are determined in a simple general equilibrium setting.

We assume that the representative household has instantaneous utility function over consumption, c, and labor supply, L^s, of $U(c, L^s) = c - (L^s/\chi)^{1+1/\sigma}/(1+1/\sigma)$ and discount rate ρ. This gives the equilibrium interest rate as a fixed constant $r = \rho$. The household's labor supply condition gives labor supply as a simple function of the wage: $L^s = \chi w^\sigma$. Finally, we suppose that investment goods can be produced one-for-one from the final good, giving a fixed equilibrium capital price of $p_K = 1$.

4.2 Result 1: Performance of a "steady state" calibration

In this section we describe what we call the "steady state" calibration, and show that it cannot match our new facts on cyclicality by joint firm age-size bin. This calibration generates a simple heterogeneous-firm model with financial frictions. The key idea is that this calibration targets only "steady state" moments of the firm distribution, and we will later contrast it with a "cyclical" calibration which additionally targets our new facts on cyclicality by age and size.

We loosely follow the calibration strategy of Khan and Thomas (2013), and so turn off three novel features of our model, which we will use later in our "cyclical" calibration. Firstly, we suppose that all firms have the same (constant) returns to scale, and so set $\eta_s = 1$ for all size types. Secondly, we attribute all employment growth of young firms to financial frictions, setting $z_1^G = z_2^G = 1$ so that the lifecycle component of productivity is constant. Finally, we do not explore how different firms may enter with different degrees of financial frictions, and suppose that $n_s^e = n^e$, so that all firms enter with the same fraction of unconstrained net worth.

4.2.1 "Steady state" calibration details

We start by describing our relatively standard parameters. We take one unit of time to be one year. We set the interest rate r to a 2% annual real interest rate, in line with the lower real interest rates seen in recent years. The capital depreciation rate δ is set to a 10% annual rate. We set θ to 0.9 to give a 10% markup in a frictionless model, as is standard in the New Keynesian literature. Firms have decreasing returns to scale in revenue, and so have well defined optimal sizes despite having $\eta_s = 1$. We choose the labor to capital ratio α to control the equilibrium quantity of employment, which is set to match the average firm size (total employment over total number of firms) in Denmark. The labor supply disutility χ is chosen to match a labor share of income of 60%. The labor supply elasticity η is set to 0.3, which implies that wages fall by 30% for a given change in aggregate employment. The entry rate of firms μ_0 is chosen to normalize the mass of firms in steady state to one.

We set the leverage constraint to $\bar{\phi} = 3$. This implies that firms remain financially constrained only until around age 3 on average and therefore represents a relatively loose borrowing limit.[19] We set the level of net worth at which firms start paying out dividends to a large number.[20]

Since we are interested in investigating cyclicality across the age and size distributions, a major goal of our calibration is to match these distributions well in steady state.[21] For the size distribution, we use our size types, s, to flexibly match the data. Specifically, we use $S = 4$ size bins to target the 0-30%, 30-60% 60-90%, and 90%+ percentile size bins in the data. We suppose the probability of being born in group $s = 1, 2, 3$ is 30% each, and $s = 4$ is 10%. By choosing the productivity levels appropriately, so that firms with $s = 2$ are on average larger than those with $s = 1$ and so on, each size type $s = 1, 2, 3, 4$ is therefore assigned to form the predominant mass of firms in each of the 0-30%, 30-60%

[19] In our data, the highest average Debt/Asset ratios for any firm-age groups are around 0.8, for firms aged 0-3. This correspond to a leverage ratio of $1/(1-0.8) = 5$, so our choice of a maximum leverage of 3 is conservative in that we allow firms to take on slightly less data than in the data.

[20] Above the "minimum saving policy" (see Khan and Thomas (2013)), exactly whether or not firms pay out dividends has no effect on firms choices of employment and so on in steady state. We therefore choose that firms pay out for some $\bar{n}$ such that even the most productive firms can fund their unconstrained optimal capital with no debt ($\phi = 1$).

[21] For the data used to calibrate distribution of number of firms and employment by firm age-size the model, we do not drop firms for whom we are missing data on debt. This ensures that we capture the number and size of firms in each age-size bin correctly, regardless of whether they have missing data on debt.

60-90%, and 90%+ size bins respectively.[22] We calculate these percentile-based size bins in our model exactly as in the data. We choose z_4^S to normalize aggregate GDP to 1, and choose the relative values of z_1^S, z_2^S, and z_3^S to match the average employment inside the 0-30%, 30-60%, and 60-90% size bins respectively. The fraction of firms in each percentile-based size bin is simply given by their definition.[23]

Table 3: Firm distributions in the model and data

	Fraction of firms				Average employment			
Size	0-30	30-60	60-90	90+	0-30	30-60	60-90	90+
Model (s.s. cali)	0.30	0.30	0.30	0.10	2.02	5.73	16.18	138.05
Model (b.c. cali)	0.30	0.30	0.30	0.10	1.99	5.81	16.02	137.86
Data	0.37	0.25	0.28	0.10	1.94	5.54	15.51	146.76

(a) Size distribution

	Fraction of firms					Average employment				
Age	0	1-3	4-8	9-19	20+	0	1-3	4-8	9-19	20+
Model (s.s. cali)	0.08	0.19	0.21	0.27	0.25	8.71	13.2	18.9	20.46	32.79
Model (b.c. cali)	0.08	0.19	0.21	0.27	0.25	8.71	12.89	18.11	21.54	32.34
Data	0.05	0.18	0.25	0.24	0.28	8.67	11.40	15.81	22.27	32.33

(b) Age distribution

Note: Firm age and size distributions in the model and data. "Model (s.s. cali)" refers to the model calibrated using the "steady state" calibration, and "Model (b.c. cali)" refers to the "cyclical" calibration. Size bins refer to percentile groups and age bins to age in years since birth. Average employment refers to total employment in the bin divided by the number of firms in the bin. In the data, the number of firms in, e.g., the 0-30% percentile bin is not exactly 30% of firms due to the fact that many small firms have exactly the same number of employees in the data, and hence lie on the boundaries of the sets.

Moving on to the firm age distribution, we target both the distribution of the number of firms by age (i.e. the exit rates) and the distribution of total employment by firm age.

[22] Note that since firm grow over their lifetime due to financial frictions, not all firms in, for example, the 30-60% percentile bin in the model will be from the $s = 2$ type. However, type $s = 2$ firms form the vast majority of firms in that bin, which allows us to choose z_2^S to target average features of firms in that percentile bin. In the calibration, 100% of firms in the 0-30% size bin have z_1^S, 82% of firms in the 30-60% size bin have z_2^S, 94% of firms in the 60-90% size bin have z_3^S, and 96% of firms in the 90%+ size bin have z_4^S.

[23] In the data, the fraction of firms in, for example, the 0-30% bin is not exactly 30%, but is instead 37%. This is due to rounding, as we define our size bins based on the number of employees at the firm, which is an integer number, meaning that a discrete mass of firms may sit at the boundary.

To target the exit rate we use data from Andersen and Rozsypal (2021), who calculate exit rates by firm age for the Danish economy. Using their data, we calculate that firms aged 0 have an exit rate 2.16 times higher than firms aged 16+, and firms aged 6 have an exit rate 1.33 times higher. We target these ratios, as well as an overall average exit rate of 8% per year, using the exit rates ζ_y and ζ_o and the speed at which firms transition from young to old, α_G. We target the distribution of employment by firm age in two ways: we match the average size of firms aged 0, and aged 20+ years old. For the former, we follow Khan and Thomas (2013) and use the initial net worth of entrants, here our parameter n_0, to target the size of firms at age 0. For the latter, we use the superstar firms to target the high employment share of very old firms. Their productivity level, $z_\star$, is used to target the average employment of firms aged 20+ years old. Intuitively, the superstar shock is very rare, and therefore only occurs for firms on average when they are very old.

A complete list of parameters is given in Table 10 in the appendix, as well as further details of firm policies in steady state.[24] Each parameter is adjusted to hit one moment, and we are able to hit all moments exactly, stopping when the error between all model and data moments falls below 5%. The model fits the firm age and size distributions extremely well, including in age bins which we did not target, as shown in Table 3. Several features of the calibration are of note in helping us hit these distributions. Firstly, the permanent productivity heterogeneity across size types s allows us to match the very wide firm size distribution in the data, where 30% of firms have on average 1.94 employees, while the largest 10% have on average 146.76 employees. Secondly, financial frictions are used to explain firm growth in the early years of the lifecycle: age 0 firms have 8.67 employees on average in the data, while firms aged 4-8 have 15.81. The model generates this growth because entrants are financially constrained (they start with only 39% of the net worth needed to reach their optimal size) and grow as they overcome this friction. Finally, the superstar shock is needed to explain why the oldest firms (20+) are so large, which the model explains by selecting a few superstars whose higher productivity leads to them having a size of nearly 700 employees each later in life. Further details of the lifecycle dynamics can be seen in Figures 15 and 16 in the appendix, where we plot average employment and the fraction of firms who are constrained by age and size.

[24] Specifically, in Figure 15 we plot log employment by age-size bin and compare it to the data, and in Figure 16 we plot the fraction of firms financially constrained by age and size bin.

4.2.2 Cyclical performance of the "steady state" calibration

We now show that the "steady state" calibration cannot match our new facts on cyclicality by joint age-size bin. As the cyclicality of different firm groups may depend on the shock hitting the economy, we begin by exploring the response of the economy to three different shocks. Given that we are primarily studying the Financial Crisis, we choose two financial shocks, and one real shock. In all cases, the size of the shock is chosen to generate a 1% fall in GDP which mostly dies out within three years, with the focus instead being on how the response to the shock differs across different firm groups.[25]

The results of this exercise are plotted in Figure 5, where we display the cyclicality of our joint firm age-size groups using the exactly the same regression approach that we previously applied to the data. We use employment as our firm-level outcome measure, which is regressed on real GDP growth using the specification (2). In Figure 5(d) we plot the data,[26] and in Figures 5(a) to (c) we plot results from the model. We plot the *relative* regression coefficients, defined as the regression coefficient for each age-size bin divided by the absolute value of the regression coefficient for the oldest-largest (age 20+ size 90%+) bin. Thus each value gives the bin's cyclicality relative to the oldest-largest age-size bin.[27]

We consider three aggregate shocks in the model. Panel (a) gives the results of an exogenous tightening of the borrowing constraint, represented as a reduction in the parameter $\bar{\phi}$. This shock was previously used in Khan and Thomas (2013), for example, to represent a financial crisis. Panel (b) gives the results an increase in the interest rate, r, charged to all firms. This represents either an increase in discount rates (Hall (2017)), or an increase in spreads charged to firms due to (for example) problems in the banking sector, modelled as an (unchanged) risk-free rate plus an increased spread charged to firms for borrowing.[28] Finally, Panel (c) gives the results of an aggregate TFP shock, which we

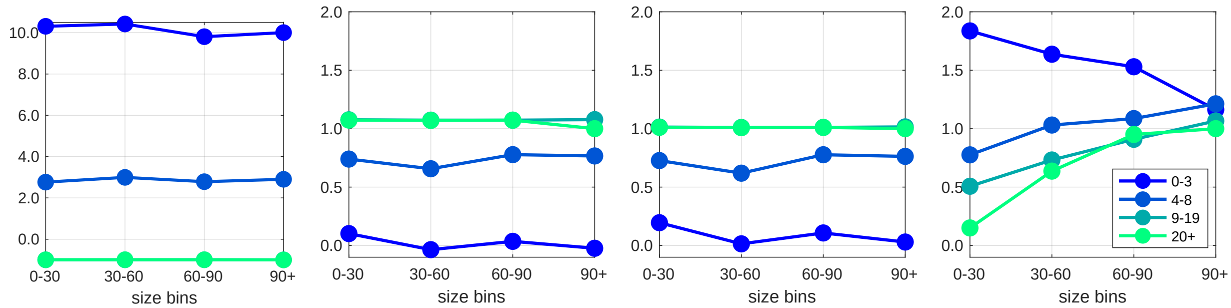
[25] Specifically, suppose we shock a parameter x, by allowing it to vary with time, t. Then at time 0 the parameter unanticipatedly jumps to its new value, x_0, and then recovers back to its original steady-state value, x_{ss}, according to the deterministic process $\dot{x}_t = -\rho_x(x_t - x_{ss})$. We set $\rho_x = 0.9$, and compute the perfect foresight transitions of the economy to this shock using the so-called "MIT shock" approach. The shock has mostly died out within three years, and we simulate the transition for 20 years, confirming that raising this number, or choosing a finer time grid, has no effect on the results. Sample paths for the shocks and aggregates are plotted in Figure 17 in the appendix.

[26] This simply repeats the previously-shown plot Figure 3(b).

[27] For the non-relative values of the regression coefficients see Figure 3 for the data, and Appendix B.3 for the model experiments.

[28] See Del Negro et al. (2017) and Gertler and Karadi (2011) for examples of representative agent mod-

Figure 5: Cyclical response of age-size groups to shocks in the "steady state" calibration



(a) Collateral ($\bar{\phi}$) (b) Discounts (r) (c) TFP (z) (d) Data

Note: Panels give relative regression coefficients from regressions of firm-level growth rates of employment on aggregate GDP growth, computed from model simulated data for our recession experiments. Values are the regression coefficient for that age-size bin divided by the (absolute value of) the regression coefficient for the oldest-largest (age 20+ size 90%+) bin.The regressions are on firm-level data aggregated to the yearly level and treated in the same way as the data. Size bins are percentiles, and each line refers to a different firm age group. The final panel gives the results from real-world data, and the remaining panels from model data.

model as a proportionally equal reduction in TFP at all firms. We include this shock in order to consider non-financial shocks, and how they compare to the two financial shocks.

Inspecting Figure 5 we see that this calibration of the model does not replicate the data in response to any of the three shocks considered. The collateral constraint shock (Panel (a)) comes closest, since in response to this shock young firms are more responsive than old firms, for a given size group. This is also true in the data: For example, in panel (d) we see that age 0-3 firms (blue line) are more cyclical than age 20+ firms (green line) for all size groups. In the model, young firms are more responsive to the collateral constraint shock than old firms because young firms are more likely to be financially constrained than old firms. For young firms, who have limited net worth, a reduction in borrowing forces them to reduce their investment in capital, and hence also their employment and output. For older firms, who have accumulated sufficient net worth to become financially unconstrained, a tightening of the borrowing constraint has no effect on their real variables, as they are already away from their borrowing constraints. However, the collateral constraint shock has an important failing relative to the data, which is that older firms are not responsive to the shock at all, whereas in panel (d) old and large firms (e.g. age 20+

els where a financial recession is modelled as an increase in financial spreads charged to all firms, either exogenously or endogenously.

and in the 90%+ size bin) are very cyclical in the data.[29]

This gradient within older firms—that larger older firms are very cyclical, while smaller older firms are not—is something that none of the three shocks can match. The discount rate (Panel (b)) and TFP (Panel (c)) shocks both generate that older firms are more cyclical than young. This is because these shocks affect the marginal incentives of firms, encouraging unconstrained firms to shrink in response to higher costs or lower productivity. This affects old firms more, since young firms are up against their borrowing constraints and hence unresponsive to such marginal incentives. But in response to these shocks all old firms (large or small) are equally cyclical, in contrast to the data. In fact, there is no size gradient within any age bin in all three of the model panels, in contrast to the data in Panel (d) where large firms tend to be more cyclical, conditioning on age. This implies that no combination of the three shocks can fully replicate the data in Panel (d), which represents a failing of the basic calibration according to our new data.[30] It is for this reason that we turn to our extended "cyclical" calibration, in order to understand the features needed to match the data.

4.3 Result 2: A "cyclical" calibration

In this section we describe what we call the "cyclical" calibration. This extends the model to include heterogeneous returns to scale and entrant net worth by firm size, to allow it to match our new facts on the cyclicality of firms by joint age-size bin. Most calibration targets remain the same as in the "steady state" calibration, and we describe the new features in the section below.

[29] Additionally, young firms are much too cyclical in response to the collateral shock: note that the coefficients on the youngest firms in Panel (a) are around 6, while in the data they are around 1.2. This is because a pure collateral constraint shock only affects young firms, while leaving older (and hence larger) firms unaffected, making the change in employment at young firms much larger than the change in aggregate GDP.

[30] While other types of shocks are also possible, we believe that these three shocks span quite well the types effects that shocks have on the firm age-size distribution in this model. In particular, we considered both shocks to the quantity (collateral constraint) and price (spread) of borrowing. We then considered a generic TFP shock, whose effects are similar (in terms of the firm age-size distribution) as any shocks to demand, productivity, or factor prices which affect firms in the same proportional manner. Since all firms in the basic model face the same factor prices, any shocks which transmit via TFP, demand, or factor prices will therefore have the same basic effects on the firm age-size distribution as the TFP shock.

4.3.1 "Cyclical" calibration details

Our "cyclical" calibration approach consists of choosing parameters of the model to match *both* steady state moments, *and* the cyclicality of firm age-size groups in the data from our regression results. Since the cyclicality of age-size groups depends on which shocks hit the economy (as we showed in the last section) we therefore jointly estimate the parameters of the model and the shocks which drive the recession episode.

Along with our previous targets, our calibration exercise targets 1) the relative cyclicality of firms by age and size (Section 3.2), 2) the size of the recession, and 3) the average growth rate of young-small firms. We choose two shock processes and two novel features of the model to match these firm level outcomes in this exercise. We jointly choose these nine new parameters to minimise the distance to all of the nine new moments discussed in this section and discuss the moments most closely related to each parameter in the text below.[31] When targeting the cyclicality of any firm age-size bin, we target the cyclicality *relative* to the largest, oldest firm group (90%+ size, age 20+) by targeting the ratio of their regression coefficients.

Firstly, we modify the calibration of the steady state of the model. In our "steady state" calibration, we used a common parameter n^e to adjust entry net worth to match the average size of aged 0 firms. In the "cyclical" calibration, we will use entry net worth to target cyclical moments, and so take a different approach. We now incorporate within-firm productivity growth with age, and instead calibrate $z_1^G < 1$ to match the average size of aged 0 firms.

We next describe the estimation of the shocks. In this section, we choose to focus only on financial shocks, and assume that the recession is driven jointly by the collateral and discount rate shock. We estimate the initial values of these shocks — denoted $\bar{\phi}_0$ and r_0 respectively — and allow them to fade back to steady state at the same rate of 0.9, as described in Footnote 25.[32] As small-young firms will be the most financially constrained

[31] Since calculating these moments requires simulating the recession experiment, we split the estimation into a two-layer procedure which exactly hits the steady state moments in an inner loop, and then uses a simulated minimum distance routine in the outer-loop, to minimize the sum of squared errors of the cyclical moments. More details are in Appendix B.1.

[32] As can be seen from Figure 5 panels (b) and (c), the discount rate shock and TFP shock generate very similar cyclicality across the age-size distribution. In this sense, the cyclicality data does not precisely identify the discount rate shock from the TFP shock, and we focus on the discount shock to provide a simple interpretation of the recession as a purely financial shock. However, other data do suggest that there was no great cyclicality of TFP during this period. In particular, we investigate the cyclicality of labor

and hence affected by the financial shock, we choose the size of the collateral constraint shock in order to match the relative cyclicality of small-young (0-30%, 0-3) firms, which is $2.69/1.46 = 1.84$ in the data in Figure 5(d). We also aim for a 5% total GDP fall an impact, comparable to the fall in GDP in Denmark in the first year of the Great Recession, and choose the size of the discount rate shock to match this fall, for a given size of collateral constraint shock.

We now turn to estimating cyclical dynamics by joint firm age-size bin. We target two key features of the data. Firstly, *among older firms* in the data, large firms are more cyclical than small firms. We argued that this is unlikely to be driven by financial frictions, and so we need another model feature to match this fact, which we match instead by calibrating the differing degrees of returns to scale across size types s. Crucially, differing degrees of returns to scale also imply different responsiveness to shocks, in a way that very natural meshes with our empirical findings. Small firms are likely to be small not because they are unproductive in a TFP sense, but because they are fundamentally very different businesses to larger firms (think a local shop versus Carlsberg Group), with a smaller business scope which could imply lower returns to scale. In terms of economic theory, firms with more decreasing returns to scale are also endogenously less responsive to shocks, which then gives a natural explanation for why larger firm are more cyclical in our data.[33] To reduce the number of free parameters, η_4 is chosen to normalize the average returns to scale to one.[34] We choose η_1, η_2, and η_3 to match the relative cyclicality of size bins 0-30%, 30-60%, and 60-90% among the oldest firm group (aged 20+).

Secondly, *among younger firms* in the data, large firms are less cyclical than small firms. We use this data to discipline the degree of financial frictions faced at birth by firm size, by allowing for differing net worth at birth for firms of different size types s. Specifically, we set n^e_2, n^e_3, and n^e_4 to match the relative cyclicality of size bins 30-60%, 60-90%, and 90%+

productivity by firm, and find that the labor productivity of large firms is essentially acyclical, despite the high cyclicality of their value added and employment. We take this as evidence that a TFP shock is unlikely to be driving the employment of this firm group.

[33] To see this, consider a simple static model where a competitive firm produces output using capital only, with returns to scale α: $y = zk^\alpha$. They rent capital at price r. Their profit maximization problem $\max_k zk^\alpha - rk$ implies optimal capital $k = (\alpha z/r)^{\frac{1}{1-\alpha}}$. The elasticity of their capital choice to a change in productivity is $\frac{\partial \log k}{\partial \log z} = \frac{1}{1-\alpha}$. Thus the more decreasing returns to scale (lower α), the less responsive is the firm to changes in TFP (lower $\partial \log k / \partial \log z$). The same is true for a change in the factor price, r.

[34] By this we mean that, at the ergodic distribution, exogenously increasing all inputs proportionally at all firms by a factor λ raises aggregate output by a factor λ. Intuitively, this means that an appropriately weighted average of η_s across firms is equal to one.

among the youngest firm group (aged 0-3). By making n_2^e greater than n_1^e, for example, this makes type 2 entrants less financially constrained than type 1, and hence reduces cyclicality for larger firms within the youngest age group. Recall that the cyclicality of the aged 0-3, size 0-30% group was already "used" to calibrated the size of the financial shock. We therefore introduce one final moment to calibrate n_1^e. We choose this parameter to match the average employment growth rate of young (0-3) firms in the 0-30% size bin, since lower initial net worth implies that they start further below their optimal size and therefore will grow faster in their youth.[35]

A complete list of parameters is given in Table 10 in the appendix. The model continues to give a very close match to the firm age and size distributions, as shown in Table 3. The values of the outer-loop moments in the data and the estimated model are given in Table 9, and the average error (computed as the square root of the mean squared error) is equal to 2.53%. We provide additional discussion in support of our heterogeneous returns to scale and initial net worth assumptions in Appendix B.

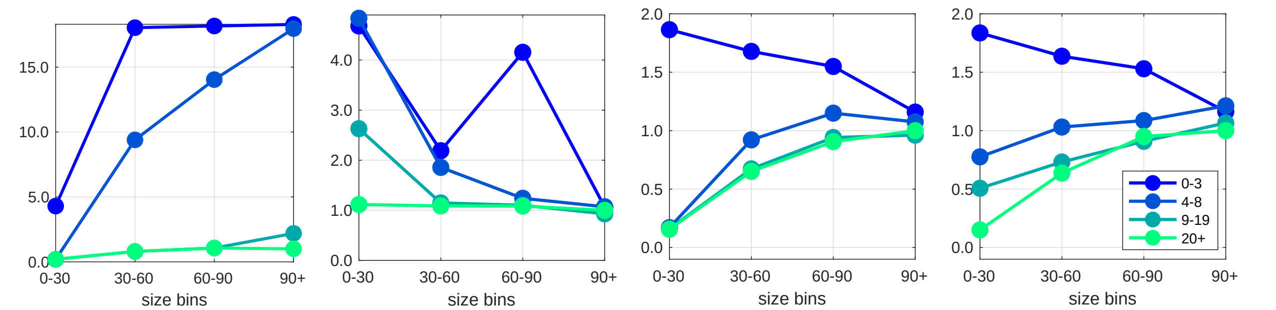
4.3.2 Cyclical performance of the "cyclical" calibration

We plot the results of our "cyclical" calibration in Figure 6, again using relative regression coefficients. Panel (d) again gives the data, and panel (c) plots the regressions from the cyclical calibration. The model is able to match the key features of the data, which we targeted in the calibration. In particular, 1) young firms are more cyclical than old firms, and 2) among older firms, large firms are more cyclical than small firms, while the opposite is true among younger firms. Sample paths for the shocks and aggregates are plotted in Figure 18 in the appendix.

The "cyclical" calibration is able to match this data well due to the novel features we added to the model, and the mechanisms by which they work were described in the previous section. To show that these features contribute to cyclicality as discussed, in panels (a) and (b) we plot the cyclicality results for two recalibrated models which incorporate only one feature each. Panel (a) shows that adding heterogeneity in returns to scale makes

[35] This data is plotted in Figure 1(e). However, since that data excludes firms for whom we are missing data on debt, we target a slightly different version of this figure which includes all firms regardless of whether they have available debt data. This alternative sample is given, along with the model results, in Figure 7. Note that we can alternatively think of n_1^e being used to target the relative cyclicality of aged 0-3, size 0-30% firms, symmetrically with how n_2^e to n_4^e are chosen, since all parameters are jointly used to minimize the distance to all moments.

Figure 6: Cyclical response of age-size groups in the “cyclical” calibration



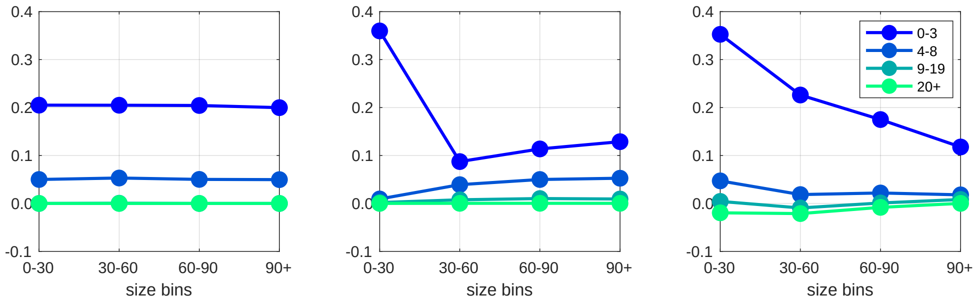
(a) Het. η_s (b) Het. n_s^e (c) Full model (d) Data

Note: Panels give relative regression coefficients from regressions of firm-level growth rates of employment on aggregate GDP growth, computed from model simulated data for our recession experiments. Values are the regression coefficient for that age-size bin divided by the (absolute value of) the regression coefficient for the oldest-largest (age 20+ size 90%+) bin. The regressions are on firm-level data aggregated to the yearly level and treated in the same way as the data. Size bins are percentiles, and each line refers to a different firm age group. The final panel gives the results from real-world data, and the remaining panels from model data.

large firms more cyclical. However, alone it also means that large-old firms are much too cyclical, as the excess cyclicality of the young does not decrease with size, as it does in the data. Panel (b) shows that adding heterogeneity in initial net worth makes the excess cyclicality of the young decrease with size. However, among old firms it does not make large firms more cyclical than small, as they are in the data. Panel (c) shows that putting these together in the full calibration yields the required match to the data.

One important feature of the data was not targeted, and instead serves as an untargeted test of the model. This is the fact that, among young firms, average growth rates are higher for smaller than for larger firms. In Figure 7(c) we plot the data, and in panel (b) we plot the results from our model, where only the value for age 0-3 size 0-30% firms was targeted. We see that the “cyclical” calibration captures the main pattern in the data, in particular the average growth rates of the smallest and largest firms in the age 0-3 group. In contrast, panel (a) shows that the “steady state” calibration fails to match this data because it generates equal growth rates when young for all firm size groups. In our new calibration, smaller firms are born with relatively less net worth, and will be more financially constrained early in life than larger firms.

Figure 7: Average growth rate of employment by age-size bin



(a) "Steady state" cali (b) "Cyclical" cali (c) Data

Note: Panels give regression coefficients from regressions of firm-level growth rates of employment on firm age-size dummies, computed from model simulated data. The regressions are on firm-level data aggregated to the yearly level and treated in the same way as the data. Size bins are percentiles, and each line refers to a different firm age group. The first panel gives results from the "steady state" calibration of the model, the second from the "cyclical" calibration, and the final from real-world data.

4.4 Result 3: Policy implications

Our final exercise is to investigate the implications of our results and model for business cycle policy. These are informed both by our empirical findings on the cyclicality of different firm groups, and the twists they implied to our calibrated model. In particular, we compare policy exercises in both the original "steady state" calibration, and in our new "cyclical" calibration model, which introduced the two new features of i) heterogeneous entrant net worth by size, and ii) heterogeneous returns to scale.

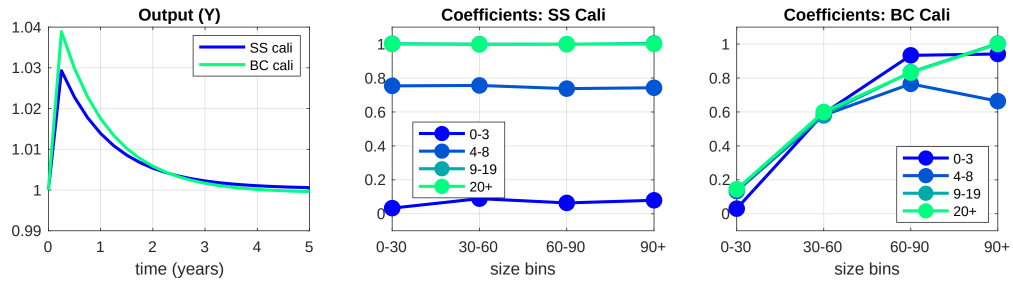
We consider two simple policies, meant to capture two different styles of possible policy intervention during a recession. The first policy we call an "incentive" type policy, which consists of a temporary subsidy to the firm's wage bill. In particular, we introduce a subsidy so that the government pays a fraction τ of all firms' wage bills, with $\tau = 0$ in steady state. We consider a temporary increase of the subsidy to 1% of the wage bill, which fades at rate 0.9 as did our business cycle shocks. We call this policy an incentive type policy because it changes the effective marginal cost of production for firms, and hence incentivises them to expand production. The second policy we call a "balance sheet" policy, which consists of giving debt relief to firms. In this policy, at time 0 the policymaker pays off a fraction x of all firms' debts, reducing their debt from b to $(1-x)b$, and hence increasing their net worth. This policy does not affect the marginal cost of finance for firms, but it does increase the net worth of firms, and thus the access to debt for financially constrained firms. We consider a one-off 20% debt forgiveness at time 0.

We start with the labor subsidy policy, shown in Figure 8(a). The left panel gives the response of aggregate output to the policy, showing that the policy is more effective in our new model than in the original calibration (3.9% output rise on impact vs. 2.9%). The centre and right panels give the regression coefficients measuring responsiveness by age-size groups. These reveal why the policy is more powerful in the new calibration, as the age-size responses are markedly different between the two models. In particular, in the new calibration, the policy now has a clear firm size dimension, with large firms being more responsive to the policy even conditioning on age. This creates a composition effect which boosts the aggregate response, as the firms who respond the most also happen to be large and hence more important for aggregate output. As with earlier results, this follows directly from the fact that large firms having less decreasing returns to scale makes them more sensitive to changes in their marginal costs.

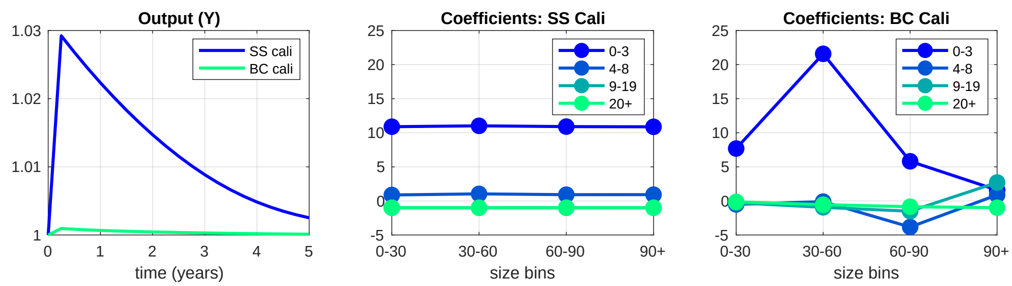
We now turn to the debt forgiveness policy, shown in Figure 8(b). The response of aggregate output in the left panel now shows that the policy is instead less effective in our new model (0.2% output rise on impact vs. 3%). In both calibrations, the policy has a persistent effect on output which lasts many years, despite the policy being enacted only at time 0. Firm's financial positions are slow moving, and so by helping firms at time 0 they remain less financially constrained for the rest of their lifecycle. The regression results in the centre and right panels reveal why the policy is weaker in the new model. In both models, the policy has the largest effect on young firms, as these firms are more likely to be financially constrained and hence benefit from debt relief. However, in our new calibration, the data suggested that *large* young firms are less financially constrained. Hence the right panel finds that responsiveness to the policy is declining with size, from the 30% percentile and above, among the young (age 0-3) group. This creates a composition effect which dampens the aggregate response, because the firms responding to the policy are now smaller on average than in the steady state calibration.

In conclusion, our empirical results inform changes to a standard heterogeneous firm model which have important policy implications. Some policies (incentive based) are more effective, and others are less (debt forgiveness), with the changes due to the fact that firms of different ages and sizes now respond differently to these policies.

Figure 8: Effect of two policies in standard vs calibrated model



(a) Labor subsidy



(b) Debt relief

Note: The left panel gives the response of aggregate GDP to the policy experiment, in both the "steady state" and "cyclical" calibrations. The center and right panel give relative regression coefficients (relative to the oldest largest age-size bin) from regressions of firm-level growth rates of employment on aggregate GDP growth, computed from model simulated data for each calibration respectively.

5 Conclusion

In this paper we documented novel facts about the cyclicality firms by age and size, with particular attention paid to the interaction between the two, and the role of finance. Using high quality registry data from the universe of Danish firms, we first document that employment, turnover, and assets are more sensitive to the business cycle at younger firms than older firms, but that the relationship between size and cyclicality is more complicated. Among older firms, large firms are more cyclical than small, while among young firms, small firms are more cyclical than large.

These results are possible because our dataset contains explicit information about when firms are formed, allowing us to construct a high quality measure of firms actual age from legal inception. This distinguishes our dataset from other sources where it is either not possible to measure age, or only to do so from the age that firms go public. This allows us to look at the cyclicality of very young firms, which is where we find the strongest excess cyclicality. We additionally have data for firms of all sizes, allowing us to investigate cyclicality for even the smallest of firms. We use this data to additionally provide a detailed investigation of firm outcomes and growth rates across different size and age groups.

Given that our dataset contains detailed financial variables, we then investigate the role of finance in driving the excess cyclicality of different firm groups. We find that young firms have higher leverage than old firms, and hence are more likely to be financially constrained. They additionally are typically trying to expand their leverage, while leverage is typically shrinking at older firms. On the other hand, leverage ratios are remarkably similar across firm size groups, after controlling for firm age. Studying cyclicality by leverage, high leverage firms are more cyclical than low leverage firms. Taking these results together, we argue that the excess cyclicality of young firms is plausibly linked to financial frictions, while the same is less likely to be true for larger old firms.

We then use these insights to build a quantitative heterogeneous firm model, and investigate the extensions to a standard calibration needed to replicate our new facts. We find that standard calibrations struggle to match cyclicalities across age, size, and joint age-size bins at the same time. Part of the problem is that in standard models age and size are too closely linked, as young firms tend to be financially constrained and, hence, smaller. Two extensions bring the model closer to the data. Firstly, we introduce heterogeneous returns to scale, so that large firms have less decreasing returns to scale. This can parsimoniously explain why they are larger and more cyclical. Secondly, we allow larger firms to be born richer, and hence less financially constrained. This explains why, among large firms, cyclicality does not depend on firm age, as in the data. Together, these extensions bring the model's implications for cyclicality by joint age-size bin in line with the data. We finally use our model to investigate the effect of recession-fighting policies, and how they transmit through the firm age and size distributions. A key implication of these exercises is that properly matching the responsiveness of firms by age and size can have large effects on the policy implications of our models.

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APPENDICES

FOR ONLINE PUBLICATION ONLY

A Data Appendix

A.1 Additional information on dataset building process

Subject to some minimal threshold on economic activity,[36] all firms are legally obliged to report data to SKAT or DST, which are then collected in these databased. We drop all observations that we deem as inactive by our definition, that is firms that provide no information about employment, sales, value added, or profits.

We also drop all firms that never in their life employ more than one worker.[37] Finally, we also drop firms listed as non-profits as well as entities controlled by government at any level. In our baseline exercises, we include only firms that do not exit in the current or the next year. We thus do not separately investigate the role of firm entry or exit in driving cyclicality.

Sometimes, information about a particular variable for a given firm is missing in the aforementioned registers. This is more likely for for financial rather than real variables, for smaller firms and for firms in the process of exiting. The year of exit also causes problems for variables that measure stock at a given point in time, rather than annual average. For these reasons, we only consider observations for firms that are not exiting in a given year. Additionally, we require lagged information about assets and debt to be present for the regressions. This way we make sure that the estimated effects of including or excluding leverage controls are not the result of changing the set of firms in the sample.

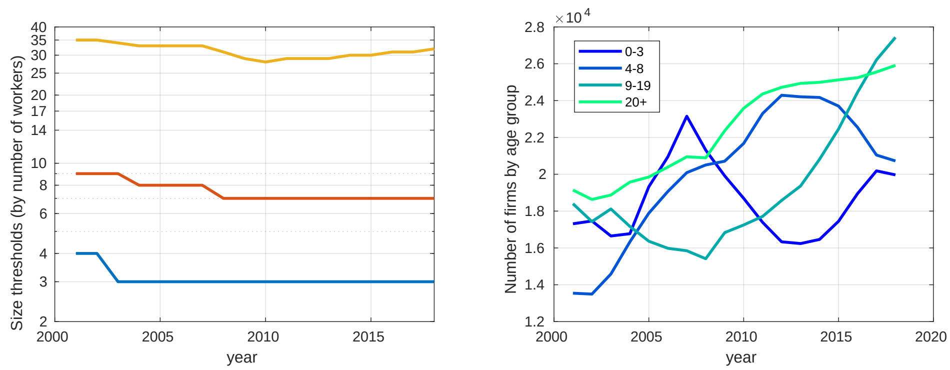
[36] In most situation, firms that report employment that corresponds to less than 0.5 full time worker are considered inactive by DST, but still present in our data.

[37] We do this to eliminate sole proprietorship firms and also firms that exist due to tax optimisation purposes.

A.2 Evolution of size thresholds and age distribution

While the size thresholds are relatively stable over time, the number of firms in different age groups changes over time, both in absolute numbers but also in group size ranking. Pro-cyclical firm entry generates stronger and weaker cohorts which propagates over time to higher age groups. While interesting on its own (Sedláček and Sterk, 2017), we ignore the potentially link between average firm quality and the state of the business cycle at the time of entry.

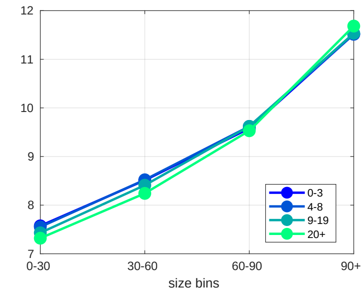
Figure 9: Size Thresholds and Number of Firms in Different Age Bins.



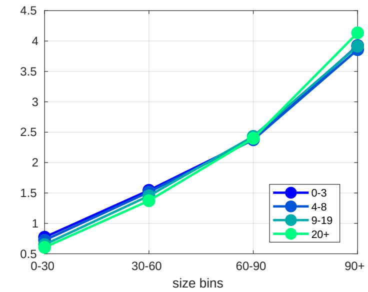
(a) Size thresholds (log scale on y-axis) (b) Number of firms in each age group over time

A.3 Results with assets as the size sorting variable

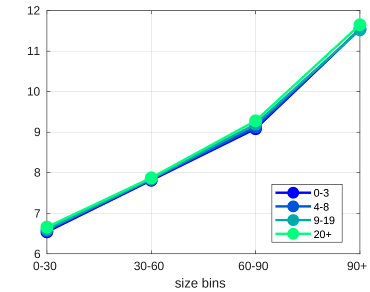
Figure 10: Average Levels with Full Interaction of Size and Age (Regression (1)).



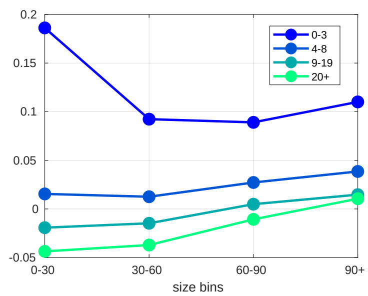
(a) Sales



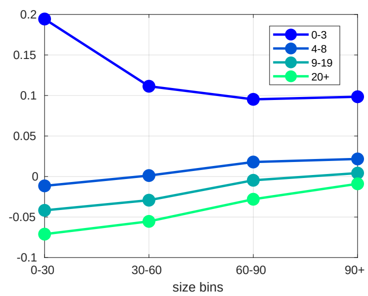
(b) Employment



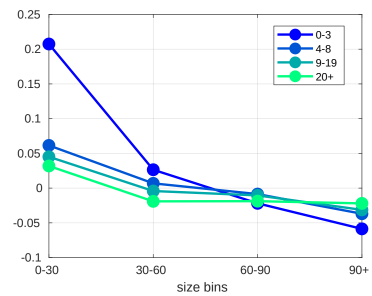
(c) Assets



(d) Growth rate of sales



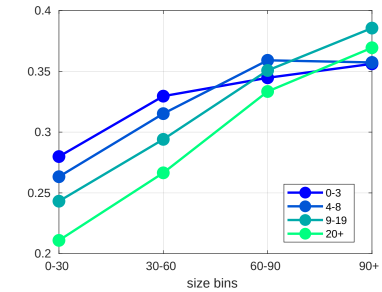
(e) Growth rate of employment



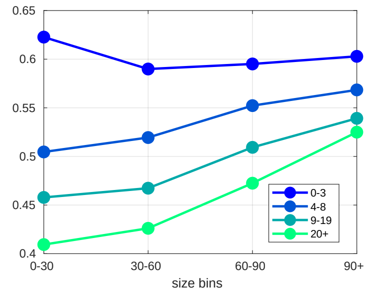
(f) Growth rate of assets

Note: Average level is computed as coefficient α_{lk} from regression (1). Different lines correspond to firms of different size bins whereas firm age bin is on x axis.

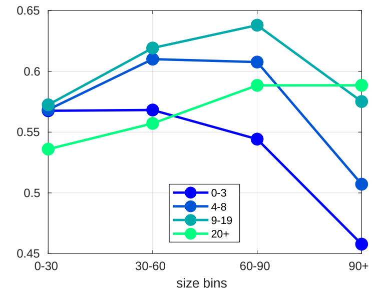
Figure 11: Average Growth Rates, Full Interaction of Size and Age (Regression (1)).



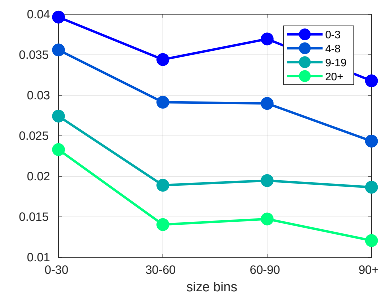
(a) Share with positive investment



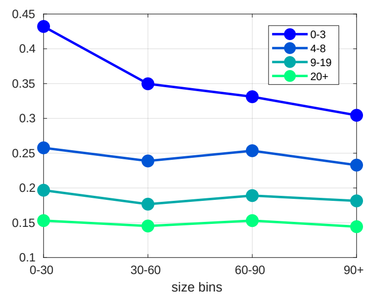
(b) Share with above median growth in sales



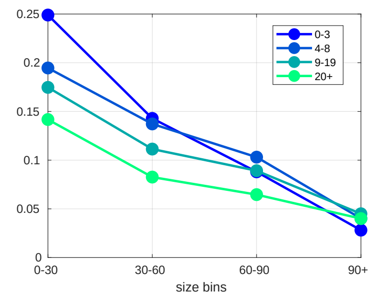
(c) Share with mildly positive profits



(d) Share with high investment



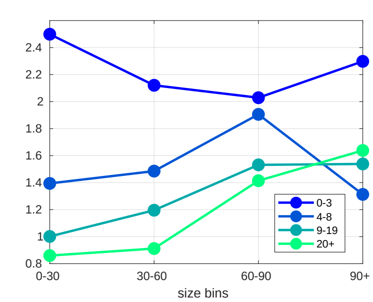
(e) Share with high sales growth



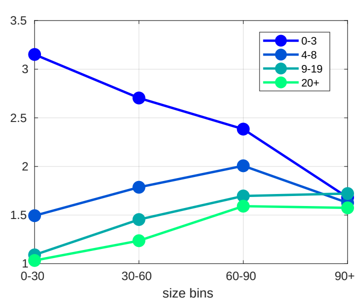
(f) Share with high profits

Note: high investment defined as investment/assets>0.25. Mild profits are defined as profits/assets>0.03, high profits are defined as profits/assets>0.25. High sales growth is defined as sales growth larger than 75th percentile of sales growth distribution.

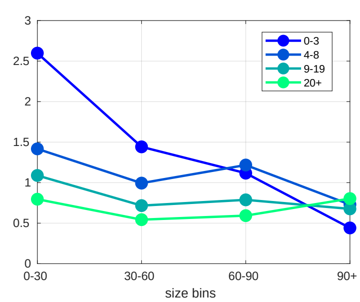
Figure 12: Cyclicality by Size and Age.



(a) Sales



(b) Employment



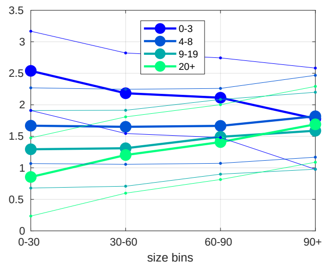
(c) Assets

Note: Cyclicality is computed as coefficient β_{lk} from regression (2). Size bin defined by assets.

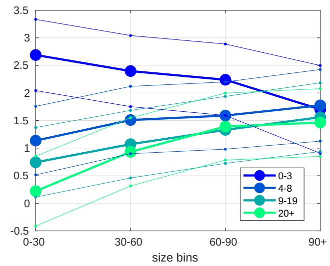
A.4 No size-age interactions

As Figure 13 clearly demonstrates, ignoring the interaction between size and age leads to finding that the cyclicality gradient with respect to size is positive across all firms. This mistake is the largest for the youngest groups of firms and gets smaller with firm age. This heterogeneity also implies that results that are based on dataset that only include some type of firms (i.e. large publicly trded firms) might not be externally valid for the whole population of firms.

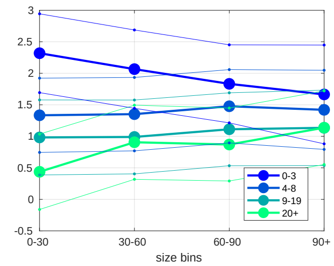
Figure 13: Comparison between the baseline specification (regression (2)) with the interaction between size and age (first row), a specification without the interaction (second row) and specifications with size (row 3) and age (row 4) only.



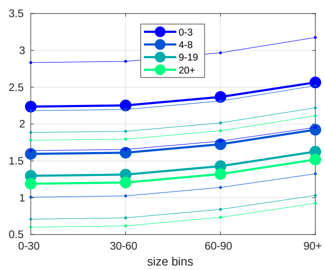
(a) Sales - baseline



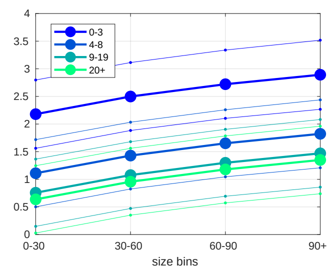
(b) Employment - baseline



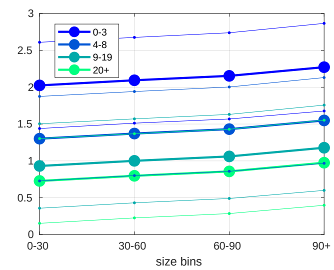
(c) Assets - baseline



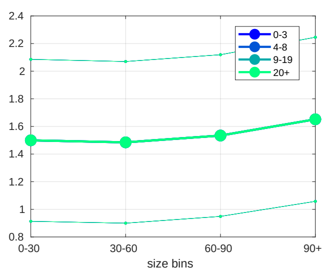
(d) Sales - no size x age interaction



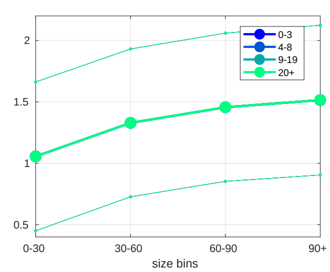
(e) Employment - no size x age interaction



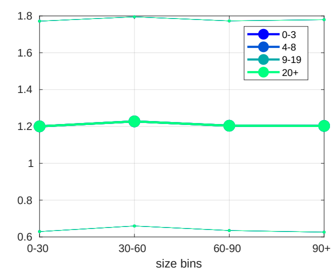
(f) Assets - no size x age interaction



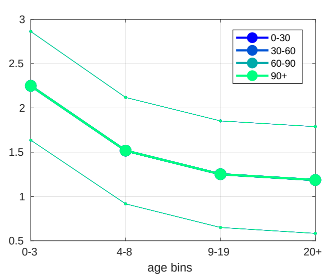
(g) Sales - only size, no age



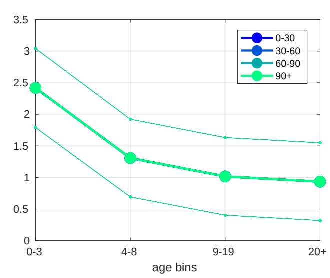
(h) Employment - only size, no age



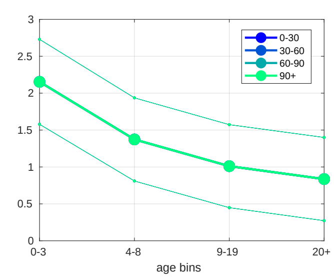
(i) Assets - only size, no age



(j) Sales - only age, no size



(k) Employment - only age, no size

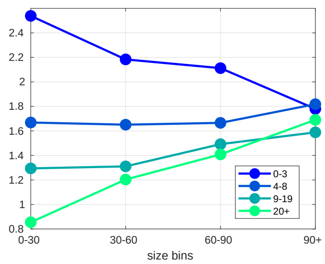


(l) Assets - only age, no size

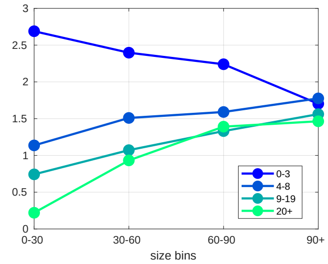
Note: Solid lines represent the mean estiamted effect and the thin lines of the corresponding color represent $\pm$ *2 standard error confidence bars*

A.5 FE vs OLS for cyclicality

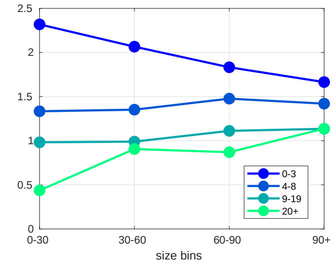
Figure 14: Baseline Specification (OLS) and Additional Firm Fixed Effect (FE) Deliver Qualitatively Similar Results for Cyclicality



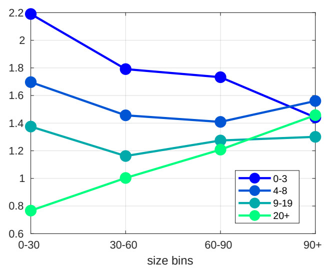
(a) Sales (OLS)



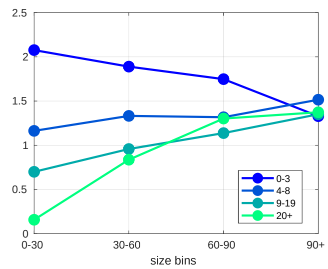
(b) Employment (OLS)



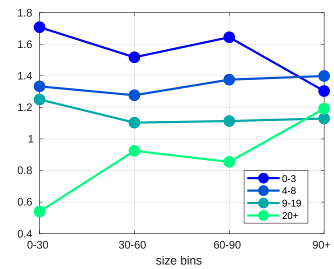
(c) Assets (OLS)



(d) Sales (FE)



(e) Employment (FE)



(f) Assets (FE)

Note: Cyclicality is computed as coefficient β_{lk} from regression (2). Comparison between the specification with interaction between size and age with baseline (OLS) in the first row and with FE on firm level in the second row. Apart from the youngest firms, cyclicality increases with age and the level is shifted down by as firms get older.

A.6 Regression results

Table 4: Cyclicality of Turnover, Employment and Assets by Firm Size and Age

	(1) sales	(2) employment	(3) assets
y	1.69***	1.46***	1.14***
	(5.62)	(4.77)	(3.89)
0-30 × y	-0.84***	-1.24***	-0.70***
	(-5.88)	(-9.94)	(-5.48)
30-60 × y	-0.49***	-0.53***	-0.23**
	(-4.05)	(-5.50)	(-2.10)
60-90 × y	-0.28***	-0.07	-0.27***
	(-2.87)	(-0.97)	(-2.93)
0-3 × y	0.09	0.24	0.53*
	(0.31)	(0.89)	(1.89)
4-8 × y	0.13	0.31**	0.28*
	(0.79)	(2.29)	(1.86)
9-19 × y	-0.10	0.10	-0.00
	(-0.85)	(0.99)	(-0.02)
0-30 × 0-3 × y	1.59***	2.23***	1.35***
	(4.72)	(7.06)	(4.11)
0-30 × 4-8 × y	0.69***	0.60***	0.61***
	(3.13)	(3.15)	(2.99)
0-30 × 9-19 × y	0.54***	0.43**	0.55***
	(2.73)	(2.40)	(2.93)
30-60 × 0-3 × y	0.89***	1.23***	0.63**
	(2.69)	(4.04)	(1.96)
30-60 × 4-8 × y	0.32	0.27	0.16
	(1.58)	(1.58)	(0.84)
30-60 × 9-19 × y	0.21	0.04	0.09
	(1.19)	(0.30)	(0.50)
30-60 × 20+ × y	0.00	0.00	0.00
	(.)	(.)	(.)
60-90 × 0-3 × y	0.61*	0.61**	0.43
	(1.91)	(2.07)	(1.38)
60-90 × 4-8 × y	0.13	-0.11	0.32*
	(0.70)	(-0.73)	(1.83)
60-90 × 9-19 × y	0.19	-0.16	0.24*
	(1.24)	(-1.33)	(1.66)
Observations	595547	674084	594646
Adjusted R^2	0.039	0.052	0.028

t statistics in parentheses
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Note: Standard error clustered at firm level. Coefficients related to sectors omitted. The coefficients relative to the base groups defined as the oldest firms (20+) and the largest groups (size percentile 90+).

Table 5: Cyclicality of Turnover, Employment and Assets by Firm Size and Age, Alternative Specifications

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
	sales	employment	assets	sales	employment	assets	sales	employment	assets	sales	employment	assets
y	1.68***	1.59***	1.29***	1.52***	1.35***	0.98***	1.19***	0.93***	0.84***	1.65***	1.51***	1.20***
	(5.66)	(5.69)	(4.19)	(5.11)	(4.42)	(3.38)	(3.94)	(3.03)	(2.97)	(5.56)	(4.97)	(4.17)
0-30 × y	-0.69***	-1.22***	-0.65***	-0.33***	-0.71***	-0.25***				-0.15*	-0.46***	-0.00
	(-4.73)	(-9.59)	(-4.92)	(-4.02)	(-9.87)	(-3.16)				(-1.90)	(-6.45)	(-0.03)
30-60 × y	-0.45***	-0.54***	-0.27**	-0.31***	-0.39***	-0.18**				-0.17**	-0.19***	0.02
	(-3.71)	(-5.60)	(-2.38)	(-4.11)	(-6.08)	(-2.41)				(-2.24)	(-2.92)	(0.34)
60-90 × y	-0.25**	-0.07	-0.34***	-0.20***	-0.17***	-0.12*				-0.12*	-0.06	0.00
	(-2.50)	(-0.96)	(-3.63)	(-2.92)	(-3.04)	(-1.80)				(-1.76)	(-1.05)	(0.02)
0-3 × y	-0.02	-0.04	0.11	1.05***	1.54***	1.30***	1.06***	1.48***	1.32***			
	(-0.05)	(-0.16)	(0.38)	(12.01)	(19.24)	(14.96)	(12.33)	(18.27)	(15.36)			
4-8 × y	0.10	0.14	0.21	0.40***	0.47***	0.57***	0.33***	0.37***	0.54***			
	(0.61)	(1.03)	(1.30)	(6.59)	(8.86)	(9.58)	(5.48)	(6.97)	(9.14)			
9-19 × y	-0.16	-0.02	-0.06	0.11*	0.12**	0.20***	0.07	0.08	0.18***			
	(-1.26)	(-0.23)	(-0.52)	(1.79)	(2.35)	(3.54)	(1.09)	(1.60)	(3.07)			
0-30 × 0-3 × y	1.44***	1.96***	1.06***									
	(3.98)	(5.88)	(2.98)									
0-30 × 4-8 × y	0.83***	0.86***	0.59***									
	(3.59)	(4.36)	(2.70)									
0-30 × 9-19 × y	0.77***	0.56***	0.77***									
	(3.67)	(3.09)	(3.89)									
30-60 × 0-3 × y	0.80**	1.10***	0.48									
	(2.29)	(3.45)	(1.39)									
30-60 × 4-8 × y	0.35*	0.35**	0.15									
	(1.66)	(2.06)	(0.72)									
30-60 × 9-19 × y	0.32*	0.14	0.24									
	(1.71)	(0.99)	(1.36)									
60-90 × 0-3 × y	0.54	0.49	0.68**									
	(1.59)	(1.59)	(2.03)									
60-90 × 4-8 × y	0.10	-0.13	0.31*									
	(0.51)	(-0.82)	(1.69)									
60-90 × 9-19 × y	0.22	-0.14	0.32**									
	(1.43)	(-1.19)	(2.10)									
Observations	595547	674084	594646	595547	674084	594646	603594	680997	602584	595547	674084	594646
Adjusted R^2	0.030	0.037	0.016	0.038	0.049	0.026	0.040	0.050	0.029	0.020	0.017	0.018

t statistics in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Note: Standard errors clustered at firm level. The coefficients relative to the base groups defined as the oldest firms (20+) and the largest groups (size percentile 90+).

Table 6: Effects of Leverage: Leverage Intensity Defined Within Age-Size Bin

	(1) sales	(2) sales	(3) employment	(4) employment	(5) assets	(6) assets
Constant	0.011	0.007	-0.008	-0.037***	0.028***	0.020**
	(1.75)	(1.07)	(-1.18)	(-5.69)	(4.79)	(3.28)
y	1.551***	1.731***	1.265***	1.448***	1.308***	1.221***
	(5.32)	(5.73)	(4.18)	(4.70)	(4.60)	(4.15)
0-20	-0.033***	-0.033***	-0.016***	-0.015***	0.016***	0.016***
	(-21.03)	(-21.04)	(-11.30)	(-11.01)	(10.28)	(10.50)
20-40	-0.011***	-0.011***	-0.002	-0.001	0.012***	0.013***
	(-7.45)	(-7.32)	(-1.41)	(-1.05)	(7.98)	(8.16)
60-80	0.002	0.002	-0.010***	-0.010***	-0.026***	-0.026***
	(1.37)	(1.41)	(-7.26)	(-7.48)	(-15.58)	(-15.58)
80+	-0.008***	-0.006***	-0.060***	-0.058***	-0.034***	-0.032***
	(-4.59)	(-3.65)	(-38.76)	(-38.12)	(-17.79)	(-17.27)
0-20 $\times$ y	-0.088	-0.087	-0.167**	-0.166**	-0.466***	-0.468***
	(-1.21)	(-1.21)	(-2.62)	(-2.65)	(-6.82)	(-6.86)
20-40 $\times$ y	-0.065	-0.060	-0.057	-0.053	-0.171*	-0.166*
	(-0.92)	(-0.86)	(-0.92)	(-0.87)	(-2.51)	(-2.44)
60-80 $\times$ y	-0.068	-0.066	0.144*	0.143*	0.072	0.071
	(-0.93)	(-0.92)	(2.22)	(2.23)	(0.98)	(0.97)
80+ $\times$ y	-0.050	-0.044	0.280***	0.276***	0.265**	0.272**
	(-0.63)	(-0.56)	(3.88)	(3.83)	(3.14)	(3.22)
Observations	595547	595547	674084	674084	594646	594646
adj-r2	0.015	0.035	0.017	0.053	0.011	0.021
Sectors	yes	yes	yes	yes	yes	yes
Clustering level	firm	firm	firm	firm	firm	firm
Size and Age controls	no	yes	no	yes	no	yes

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Note: This table presents results based on different Leverage group definition (within each age-size bin) compared to the baseline presented in Table 2. Firms with median leverage (between 40th and 60th percentile) are treated as the base group. This table is a version of Table 2 with alternative definition of leverage groups.

Table 7: Leverage and the Cyclicality of Employment

	(1) employment	(2) employment	(3) employment	(4) employment	(5) employment	(6) employment	(7) employment	(8) employment	(9) employment
0-3 × y	2.97***	2.94***	3.01***	3.15***	3.16***	3.18***	2.85***	2.82***	2.88***
	(8.66)	(8.54)	(8.29)	(17.78)	(17.40)	(14.66)	(8.75)	(8.61)	(8.56)
4-8 × y	0.99***	0.97***	1.04***	1.01***	1.02***	1.06***	1.17***	1.15***	1.21***
	(3.13)	(3.04)	(3.09)	(9.47)	(8.90)	(6.40)	(3.78)	(3.69)	(3.78)
9-19 × y	0.51	0.55*	0.64*	0.52***	0.60***	0.65***	0.72**	0.76**	0.83**
	(1.59)	(1.72)	(1.89)	(4.48)	(4.80)	(3.81)	(2.32)	(2.43)	(2.57)
20+ × y	-0.02	0.07	0.17	0.02	0.13	0.20	0.44	0.52*	0.59*
	(-0.06)	(0.21)	(0.51)	(0.15)	(1.06)	(1.13)	(1.43)	(1.67)	(1.84)
0-3 × y × size	-0.27***	-0.28***	-0.31***	-0.29***	-0.31***	-0.32***			
	(-3.65)	(-3.83)	(-3.67)	(-3.95)	(-4.17)	(-3.75)			
4-8 × y × size	0.21***	0.20***	0.17***	0.24***	0.23***	0.22***			
	(5.09)	(4.92)	(2.96)	(5.91)	(5.61)	(3.72)			
9-19 × y × size	0.27***	0.26***	0.22***	0.32***	0.30***	0.28***			
	(6.75)	(6.31)	(3.86)	(7.85)	(7.27)	(4.82)			
20+ × y × size	0.42***	0.40***	0.36***	0.48***	0.45***	0.42***			
	(10.92)	(10.32)	(6.43)	(12.34)	(11.61)	(7.57)			
0-20 × y		-0.19***	-0.42**		-0.27***	-0.43**		-0.21***	-0.40***
		(-2.98)	(-2.37)		(-4.23)	(-2.40)		(-3.36)	(-2.97)
20-40 × y		-0.08	-0.24		-0.10	-0.21		-0.08	-0.14
		(-1.34)	(-1.32)		(-1.62)	(-1.12)		(-1.42)	(-1.01)
60-80 × y		0.17***	0.24		0.16***	0.23		0.17***	0.23
		(2.79)	(1.21)		(2.62)	(1.13)		(2.83)	(1.50)
80+ × y		0.31***	0.25		0.18**	0.20		0.30***	0.16
		(4.27)	(1.15)		(2.38)	(0.94)		(4.16)	(0.99)
0-20 × y × size			0.11*			0.08			
			(1.71)			(1.20)			
20-40 × y × size			0.07			0.05			
			(1.10)			(0.75)			
60-80 × y × size			-0.03			-0.03			
			(-0.43)			(-0.39)			
80+ × y × size			0.02			-0.01			
			(0.32)			(-0.17)			
0-3 × y × lls							-0.27***	-0.28***	-0.31***
							(-4.18)	(-4.35)	(-4.40)
4-8 × y × lls							0.14***	0.13***	0.10**
							(4.01)	(3.85)	(2.23)
9-19 × y × lls							0.20***	0.18***	0.15***
							(6.24)	(5.77)	(3.51)
20+ × y × lls							0.25***	0.23***	0.20***
							(9.29)	(8.72)	(5.09)
0-20 × y × lls									0.10**
									(2.06)
20-40 × y × lls									0.02
									(0.54)
60-80 × y × lls									-0.02
									(-0.48)
80+ × y × lls									0.07
									(1.05)
Observations	674084	674084	674084	674084	674084	674084	674084	674084	674084
adj-r2	0.05	0.06	0.06	0.05	0.05	0.05	0.05	0.06	0.06
Sectors	yes	yes	yes	no	no	no	yes	yes	yes
Clustering level	firm	firm	firm	firm	firm	firm	firm	firm	firm

t statistics in parentheses
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Note: This table shows the results of three different specifications (baseline size specification with sector fixed effects, columns 1-3, baseline without sector fixed effects, columns 4-6, and a specification where size bin is replaced with continuous log of employment, columns 7-9), each for three levels of controlling for leverage (no leverage, basic leverage and basic leverage + leverage group dependent size slope).

Table 8: Leverage and Cyclicality of Sales, Employment and Assets

	(1) sales	(2) sales	(3) sales	(4) employment	(5) employment	(6) employment	(7) assets	(8) assets	(9) assets
0-3 × y	2.734***	2.723***	2.600***	2.972***	2.938***	3.009***	2.538***	2.609***	2.674***
	(8.06)	(8.00)	(7.20)	(8.66)	(8.54)	(8.29)	(7.50)	(7.67)	(7.43)
4-8 × y	1.616***	1.600***	1.482***	0.990**	0.965**	1.044**	1.270***	1.360***	1.422***
	(5.26)	(5.18)	(4.46)	(3.13)	(3.04)	(3.09)	(4.21)	(4.47)	(4.33)
9-19 × y	1.140***	1.118***	1.009**	0.508	0.550	0.641	0.898**	1.074***	1.123***
	(3.64)	(3.55)	(3.00)	(1.59)	(1.72)	(1.89)	(2.96)	(3.51)	(3.43)
20+ × y	0.607	0.601	0.499	-0.019	0.069	0.173	0.322	0.537	0.584
	(1.93)	(1.90)	(1.47)	(-0.06)	(0.21)	(0.51)	(1.07)	(1.76)	(1.78)
0-3 × y × size	-0.230**	-0.228**	-0.180*	-0.270***	-0.283***	-0.311***	-0.234**	-0.259**	-0.283**
	(-2.95)	(-2.92)	(-2.01)	(-3.65)	(-3.83)	(-3.67)	(-2.93)	(-3.24)	(-3.12)
4-8 × y × size	0.023	0.032	0.076	0.210***	0.203***	0.171**	0.052	0.029	0.007
	(0.51)	(0.70)	(1.19)	(5.09)	(4.92)	(2.96)	(1.14)	(0.63)	(0.10)
9-19 × y × size	0.111*	0.122**	0.160**	0.271***	0.255***	0.219***	0.061	0.028	0.010
	(2.50)	(2.74)	(2.60)	(6.75)	(6.31)	(3.86)	(1.42)	(0.64)	(0.17)
20+ × y × size	0.269***	0.278***	0.313***	0.419***	0.399***	0.357***	0.200***	0.165***	0.149*
	(6.20)	(6.35)	(5.09)	(10.92)	(10.32)	(6.43)	(5.08)	(4.16)	(2.52)
0-20 × y		0.019	0.183		-0.186**	-0.423*		-0.407***	-0.476*
		(0.26)	(0.94)		(-2.98)	(-2.37)		(-5.92)	(-2.57)
20-40 × y		-0.110	0.021		-0.078	-0.244		-0.150*	-0.219
		(-1.61)	(0.10)		(-1.34)	(-1.32)		(-2.25)	(-1.10)
60-80 × y		-0.030	-0.053		0.172**	0.245		0.088	0.116
		(-0.43)	(-0.25)		(2.79)	(1.21)		(1.24)	(0.53)
80+ × y		0.021	0.252		0.311***	0.247		0.352***	0.149
		(0.26)	(1.12)		(4.27)	(1.15)		(4.00)	(0.61)
0-20 × y × size			-0.066			0.107			0.025
			(-0.93)			(1.71)			(0.37)
20-40 × y × size			-0.049			0.069			0.025
			(-0.70)			(1.10)			(0.36)
60-80 × y × size			0.011			-0.029			-0.015
			(0.14)			(-0.43)			(-0.20)
80+ × y × size			-0.099			0.025			0.081
			(-1.19)			(0.32)			(0.89)
Observations	595547	595547	595547	674084	674084	674084	594646	594646	594646
adj-r2	0.039	0.041	0.041	0.052	0.056	0.056	0.028	0.030	0.030
Sectors	yes	yes	yes	yes	yes	yes	yes	yes	yes
Clustering level	firm	firm	firm	firm	firm	firm	firm	firm	firm

t statistics in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Note: This table compared the cyclicality among Sales, Employment and Assets for the baseline leverage effects specifications (columns 1-3 from Table 7).

B Quantitative model appendix

B.1 Numerical solution details

We solve the model using continuous time numerical methods which draw on Achdou et al. (2021). We use their finite difference methods, and discretize the state variable n with a grid of 1000 nodes. Since the n grid is wide due to the permanent cross sectional heterogeneity between small and large firms, we place these nodes in a non-uniform way to allow more nodes at the low net worth levels experienced by small firms. Ergodic distributions and the aggregate simulations are calculated using the grid based simulation procedure that forms part of the Achdou et al. (2021) method.

To be comparable with the data when running regressions on model-simulated data, we construct time-aggregated yearly data for our regressions. This is done in such a way as to be comparible to our Danish data source. We first solve the transition path of the economy to our aggregate shocks exactly, use the grid-based simulation approach of the Achdou et al. (2021) method, iterating over guesses of aggregate price paths until the economy converges to the true transition path. This ensures an accurate solution to our transition experiments, which does not rely on simulated data from a finite number of firms.

We then construct a panel of 100,000 firms, accounting for entry and exit, which we simulate in response to the aggregate shock. The policy functions of these firms are the policies solved for exactly during the grid-based transition experiment. We aggregate the data up to yearly frequency to make firm-year observations, and regress this data on the growth rate of aggregate output, as done in our data work, using the same regression specification. Since we do not have a notion of industries in our model, we omit the sector dummies from our specification in the model-based regressions. We generate 20 years of data from the model to use for our regressions, which contains the single recession event driven by our MIT shock. Specifically, we allow for 5 years of data pre shock, and then 15 years of data from the moment the shock hits and through the economic recovery.

Estimation of the “steady state” calibration: The estimation of the “steady state” calibration is relatively simple, because it involves only steady state moments, and no cyclical moments. Parameters are either pre-set to a known value, or chosen to exactly hit

Table 9: Simulated Minimum Distance Details for "Cyclical" Calibration

Moment	Data	Model	Error	Associated parameter
5% peak GDP fall during recession	0.05	0.05	1.11%	r_0
Average employment growth age 0-3, size 0-30%	0.35	0.36	2.02%	n_1^e
Relative cyclicality age 0-3, size 0-30%	1.84	1.86	1.51%	$\bar{\phi}_0$
Relative cyclicality age 0-3, size 30-60%	1.64	1.68	2.53%	n_2^e
Relative cyclicality age 0-3, size 60-90%	1.53	1.55	1.29%	n_3^e
Relative cyclicality age 0-3, size 90%+	1.16	1.16	-0.31%	n_4^e
Relative cyclicality age 20+, size 0-30%	0.15	0.16	3.88%	η_1
Relative cyclicality age 20+, size 30-60%	0.64	0.65	2.26%	η_2
Relative cyclicality age 20+, size 60-90%	0.95	0.91	-4.64%	η_3
Average error (sqrt. of mean squared error)	–	–	2.53%	–

Note: Targeted moments in the outer-loop simulated minimum distance estimation for the "cyclical" calibration. Associated parameter is illustrative only, as all parameters are jointly chosen to minimise the mean squared error of all moments.

one moment using an associated parameter. We use an iterative updating scheme, and stop once all moments are hit with 1% tolerance or less. There are 25 parameters of the model, which are given in the "Steady state" column of Table 10, with each associated moment given in the Source column.

Estimation of the "cyclical" calibration: The estimation of the "cyclical" calibration is more complicated, because we additionally target cyclical moments. For any parameter guess we must 1) solve the steady state of the model, 2) simulate the business cyclical experiment using an MIT shock, and 3) simulate a panel of firms to perform our regressions. We speed up the estimation using a two layer procedure.

In the "outer loop" we choose all parameters which are estimated on cyclical moments. These parameters are jointly chosen to minimize the distance from the cyclical moments using a numerical minimization routine (we use a pattern search algorithm). The nine parameters chosen in the outer loop are $(\eta_1, \eta_2, \eta_3, n_1^e, n_2^e, n_3^e, n_4^e, \bar{\phi}_0, r_0)$. Here $\bar{\phi}_0$ and r_0 refer to the value of the collateral constraint and discount rate shock at time 0. These are chosen to hit the following nine moments: 1) 5% aggregate output fall, 2) average growth rate of "age 0-3, size 0-30%" firm bin, 3) relative cyclicality of age 0-3 firms in all four size bins, and 4) relative cyclicality of age 20+ firms in the 0-30%, 30-60%, and 60-90% size bins.

In the "inner loop" we choose all parameters which are estimated on steady state moments. For any guess of the outer loop parameters, the inner loop chooses the inner loop parameters to exactly hit the inner loop moments (to a 5% tolerance). All parameters in the "Cyclcal" column of Table 10 are chosen in the inner loop (apart from the nine outer loop parameters) with associated moment given in the Source column. Note that η_4 is chosen to impose aggregate constant returns to scale, which is done in the inner loop. Similarly, z_1^G is chosen to hit the average employment of aged 0 firms, which is also done in the inner loop.

The values of the moments in the data and the estimated model are given in Table 9. The estimation successfully matches all moments with errors of less than 5%, and the average error (computed as the square root of the mean squared error) is equal to 2.53%.

B.2 Evidence in favour of heterogeneous η_s and n_s^e

In this section we provide independent evidence that the heterogeneous η_s and n_s^e that allow the model to replicate the age-size cyclicality facts are in fact reasonable. In the main text we described why these two forces — heterogeneous returns to scale and initial net worth — are natural candidates to explain these features. However, as an independent test, we also show that they naturally fit some other features of the data, which are not targeted.

Firstly, consider heterogeneous returns to scale. Our "cyclical" calibration sets the degree of returns to scale for each size group, η_s, to match their cyclicality, and chooses their productivities, z_s^S, to match their average employment. If the calibration still required large differences in z_s^S across the four groups this would suggest that returns to scale differences were not the true cause of the size differences across groups. These values of z_s^S are given in the calibration table, Table 10. There we see that the differences in z_s^S across groups are greatly reduced relative to the "steady state" calibration. The steady state calibration uses values from $z_1^S = 0.3349$ to $z_4^S = 0.5195$ to explain the differences in average employment across groups, given that it assumes all firms have constant returns to scale. The cyclical calibration only needs to use values from from $z_1^S = 0.3214$ to $z_4^S = 0.4310$. This shows that the differences in returns to scale required to explain the cyclicality differences across size groups also help to explain the size differences across these groups, requiring a less disperse exogenous TFP distribution.

Another way to interpret our heterogeneous returns to scale assumption is as a hetero-

Table 10: Model Parameters and Calibration

	Interpretation	"Steady state"	"Cyclical"	Source
	Parameters used in both calibrations:			
r	Discount rate	0.0202	0.0202	2% yearly real interest rate
δ	Depreciation rate	0.1054	0.1054	10% annual rate
θ	Substitution across varieties	0.9	0.9	10% markup in frictionless model
α	Labor-capital ratio in prod fun	9.0804	8.4854	Aggregate L
μ_0	Firm entry rate	0.0834	0.0834	Normal total mass of firms to one
$\bar{\phi}$	S.s. collateral limit	3	3	Maximum leverage
$\bar{n}$	Net worth where start paying dividends	51.4688	62.8036	Normalisation
χ	Labor disutility shifter	0.0115	0.0115	Labor share of income
η	Labor supply elasticity	0.3	0.3	Real wage flexibility
α_s	Rate transition to superstar firm	5.1e-5	5.1e-5	0.5% of firms are superstar
$z_\star$	Superstar productivity	0.6361	0.4785	Employment share of firms age 20+
ζ_y	Exit rate when young ($g = 1$)	0.1415	0.1415	Exit rate age 0
ζ_o	Exit rate when old ($g = 2$)	0.0647	0.0647	Average exit rate 8% per year
α_G	Transition rate young to old	0.1964	0.1964	Exit rate age 6
z_1^S	Productivity for type $s = 1$	0.3349	0.3214	Av. emp. size 0-30%
z_2^S	Productivity for type $s = 2$	0.3784	0.3648	Av. emp. size 30-60%
z_3^S	Productivity for type $s = 3$	0.4212	0.3858	Av. emp. size 60-90%
z_4^S	Productivity for type $s = 4$	0.5195	0.4310	Normalise $Y = 1$
γ_1^S	Fraction born type $s = 1$	0.3	0.3	Firms for 0-30% size bin
γ_2^S	Fraction born type $s = 2$	0.3	0.3	Firms for 30-60% size bin
γ_3^S	Fraction born type $s = 3$	0.3	0.3	Firms for 60-90% size bin
γ_4^S	Fraction born type $s = 4$	0.1	0.1	Firms for 90%+ size bin
	Parameters used "Steady state" calibration:			
η	Returns to scale (all firms)	1	–	All firms CRS
n^e	Net worth fraction of entrants	0.3880	–	Average employment of age 0 firms
z_1^G	Relative productivity of young	1	–	Not used
	Parameters used in "Cyclical" calibration:			
η_1	Returns to scale ($s = 1$)	–	0.6231	SMM (see Table 9)
η_2	Returns to scale ($s = 2$)	–	0.9941	SMM (see Table 9)
η_3	Returns to scale ($s = 3$)	–	1.0266	SMM (see Table 9)
η_4	Returns to scale ($s = 4$)	–	1.0412	Impose agg. economy has CRS
n_1^e	Net worth fraction of entrants ($s = 1$)	–	0.1453	SMM (see Table 9)
n_2^e	Net worth fraction of entrants ($s = 2$)	–	0.6484	SMM (see Table 9)
n_3^e	Net worth fraction of entrants ($s = 3$)	–	0.8148	SMM (see Table 9)
n_4^e	Net worth fraction of entrants ($s = 4$)	–	0.9563	SMM (see Table 9)
z_1^G	Relative productivity of young	–	0.9225	Average employment of age 0 firms
$\bar{\phi}_0$	Size of collateral constraint shock	–	-20%	SMM (see Table 9)
r_0	Size of discount rate shock	–	+15%	SMM (see Table 9)

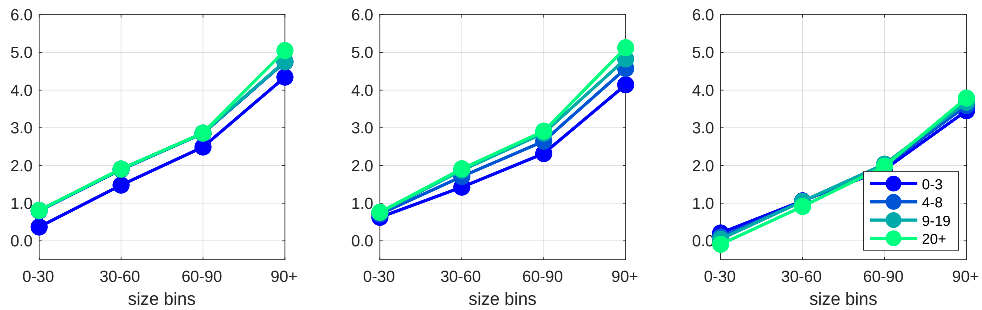
Note: Parameters and calibration targets for the quantitative model. "Steady state" refers to the calibration of the model to steady-state moments only, from Section 4.2.1. "Cyclical" refers to the calibration to both steady state and business cycle moments, from Section 4.3.1.

geneous *demand elasticity* assumption. In particular, with our CES demand curve, firms face overall returns to revenue $\eta_s\theta$, which is the combination of returns to scale in production, η_s, and the demand elasticity parameter, θ. Therefore, to give smaller firms more decreasing returns to scale, we could equivalently held η_s equal across firms, and used heterogeneous demand elasticities, θ_s, across size groups. This would require setting $\theta_1 < \theta_2 < \theta_3 < \theta_4$, meaning that demand is more inelastic for small firms than large ($\epsilon = 1/(1-\theta)$). In our model, this would mean that small firms charge higher *markups* than large firms, based on the usual result that inelastic demand leads to higher markups: recall the standard static result that the optimal markup is equal to $\mu = \epsilon/(1-\epsilon) = 1/\theta$. This interpretation allows us to use data on average markups by firm size to interpret our assumption of more decreasing returns to scale at small firms. If markups are higher at small firms than large firms, this would provide additional support. Indeed, this appears to be the case in the data. Díez et al. (2021) compute markups for both private and public firms using the global Orbis dataset, for a set of firms accounting for 70% of global GDP. They find that there is a U-shaped relationship between markups and firm size, and that markups are decreasing with firm size for most of the size distribution: *"Contrary to common wisdom, we find that, unconditionally, smaller firms have higher markups even within narrowly defined industries—only when we focus on very large firms we do find a positive relation... markups first decrease with firm size and only when a (fairly large) size threshold is reached, markups start increasing with firm size"* (p2).

Secondly, consider heterogeneous initial net worth. Through the lens of the model, this is calibrated to match the cyclicalities of firms aged 0-3 of different sizes. On top of this, as discussed in the main text in Section 3.3, the data suggests that young large firms are not much more financially constrained than old large firms. This is especially true compared to the difference between young and old firms within the small firm group. Specifically, we highlight three pieces of evidence in favour of this idea. First, Figure 4(d): Among large firms, Debt/Assets is higher for young firms than old, just like all size groups. However, the gap is smaller for large firms: Leverage of young (0-3) minus old (20+) is $0.76 - 0.6 = 0.16$ for the largest firm group. For the smallest firm group it is $0.82 - 0.57 = 0.25$. This shows that among large firms, the differences in leverage across age are smaller than among small firms. Second, Figures 4(b) and (e): The results by growth rates of debt are even clearer. Among large firms (90%+) the growth rates of Debt and Debt/Assets are almost identical for all age groups. This is in contrast to small firms (0-30%) where the growth rate of debt is much higher for young firms. This suggests that the financing needs, measured by whether firms are trying to increase or reduce their debt, are very similar among large firms, regardless of their age. Finally, Figure 4(c): among large firms (90%+) the cyclicality of debt is the same for all age groups. This suggests the cyclicality of financial conditions for large firms similar regardless of their age, in contrast to small firms where debt is more cyclical for young than old. Overall, these data all point towards age mattering less for the degree of financial frictions among larger firms. This justifies our focus on using differences in n^e_s across size-types to explain the differing age gradient of the cyclicality of employment between small and large firms.

B.3 Additional model tables and figures

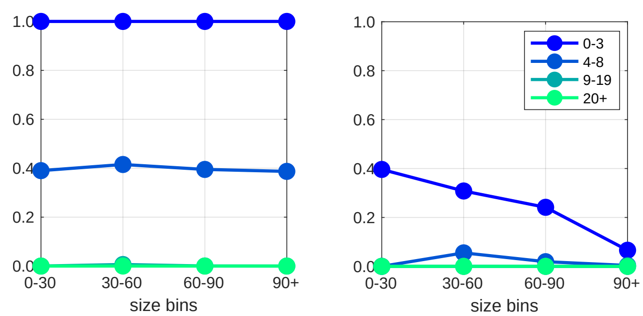
Figure 15: Average of Log Employment by Age-Size Bin



(a) "Steady state" cali (b) "Cyclical" cali (c) Data

Note: Panels give regression coefficients from regressions of firm-level log employment on firm age-size dummies, computed from model simulated data. The regressions are on firm-level data aggregated to the yearly level and treated in the same way as the data. Size bins are percentiles, and each line refers to a different firm age group. The final panel gives the results from real-world data, and the remaining panels from model data.

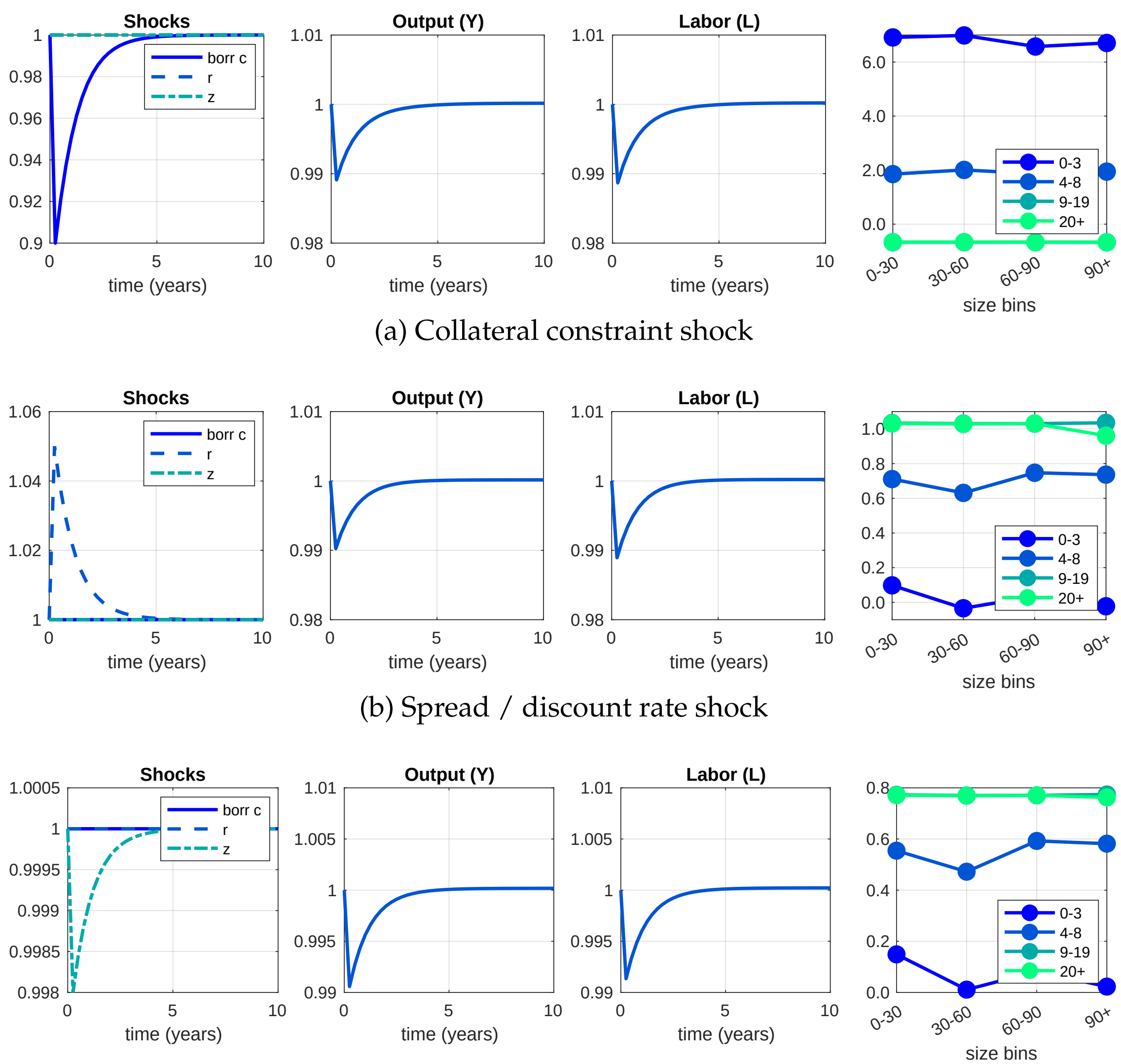
Figure 16: Fraction of Firms Constrained by Age-Size Bin in the Model



(a) "Steady state" cali (b) "Cyclical" cali

Note: Panels give regression coefficients from regressions of a dummy of whether a firm is financially constrained ($\phi = \bar{\phi}$) on firm age-size dummies, computed from model simulated data. The regressions are on firm-level data aggregated to the yearly level and treated in the same way as the data. Size bins are percentiles, and each line refers to a different firm age group.

Figure 17: Effect of Various Shocks in the "Steady State" Calibration



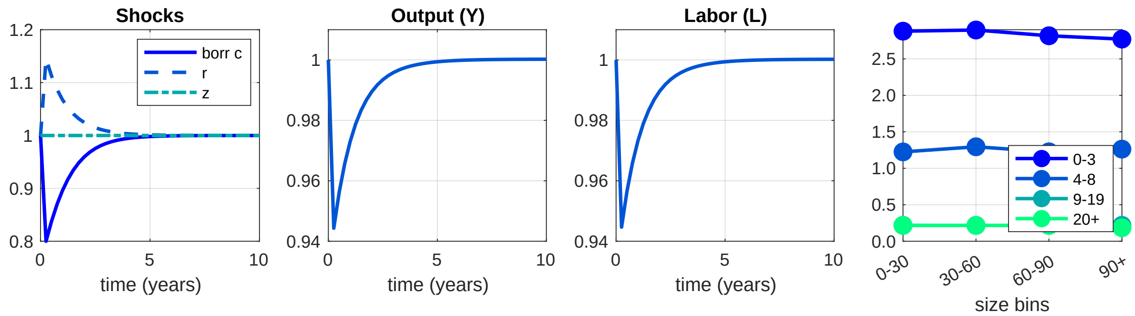
(a) Collateral constraint shock

(b) Spread / discount rate shock

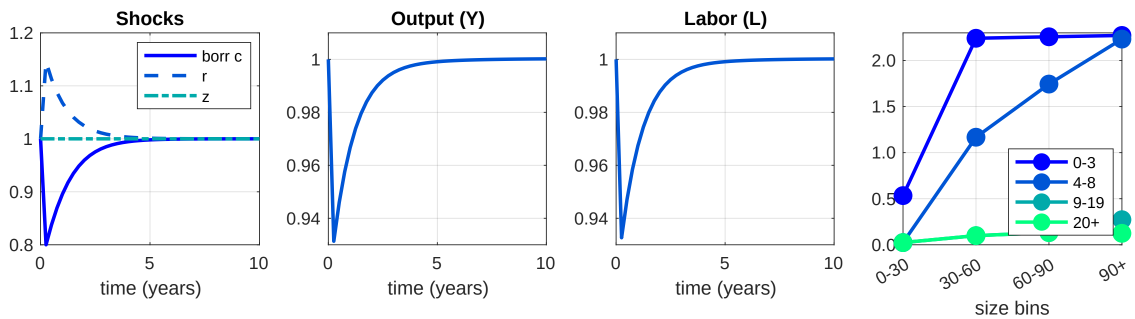
(c) TFP shock

Note: This figure gives simulated aggregate paths and regression coefficients for various recession experiments. In each panel, the left plot gives the shock paths, the center two panels give the paths for aggregate output and labor, and the right panel gives the cyclicality of firm age-size groups computed using our regression approach. In the left panel, "borr c" refers to the path for $\bar{\phi}$, "r" to the path for r, and "z" to the path of the aggregate TFP shock.

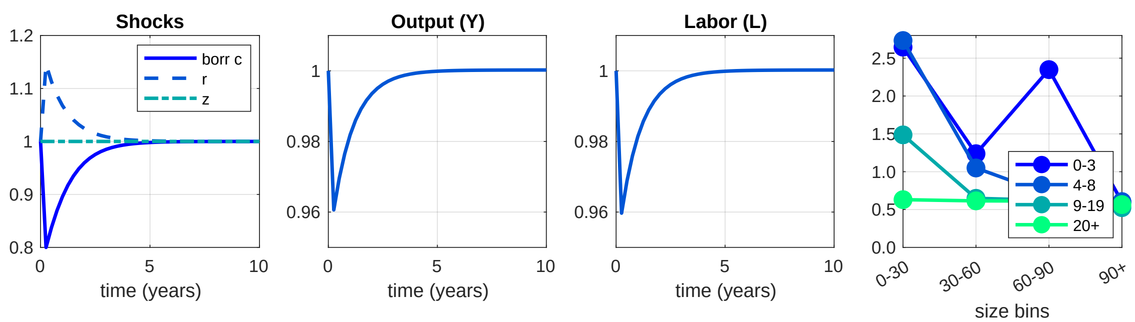
Figure 18: Effect of Calibrated Shock Combination in Various Models



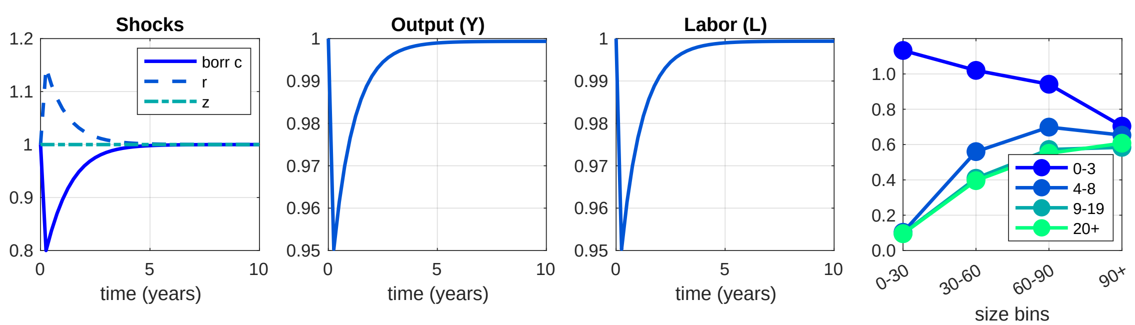
(a) "Steady state" calibration



(b) Adding heterogeneous returns to scale only



(c) Adding heterogeneous initial net worth only



(d) Full "cyclical" calibration

Note: This figure gives simulated aggregate paths and regression coefficients for various recession experiments. In each panel, the left plot gives the shock paths, the center two panels give the paths for aggregate output and labor, and the right panel gives the cyclicality of firm age-size groups computed using our regression approach. In the left panel, "borr c" refers to the path for $\bar{\phi}$, "r" to the path for r, and "z" to the path of the aggregate TFP shock.