The constraint on public debt when $r < g$ but $g < m$

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Abstract

With real interest rates below the growth rate of the economy, but the marginal product of capital above it, the public debt can be lower than the present value of primary surpluses because of a bubble premia on the debt. The government can run a deficit forever. In a model that endogenizes the bubble premium as arising from the safety and liquidity of public debt, more government spending requires a larger bubble premium, but because people want to hold less debt, there is an upper limit on spending. Inflation reduces the fiscal space, financial repression increases it, and redistribution of wealth or income taxation have an unconventional effect on fiscal capacity through the bubble premium.

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1 Introduction

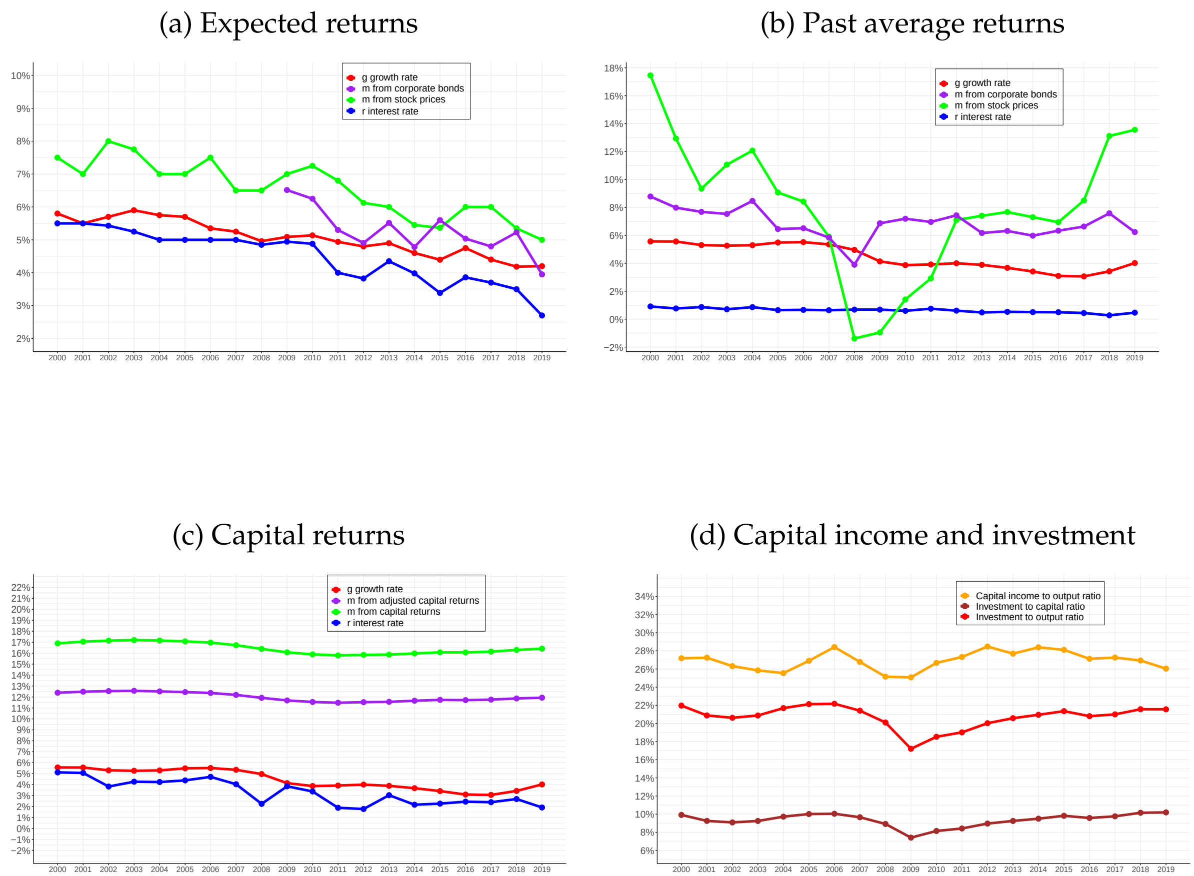
Almost every year in the past century (and maybe longer), the long-term interest rate on US government debt (r) was below the growth rate of output (g). In the last decade, the gap between them has increased. At the same time, the US data also strongly suggest that the marginal product of capital (m) has stayed relatively constant, well above the growth rate of output, so $g < m$. Panel (a) of Figure 1 shows expected long-run values of these three rates, while panel (b) instead uses geometric averages over the past 10 years. Panel (c) measures the marginal product of capital using capital income, as opposed to asset prices, which after subtracting for depreciation is higher than the growth rate of the economy. Panel d) shows that capital income exceeds investment, the dynamic efficiency counterpart to $g < m$ (Abel et al., 1989), as well as the investment to capital ratio, which would be a lower bound for m, and is well above g.[1]

This paper investigates the implications for the government budget constraint of having $r < g < m$. Section 2 goes through simple, yet general, debt arithmetics to show that the government can run a perpetual budget deficit in this case. Yet, there is still a well-defined budget constraint whereby the debt equals the present value of the ratio of primary surpluses to output discounted by $m - g$ plus the discounted bubble premia earned on the debt that equals $m - r$. It is not the gap $r - g$, but rather the gaps $m - g$ and $m - r$ that matter for public finances. These arithmetics open up several questions: why is $m > g > r$ and so what drives the two gaps? How does more government spending affect the bubble premium in equilibrium? Is there an upper bound on the amount of spending for the bubble to be sustainable? How do monetary and fiscal policies affect the bubble premium, and through it do they tighten or loosen the government budget constraint?

Section 3 offers a model that answers these four questions by jointly determining r, g and m. Private investment is subject to idiosyncratic risk and to borrowing constraints. Public debt provides a safe haven from that risk, and an alternative store of value beyond the limits of private credit. These two properties are the most commonly estimated reasons for the $r - g$ differences that we observe in the data. A simpler version of the model that has no risk, but only borrowing constraints, show sthat the misallocation of private capital by itself creates a demand for public debt as an alternative form of savings, and this creates a bubble premium.

[1] For further discussions on the measurement of r, g, m, and for the international evidence, see Gomme, Ravikumar and Rupert (2011), Geerolf (2018), Barrett (2018), Rachel and Summers (2019), Mauro and Zhou (2020), Jordà et al. (2019).

Figure 1: The US marginal product of capital, growth rate, and interest rate since 2000



Notes: All series are annual. Panel a) plots 10-year ahead expectations on: treasury bond yields, nominal GDP growth (measured as the sum of expected real GDP growth and expected PCE inflation rate), stock returns, and returns on Baa rated corporate bonds, according to the median respondent to the Survey of Professional Forecasters. Panel b) plots the 10-year geometric averages of: 10-year treasury bond returns, nominal GDP growth, SP500's nominal returns, and returns on an index of Baa rated corporate bonds. Panel c) plots the 10-year geometric average of: an index of 10-year maturity treasury bonds, the same output growth rate as in panel b), the ratio of net value added (excluding labor expenditures) to the corporate capital stock in the non-financial corporate sector, obtained from the Bureau of Economic Analysis' Survey of Current Business, and an adjusted return on capital that takes away 5% of GDP from capital income to account for land income, and 2/3 of proprietary income, attributed to a remuneration for labor. Panel d) plots point-in-time capital income series, now as a ratio of GDP, and the investment to capital and investment to output ratios using the BEA's data for non-financial corporate investment, capital stock and value added, all in real terms.

Section 4 shows that, in this model, higher public spending as a ratio of the debt raises the bubble premium $m - r$, but it lowers the amount of debt held by the public as a ratio of private capital. There is a maximal amount of public spending after which the bubble is not sustainable. This limit is tighter in economies that are more financially developed, have less undiversifiable risk, and less inequality. Section 5 considers various extensions of the model—a different fiscal rule for spending, aggregate risk, foreign demand for public bonds, transition dynamics—and shows that the results are robust, but come with some new insights. The exercises in these two sections make clear how useful it is to think in terms of the bubble premium $m - r$ derived in the debt arithmetics.

Section 6 shows that monetary and fiscal policies, by affecting r, g and m, will change the bubble premium and so have surprising effects on the fiscal space and capacity of the government. Expected inflation is neutral, but inflation volatility lowers the safety of the public debt, and so it tightens the government budget constraint. There is no conflict in the mandates of the central bank and the fiscal authority, since delivering stable inflation is what creates the most fiscal space to raise public spending. Financial repression that coerces the private sector to hold government bonds at a below-market rate creates fiscal space through an additional repression premium on the debt. However, it lowers growth because it worsens the allocation of capital. Perhaps more surprisingly, a tax-transfer system that redistributes wealth to those that have less income raises the bubble premium, keeping spending fixed, or lowers spending, keeping the premium fixed. It lowers the maximum spending before the bubble bursts. Therefore, there is a conflict between a fiscal authority that wants to spend more, and one that wants to redistribute more. Finally, a higher proportional income tax directly raises revenue, but indirectly reduces private credit. It shrinks the bubble in the public debt, even as it raises primary surpluses. In some cases, the effect on the bubble is larger, so that tax cuts can pay by themselves by raising economic activity and increasing the bubble premium on the debt.

All combined, the conclusion is: in an economy that is dynamically efficient, but with a bubble in the public debt, there is still a constraint on how much the government can spend, and policies can loosen or tighten this constraint through their separate effect on $m - r$ and on $m - g$.

2 Debt arithmetics and the literature around it

Government budgets are not easy reads: borrowing comes through multiple instruments with different payment profiles and maturities, and spending and revenue lines depend on different bases and commitments. There are multiple r's and g's. Yet, some mild simplifications provide a clear statement of how debt will evolve over time. First, let s_t be the (net) public spending, or the (primary) public deficit. This could include both the flow of resources used by the government, as well as tax revenues and transfer spending. The question I ask is how large s_t can be. Therefore, most of the paper assumes it is exogenous.

Second, let the real market value of outstanding government debt be denoted by b_t. The return that private agents earn (and the government pays) on this debt is r_t. This need not correspond to the promised yield on the debt, as there may be capital gains on long-term debt, or inflation affecting nominal debt.[2] The second assumption is that there is no aggregate uncertainty affecting either r_t or s_t. I will return to introducing uncertainty at different parts of the paper, showing it does not materially affect the main results.

The law of motion for the evolution of the public debt then is:

$$db_t = s_t dt + r_t b_t dt. \tag{1}$$

Debt increases by the sum of spending and interest paid on the debt (the public deficit).

The third and final assumption is that output y_t grows at the rate g_t, that is also deterministic. It may vary over time, but asymptotically it converges to a constant g.

The key endogenous variables are then: b_t, r_t, g_t. Throughout, the empirically relevant case is when $r_t \leq g_t$ and $b_t \geq 0$. Further, I focus on a balanced growth path where the exogenous spending s_t and the endogenous debt asymptotically grow at the same rate as output, g.[3]

[2]To see this clearly, assume that there is a single nominal government bond, of which every instant a fraction ξ expires giving its holder a principal payment of 1, while the remaining $1-\xi$ pays no coupon but survives until next period. The expected maturity of government debt is $1/\xi$, matching the actual behavior of governments that perpetually roll over their debt, while keeping the maturity relatively stable. If $B_t \geq 0$ are the units outstanding of this bond, then its value in output units is: $b_t = B_t v_t / p_t$ where v_t is the nominal value (or price) of the bond, and p_t is the price level. Then, the return on the bond is: $r_t = \xi + (1-\xi)\frac{dv_t}{v_t} - \frac{dp_t}{p_t}$ where the first term is the coupon rate (or promised yield), the second term is the capital gain, and the third term is the inflation loss. Even if the government can choose the maturity ξ, or even how many bonds to sell B_t, the return on the government debt is endogenous as the market price adjusts as needed to clear markets.

[3]Could public debt ever grow at a faster rate than output? Since private consumption is bound by private gross income, then the savings of the households that hold the debt would have to grow to infinity.

Working through this law of motion produces debt arithmetics that are, on the one hand, useful to understand the links between variables, but on the other hand, ultimately unsatisfactory, since they are mere identities. This section shows that debt arithmetics answer some questions, yet raise just as many others.

2.1 Permanent deficits?

In the balance growth path, equation (1) implies that:

$$b = \max\left\{\frac{s}{g-r}, 0\right\}. \tag{2}$$

In spite of a fixed permanent deficit, with $r < g$, the government can still sustain positive debt.[4] From the opposite perspective, for a fixed amount of debt, the government can spend, as a ratio of that debt, the gap between the growth rate and the interest rate. Taking as given the value of b at the end of 2020 (127% of GDP), and a generous $r - g = 2\%$ this would imply that the permanent deficit could be 2.5% of GDP. But, surely, a change in spending would affect $r - g$ in equilibrium. The arithmetics show what is possible, and the tight link between each variable, but are not enough.

Perhaps there is no such equilibrium. If net spending is too high, the price of the debt will be zero, as the private sector refuses to hold this Ponzi scheme. Since b is the value of the debt, this corresponds to $b = 0$, which also solves equation (1). Since the capital stock must be non-negative, perhaps this limit is reached when debt is equal to net private assets. If so, then this suggests an upper limit on spending between 4.8% and 7% of GDP, not much above the average primary deficit in 2010-20 (4.8%) or the Congressional Budget Office projection of a 4.6% deficit in 2050, suggesting that almost all of the fiscal profligacy from $r < g$ has already been used.[5] Yet, as spending changes, this will surely

This is possible as long as they receive transfers from the government that rise over time as quickly as the debt. In turn, this is consistent with the government budget constraint as long as s_t/b_t converges to a constant equal to the growth rate of debt minus the real interest rate. While theoretically possible, this is not the most empirically interesting, so I focus on the balanced growth path instead.

[4] The importance of $r < g$ may be best understood when $g = 0$. In this case, thinking of government debt as a consol, since $r < 0$, the debtholder is *paying* the government a fixed stream to hold the bond. This revenue is what finances the permanent spending.

[5] The lower bound comes from using the Bureau of Economic Analysis 2019 estimate that the capital stock was 2.1 times GDP, a net international investment position of -0.5 of GDP, and privately-held public debt plus debt of the Federal Reserve of 0.8 of GDP, for total assets of 2.4 times GDP. The upper bound comes from the Federal Reserve's Financial Accounts of the United States measure of private non-financial assets of 3.5 times of GDP.

affect the desirability of holding private and public assets in the economy. Net assets are not fixed, so while the debt arithmetics point to a limit, they can not go very far in telling when it will be hit.

2.2 Recurrent deficits?

Let the average interest rate between dates 0 and t be $\bar{r}_t = (\int_0^t r_s ds)/t$, and likewise for $\bar{g}_t$. A one-off spending splurge at date 0, Δs_0, raises debt-to-output t periods later by $\Delta b_t = e^{(\bar{r}_t - \bar{g}_t)t}\Delta s_0$. Even if the temporary increase in spending is very large, by pursuing this "deficit gamble" for enough years, there is a negligible decrease in spending needed to pay for the resulting debt in the distant future. However, the initial increase in spending is limited by output in that period. At an extreme, if the spending splurge was as high as output, with nothing left to consume, marginal utility of consumption would approach infinity, driving interest rate above the growth rate to infinity. Again, endogenizing r and g is crucial to understand even temporary gambles.

If a deficit gamble can be done once, why not frequently? Solving the debt dynamics in equation (1) forward to infinity:

$$b_0 = \lim_{T\to\infty} \left[-\int_0^T e^{-\bar{r}_t t} s_t dt + e^{-\bar{r}_T T} b_T \right]. \tag{3}$$

Since asymptotically $r < g$, then even if debt is never paid as it grows at the rate of output g, the limit of the second term in the right-hand side goes to infinity. This leads to the erroneous conclusion that any initial debt b_0 can be sustained, with no limit on public debt beyond the available resources in the economy, since deficit gambles can be repeated and rolled over in a Ponzi way. This is incorrect: the limit of the sum is not the same as the sum of the limits. While the limit of the second term is plus infinity, the limit of the first term is minus infinity.

Rather, to solve the debt dynamics forward requires re-writing the flow budget constraint as $db_t - d_t b_t dt = s_t dt + (r_t - d_t) b_t dt$, for some discount rate $d_t > g_t$. Then, the limits are well defined and:

$$b_0 = -\int_0^\infty e^{-\bar{d}_t t} s_t dt + \int_0^\infty e^{-\bar{d}_t t}(d_t - r_t) b_t dt. \tag{4}$$

Even if $r < g$, there is still a mathematically well-defined limit on public debt or, equivalently, on how large can spending be. As a matter of arithmetics, a strictly higher sequence of d_t raises the first term towards zero and lowers the second one towards the initial value of the debt. But which is an appropriate d_t to use?

2.3 The bubble premium

Let m_t be the marginal product of capital in the economy. This gives the private return of investing in production as opposed to in the government debt. Since, in the data $m_t > g_t$, this is an empirically legitimate choice for d_t. Moreover, it is a sensible choice. While equation (4) was a mathematical expression with no economic interpretation, the following equation:

$$b_0 = -\int_0^\infty e^{-\bar{m}_t t} s_t dt + \int_0^\infty e^{-\bar{m}_t t}(m_t - r_t) b_t dt \tag{5}$$

has an economic meaning as public debt is the sum of two terms.

The first term is the present value of spending, using the return on private assets as the valid stochastic discount factor, as one would for the payoffs in any other asset.[6] The marginal holder of the public debt could at the margin hold a unit of capital instead, so m_t is the relevant rate through which she would discount the holdings of the public debt. In turn, discounting by m_t is consistent with the transversality condition for those agents since optimal capital investment requires the marginal utility of consumption to grow at the rate of return on private assets. The condition $m > g$ for the integrals to be well defined is then just the dynamic efficiency condition.

The second term is the present value of the implicit government revenues that arise from paying r_t in its debt below the marginal return in the private economy m_t. The spread between the two measures how special debt is: its bubble premium, or convenience yield, or seigniorage from issuing bonds that provide a service. The product of the premium and the amount of outstanding debt is then the bubble revenue, or seignorage revenue. When $m_t > r_t$, the government can now pay for outstanding debt in part through these bubble revenues, so recurrent spending can be positive in present value.

In this expression, it is not $g_t - r_t$ that matters. Rather, $m_t - r_t$ is what drives the size of the bubble premium flows, while $m_t - g_t$ is what discounts future flows of spending and bubble premium. In an economy where $r_t \to g_t < m_t$ there is still a bubble premium, allowing for persistent government spending. In the neoclassical model, $m_t = r_t > g_t$ at all dates, the bubble is zero, and the conventional result follows that debt is equal to the

[6] Recall that a valid stochastic discount factor (SDF) is one such that the expectations of its product with the market return is 1. Of course, the market return is itself a SDF.

present value of primary surpluses. It is only asymptotically that discounting by $m - g$ and the bubble premia of $m - r$ combine to cancel out m and leave $g - r$ in equation (2).

2.4 Aggregate uncertainty?

Imagine now that spending, output, and the return on debt are all uncertain, so r_t is the *ex post* return on the debt. The debt constraint in equation (1) is unchanged. The stochastic discount factor D_t is also uncertain now. Letting $\mathbb{E}_t(.)$ be the expectations operator, integrating forward just as before gives:

$$b_0 = -\mathbb{E}_t\left(\int_0^\infty \frac{D_t s_t}{D_0} dt\right) + \mathbb{E}_t\left(\int_0^\infty \frac{D_t(m_t - r_t)b_t}{D_0} dt\right), \tag{6}$$

as long as the terminal condition $\lim_{t\to\infty}\mathbb{E}_t\left[(D_t/D_0)e^{\bar{g}_t t}(b_t/y_t)\right] = 0$ holds. This replaces the previous $d > g$ condition.

Debt is still equal to the (now expected) present value of net spending plus the expected present value of bubble premium revenues. Again, choosing $D_t = e^{-\bar{m}_t t}$ is valid and economically meaningful if private agents at the margin can hold both the capital stock and public debt. The terminal condition arises from combining their transversality condition and the fact that the capital stock cannot be negative. Again also, a stochastic discount factor that is valid for asset pricing must price the capital stock, so the no arbitrage condition is $\mathbb{E}_0\left[(D_t/D_0)e^{\bar{m}_t t}\right] = 1$, and one obvious solution is $D_t = e^{-\bar{m}_t t}$. Just as with deterministic debt arithmetics, stochastic arithmetics leave open questions, such as how the discount factor endogenously co-moves with the bubble premium.

2.5 Making progress

Debt arithmetics can provide useful insights. They show that there is a clear constraint on the public debt, that debt can exceed the present value of surpluses by the value of the bubble premium, and that recurrent and permanent spending are possible. Moving further though requires a model that endogenizes spending, the marginal product of capital, growth and interest rates, to make sense of how and when they vary with each other. The next section provides one such model that focuses on the safety and liquidity roles of government debt to generate the bubble premium.

The model builds on Reis (2013) and Aoki, Benigno and Kiyotaki (2010), by generating misallocation within a sector because more productive firms cannot borrow more than a fraction of their future revenue. Those papers studied the effect of large swings in capital flows from abroad, while this paper introduces uncertainty, and focuses on bubbles and public debt. For the most part, I assume a closed economy to complement their analysis.

The focus on bubbles is shared with the production economies in Kocherlakota (2009), Martin and Ventura (2012), Farhi and Tirole (2012), Aoki, Nakajima and Nikolov (2014), Hirano, Inaba and Yanagawa (2015) that is surveyed in Martin and Ventura (2018), but it is applied here to make sense of public debt and the intertemporal government budget constraint. Therefore, I do not study bubbles in private assets, which are covered there.[7]

There is an older literature on public debt in exchange economies with overlapping generations including economies with incomplete markets (Tirole, 1985, Santos and Woodford, 1997, Kocherlakota, 2008, Hellwig and Lorenzoni, 2009) with a focus on the link between $r < g$ and the existence and optimality of bubbles. In this paper, there is production, so that there can be a marginal product of capital, and I focus on the fiscal implications of bubbles. To complement that literature, the model has agents that live forever.

Bassetto and Cui (2018), Brunnermeier, Merkel and Sannikov (2020*b*) also study the effect of idiosyncratic risk and incomplete markets on the government budget constraints. However, they apply it to a fiscal theory of inflation. Similarly, Sims (2021) studies the interaction between distortionary taxes and the bubble term, but takes the premium to narrowly to refer to seignorage due to printing money, and focuses on the inflation tax. This paper takes inflation as given, and I refer readers to these three papers for these complementary implications. Moreover, I emphasize more the misallocation of capital and liquidity premia, especially in the study of the interaction between fiscal capacity and other fiscal policies, whicshis not in these papers.[8]

Risk premia due to safety and liquidity are a major driver of the increasing wedge between m and r (Caballero, Farhi and Gourinchas, 2017, Farhi and Gourio, 2018, Mark, Mojon and Veldes, 2020, Negro et al., 2017, Ferreira and Shousha, 2020). Another part of it seems to be due to an increase in market power (Farhi and Gourio, 2018, Eggertsson, Robbins and Wold, 2020). The work of Ball and Mankiw (2021) complements the one in this paper, by writing a model where instead market power generates the bubble premium.

A different literature has focussed instead on the impact of aggregate uncertainty on

[7] I call the government revenue that results from $r < m$ a bubble premium, because of the link to this literature. The empirical literature that tries to measure it often calls it instead a convenience yield, and the theoretical literature that focuses on currency calls it seignorage.

[8] Ongoing work by Brunnermeier, Merkel and Sannikov (2020*a*), Kocherlakota (2021) is closer to this paper by focussing on the safety of debt with only idiosyncratic risk.

the government budget constraint, but assuming a representative agent. It has shown that spikes in interest rates may make one-off deficit gambles fail (Abel, 1992, Ball, Elmendorf and Mankiw, 1998), that a stochastic discount factor gives the right weights to consider different levels of spending (Barro, 2020, van Wijnbergen, Olijslager and de Vette, 2020), and that there is a stationary distribution of debt-to-GDP that may include high levels (Mehrotra and Sergeyev, 2020). Given this complementary work, for most of the paper, I abstract from aggregate uncertainty to focus on idiosyncratic risk and on borrowing constraints leading to inequality and capital misallocation.

The two more direct intellectual antecedents of this paper are Blanchard (2019) and Jiang et al. (2019). Blanchard (2019) lays out arguments (and counter-arguments) for why, given $r < g$, governments can run prolonged deficits with minimal impact on fiscal space, or aim for a larger steady state debt-to-GDP.[9] This paper re-examines these conclusions when $g < m$, and investigates how fiscal, monetary, and financial polices affect the ability to undertake deficit gambles or carry larger debt.[10]

Jiang et al. (2019) argued that the stochastic discount factor that should be used in the government budget constraint is the same that should price risky assets in the economy, which the discussion above built on.[11] They estimated that the present value of surpluses is quite small, so that the residual—the bubble term—must be very large. Since existing direct estimates of the convenience yield on the debt are an order of magnitude too low, they call this a valuation puzzle.[12] This paper can be seen as investigating the bubble premium, but doing so theoretically, endogenizing it in a general-equilibrium model, and studying what forces generate it and what policies change it.[13] Future work can take on the next step of quantifying the effects discussed here towards solving the puzzle they

[9]Earlier, Blanchard and Weil (2001) also discuss the government budget constraint as a result of aggregate uncertainty leading to a Pareto inferior equilibrium. But, they do not discuss the bubble premium, do not have a borrowing constraint causing misallocation, and do not examine how more spending, and other monetary and fiscal policies, affect the fiscal space.

[10]A slightly different perspective is that this paper reconciles Blanchard (2019) on what $r < g$ implies for public debt and spending, and Piketty (2013) on what $g < m$ implies for inequality and taxation. Both treated r, g, m as given, while this paper endogenizes them and discusses the interaction between taxation, spending, debt and inequality (see also Moll, Rachel and Restrepo (2021)).

[11]An earlier statement of this insight is in Bohn (1995).

[12]For estimates of convenience yields, see Negro et al. (2017), Jiang, Krishnamurthy and Lustig (2020), Rachel and Summers (2019)

[13]Jiang et al. (2020) also study policies in this context, but focussing on the covariation of s_t and m_t. Jiang et al. (2021) evaluate the implicit beliefs in the expectations operator of bondholders. Complementing this work focussed on aggregate uncertainty, in this paper I mostly assume a deterministic environment; I introduce aggregate uncertainty to show my conclusions are robust, but leave to these other papers the exploration of all their consequences.

identify.[14]

3 A model where safety and liquidity are scarce

The model is a version of the neoclassical growth model where, besides the government, there is a representative firm, and many households that, because of incomplete markets, are unequal in opportunities and outcomes.

3.1 The firm

Because the focus is on public debt along a balanced growth path, I consider a simple linear economy, where there are no transition dynamics or aggregate risk, and all idiosyncratic risk is iid over time. A technology, which anyone can freely access, transforms quality-adjusted capital into output with a marginal product of capital of m_t.

In the population, there is a distribution $Q(q)$ of capital quality types $q \in [0,1]$, from where each instant the household has an iid draw. For each type, there is a continuum of households who get hit by an idiosyncratic depreciation shock to their capital $\delta(q)dz_t^{qi}$, which follows a Wiener process dz_t^{qi} such that $\int dz_t^{qi}di = 0$. The standard deviation of depreciation shocks $\delta(q) \geq 0$ weakly declines with quality. Each household's capital is therefore different in two ways: ex ante, through their type, and ex post through the realized depreciation. This is the only source of uncertainty and inequality in the economy: higher-quality types have better capital both in its average value and in a lower risk of wear-and-tear. There is a positive mass of high-quality types, for whom $q = 1$ and $\delta(1) = 0$, so they reap the full marginal product of capital at no risk.

The neoclassical firm chooses how much of each capital to hire from each agent k_t^i by paying them r_t^{qi}:

$$\max\left\{\int\int\left[m_t q_t dt - r_t^{qi}dt - \delta dz_t^{qi}\right]k_t^{qi}dQ(q)di\right\}. \tag{7}$$

[14]Two important considerations in the quantification that are strongly suggested by the results in this paper are that: (i) the correlation between idiosyncratic volatility and aggregate risk will amplify the bubble premium, and (ii) fiscal policies and the state of the business cycle affect both the bubble premium and the marginal product of capital, so their covariance can be substantial. Both point to aggregate uncertainty being important for quantification: this paper instead qualitatively investigates the premium.

Therefore, for zero profits to result due to competition, each quality type gets paid:

$$r_t^{qi} dt = m_t q_t dt - \delta dz_t^{qi}. \tag{8}$$

3.2 The households

Households live forever, discounting the future at rate $\rho > 0$ and obtaining utility from their individual consumption c_t^{qi}, and from the government services. I assume that the utility function is separable in these two sources of well-being so that, regardless of how important public services are, I can leave them out of the model as they have no effect on the equilibrium.

Household assets a_t^{qi} can be used to buy government bonds, b_t^{qi}, to invest in capital k_t^{qi}, or to lend to other households l_t^{qi}. The return on this last option is given by the interest rate r_t^l because there is a single private credit market. Households cannot short public debt, or invest negative amounts in capital, but they can borrow. However, they face a borrowing constraint in that the repayment of debt cannot exceed a fraction $\gamma < 1$ of the returns from capital investment in type q. As usual, this is justified by the borrower being able to abscond with all assets but for this share of the capital stock before it is time to pay the lender. Given the ex post depreciation risk, one can think of a mutual fund that pools capital across individuals within each quality type and borrows against it. Going forward, I refer to γ as the level of financial development of the economy, since the larger it is, the larger is the private debt market.

Combining all the ingredients, each household solves the following dynamic problem:

$$\begin{aligned} &\max_{\left\{c_t^{qi}, b_t^{qi}, l_t^{qi}, k_t^{qi}\right\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_t^{qi} dt \right] \\ \text{subject to: } \; a_t^{qi} &= b_t^{qi} + l_t^{qi} + k_t^{qi} \quad \text{with} \quad b_t^{qi} \geq 0, k_t^{qi} \geq 0 \\ da_t^{qi} &= (r_t b_t^{qi} + r_t^l l_t^{qi} + r_t^{qi} k_t^{qi} - c_t^{qi}) dt \\ -r_t^l l_t^{qi} &\leq \gamma m_t q_t k_t^{qi} \end{aligned} \tag{9}$$

while taking initial assets a_0^i and the returns on investment as given.[15]

[15] In overlapping generations model, some agents are constrained in how much they can borrow from the future. It is intriguing, but left for future research, to explore how the borrowing constraints in those models (which generate a bubble premium but also dynamic inefficiency) would interact with the borrowing constraints in this paper.

3.3 Market clearing, equilibrium, and the first best

The economy is closed, so the market clearing conditions for the two assets are:

$$\int\int l_t^{qi} dQ(q)di = 0 \quad \text{and} \quad \int\int b_t^{qi} dQ(q)di = b_t. \tag{10}$$

Letting $k_t = \int\int k_t^{qi} dQ(q)di$ be the aggregate capital stock, I denote the ratio of government bonds to capital in equilibrium $\kappa_t = b_t/k_t$.

A *balanced-growth-path equilibrium* is an interest rate on government bonds r, a share of government bonds $\kappa_t \geq 0$, and a common growth rate for aggregate output, consumption, capital and public debt of g, given an exogenous choice of net spending as a ratio of the public debt s/b, such that: (i) the firm behaves competitively, so according to equation (8), (ii) the consumers behave optimally, so they solve the intertemporal problem in (9), (iii) the government debt satisfies the budget constraint in equation (2), and (iv) markets clear, as in the equations in (10).

Importantly, I restrict attention to the case this paper wishes to study: when $r \leq g \leq m$ and there is permanent spending $s/b \geq 0$. Such an equilibrium may not exist. Indexing each equilibrium by the exogenous s/b, I will later show that it exists as long as $s/b \leq S$. This upper bound on spending, beyond which there is no equilibrium with a positive value of the public debt, is the *fiscal capacity* of the economy.[16]

The first best in the economy is simple. If $\gamma = 1$ then there are no credit frictions, so all households but the highest quality types prefer to lend to the positive mass of $q = 1$ types, who invest in their superior capital stock. Therefore, we are in the textbook AK version of the neoclassical model, with $r = m > g(= r - \rho)$, contradicting the data. Since there is no bubble premium in this economy, the public debt must equal the present value of surpluses, so the fiscal capacity, defined above, is zero.

3.4 The roles of public debt

With incomplete financial markets, the most productive entrepreneurs cannot borrow as much as they would like to invest in their technology. Least productive and risky firms are in business, creating a misallocation of resources that endogenously drives the wedge between the marginal product of capital and the growth rate of the economy.

[16] In this economy, κ could in principle be very large, arbitrarily so. In an overlapping generations model without transfers, there would be a limit to it, because the initial generation cannot save more than its income, and this would provide an additional constraint on spending.

Households are unable to trade the idiosyncratic depreciation risk that they bear if they invest in capital. The public debt is safe, since its returns are uncorrelated with the returns on individual capital. Public debt therefore provides *safety*, and will earn a corresponding premium.

Moreover, public debt provides an alternative to store wealth over periods for households if the constraint on private lending is too tight. Households are aware that they can become high-types in the future, and want to store value for when this happens. Public debt therefore also provides outside *liquidity*, complementing the inside liquidity from private lending, and commanding a premium in return.

The two premia combined lead to $m > r$ creating a bubble premium. If the premia are large enough, it will also be that $g > r$ and the economy can sustain permanent net spending. The model therefore can generate the observed $r < g < m$ as a result of the desire for liquidity and safety in an unequal economy because of borrowing constraints. The misallocation of resources due to incomplete markets creates a role for public debt giving rise to the bubble.

The model gives a simple vehicle to capture two important roles of public debt, and study whether persistent spending that tries to take advantage of the bubble premium is consistent with optimal behavior and markets clearing. If it is, then the size of this spending will endogenously determine r and g, as well as the fiscal capacity S. Policies and public debt will change the relative strengths of the safety and liquidity effects, and so can move the two key spreads, $m - r$ and $m - g$, as well as the fiscal capacity of the economy. In short, the model can give answers to the four questions that the debt arithmetics posed.

3.5 Liquidity only, and more effects

A simpler version of the model provides much of the intuition, and sharply highlights the role of capital misallocation. It assumes away the ex post uncertainty ($\delta(q) = 0$), and reduces the ex ante heterogeneity to only two types of agents:

$$q_t = \begin{cases} 1 & \text{if type } H, \quad \text{share } \alpha \\ 0 & \text{if type } L, \quad \text{share } 1-\alpha \end{cases} \tag{11}$$

If the household is in the high group H, then quality is high (normalized to 1). By renting its capital, the household gets the full marginal product of capital $r_t^H = m_t$. The remaining share of low-type households L have no access to production, as their capital is useless. The probability α plays an important role: the lower it is, fewer agents have access to the good technology where all capital should be invested.

In this simple economy, there is no safety role of debt, since there is no uncertainty. All that remains is the liquidity role of public debt as a store of value, as all agents hope to be high-types in the future. This is a model solely of misallocation of capital across types because of the limits of private credit markets. The high types can only borrow up to the debt limit and invest it all in the productive capital. The misallocation creates a role for public debt because the low types will save in government bonds together with private credit. For the economy to generate $r < g < m$, it must be that $\gamma < 1 - \alpha$: the borrowing constraint must be sufficiently tight that the economy cannot reach the first best, or, alternatively, there must be enough H-types wanting to lend through imperfect markets to the few E-types.

The next section will cover the simple and more general models. I will also consider more complicated versions that extend the results in the following section. First, one can have fiscal policy instead follow a rule that makes net spending as a ratio of private assets be an exogenous s/a. Second, one can include aggregate uncertainty by having the shocks dz_t^{qi} have an aggregate component: now, $\int dz_t^{qi} di = \zeta dz_t$ so ζ measures the correlation between aggregate and individual risk. Third, one can open the economy and have a foreign demand for public bonds according to a demand function $B(r)$ that weakly falls with the interest rate paid. Fourth, one can have diminishing returns to scale by writing the production function instead as $y_t = A_t K_t^{\theta}$ and having TFP grow at an exogenous constant rate. Each of these brings an interesting new consideration to the interplay between spending, the bubble premium, and fiscal capacity.

4 Equilibrium gaps and fiscal capacity

I start by covering the simple liquidity-only model, before moving to the general case.

4.1 The simple model

Since both bonds and private lending are safe investments, they must give the same return $r^l = r$. Then, if this interest rate is too low, the high-quality types would be able to borrow enough in private credit markets to reach the first best. For an equilibrium with

$r < g < m$, it must be that $r > \gamma m$. At the same time, for any production to take place, it must be that in equilibrium $r < m$, otherwise no one would invest.

The appendix writes the dynamic problem solved by households. The high-quality types borrow as much as they can and invest it all in capital, not holding any government bonds. Their consumption and savings then grow at the rate:

$$\frac{\dot{a}_t^H}{a_t^H} = \frac{(1-\gamma)mr}{r-\gamma m} - \rho, \tag{12}$$

reflecting their limits to borrowing and the ability to leverage the m returns on investment. As for the L types, their capital is worthless so they split their assets between private lending and government bonds. The growth rate of their assets and consumption is then:

$$\frac{\dot{a}_t^L}{a_t^L} = r - \rho. \tag{13}$$

Since each type is drawn from the same population, income inequality is the difference between the two growth rates, which is $(m-r)/(r-\gamma m)$.

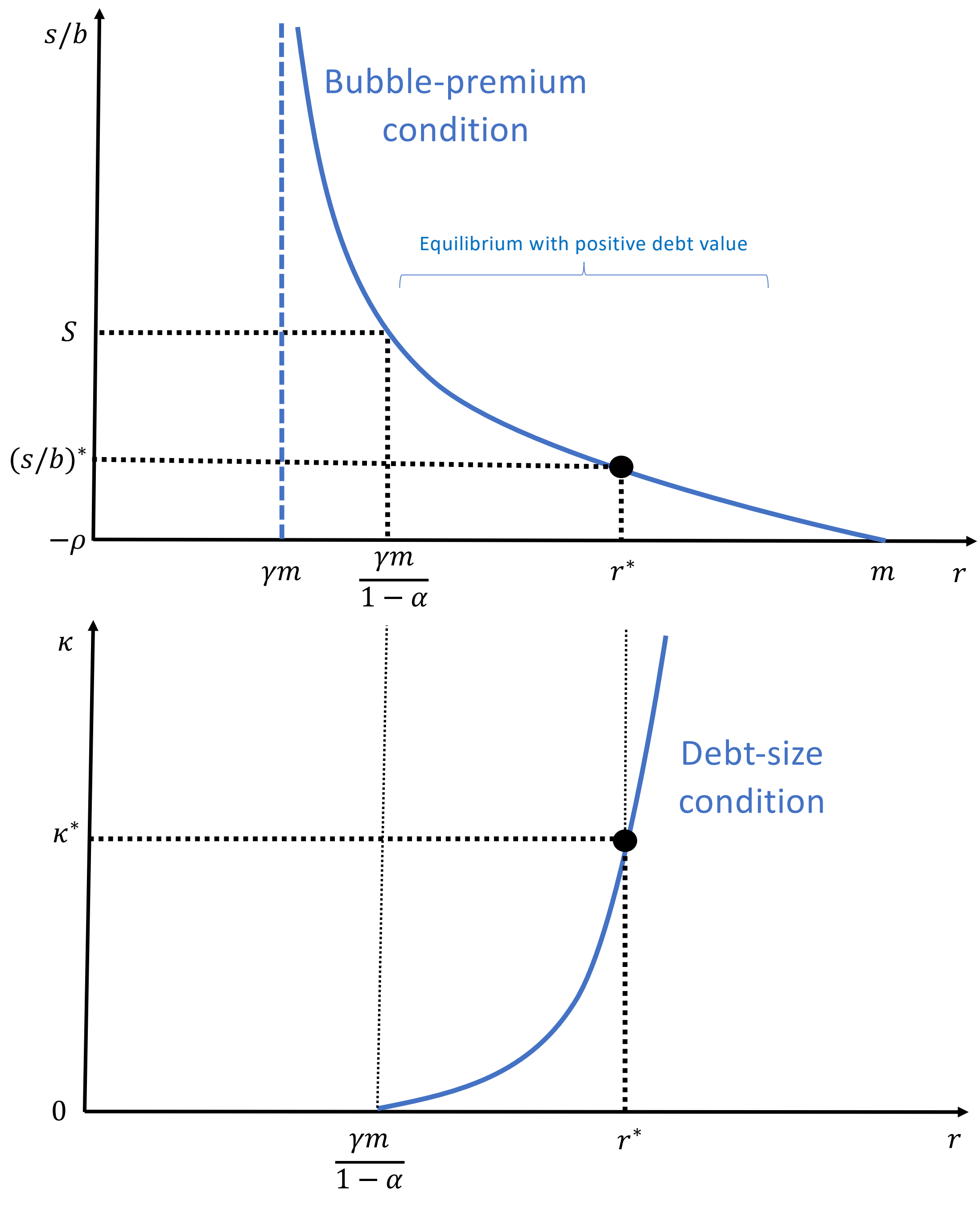
In a balanced-growth path, the growth rate of the economy is the weighted average of these two rates, with weights α and $1-\alpha$, respectively. In turn, the budget constraint of the government imposed that the growth rate is equal to $r + s/b$. Replacing out g, and rearranging gives the key equilibrium *bubble-premium condition*:

$$\frac{s}{b} = \frac{\alpha(m-r)}{1-\frac{\gamma m}{r}} - \rho. \tag{14}$$

Intuitively, the first term on the right-hand side captures the extra return that the high types earn by investing in the private capital stock, and being able to leverage those investments. Since the right-hand side is continuous and monotonic in r, this pins down the unique r solution of the model. The top panel of Figure 2 graphically represents this equilibrium.

At the same time, on aggregate $a = k + b$. Using the definition of the private capital to public debt ratio, this becomes: $a/k = 1 + \kappa$. Now, market clearing in capital implies that $k = k^H$, and since each type is an iid draw, $a^H = \alpha a$. Then: $\alpha a^H/k^H = 1 + \kappa$. The high-type assets equal $k^H - \gamma m k^H/r$, since they borrow to invest in capital and the borrowing

Figure 2: Equilibrium in the simple 2-type model



constraint is tight. The other equilibrium *debt-size condition* then is:

$$\kappa = \frac{1-\alpha}{\alpha} - \frac{\gamma m}{\alpha r}. \tag{15}$$

If no private lending is allowed, then of course, the ratio of bonds to capital is simply the ratio of the wealth of the low types $1-\alpha$ to that of the high types α. As more lending is possible, at a higher gap between the return to capital versus the return to bonds, the more private capital is held, so the lower is the bond-capital ratio. This is depicted in the bottom panel of figure 2. It uniquely solves for the size of the public debt given a solution for r from the top panel.

There is only an equilibrium if $\kappa \geq 0$. From the second equation, this is only the case when the interest rate is above $\gamma m/(1-\alpha)$, which from the first equation requires spending to not be too high. Combining all the results:

Proposition 1. *In the simple 2-type economy there is an equilibrium where government can run a permanent deficit paid for by the bubble premium and:*

- *More spending (s/b) requires a higher bubble premium $m - r$.*
- *More spending (s/b) lowers the ratio of public debt to private assets κ.*
- *More spending (s/b) increases inequality of consumption and asset growth.*
- *The fiscal capacity is:*

$$S = m\left(1 - \frac{\gamma}{1-\alpha}\right) - \rho, \tag{16}$$

so it is smaller if the marginal product of capital is lower (low m), the economy is more financially developed (high γ), or if there are more high productivity types (high α)

Some of these conclusions may seem surprising. But, they follow naturally if the specialness of public debt arises from it allowing people to store their liquidity in an environment with misallocation. When the government spends more, the bubble premium must be higher to sustain the extra permanent spending. Yet, this requires the bond holders, who are the poorer and less productive households, to earn lower returns at the same time as the equity holders, who are richer, can leverage up more and earn higher returns. Therefore, inequality rises. At the same time, the higher spending comes with lower bond holdings, as the households prefer to lend in the private credit market instead. If the spending increase is too high, then the bubble pops, and there is no equilibrium with a positive value for debt since that much spending requires such low interest rates that no one wants to lend to the government. The fiscal capacity depends on the desirability of public bonds relative to private credit. If the economy is financially developed or has many investment opportunities, the bubble premium is lower because the private economy is able to allocate resources better and provide higher returns in credit markets. There is less room to finance public spending.[17]

4.2 The general economy

In the general economy, it is still the case that $r^l = r$, and it must be that $r > \gamma m$ otherwise the economy would reach the first best.

Starting with the household problem, because each type is ex ante identical, her choices of consumption and investment are going to be the same. As the appendix shows, optimal consumption requires that $c_t^{qi} = \rho a_t^{qi}$. Rearranging the budget constraint for each agent in the economy, assets grow according to: $da^{qi} = [r - \rho + (mq - r)(k/a)^q]\, a_t^{qi} dt - \delta(q)(k^q/a^q) a_t^{qi} dz_t^{qi}$. By market clearing, the growth rate in the balanced growth path is given by:

$$gdt = \int \int \left(\frac{da^{qi}}{a^{qi}} \right) \left(\frac{a^{qi}}{a} \right) dG(q) di. \tag{17}$$

Now, the iid assumption implies that each type has the same assets at the start of the period. Moreover, from the government budget constraint, the growth rate must equal $r + s/b$. Combining all of these into the previous equation, gives:

$$\rho + \frac{s}{b} = \int (mq - r) \left(\frac{k}{a} \right)^q dG(q). \tag{18}$$

Depending on their quality, different types of agents sort into different classes according to their investment decisions. For those whose q is lower than r/m, productivity is too low. They prefer to invest zero in capital and put all their assets into either private credit markets or the public debt. The next class is made of those with quality above r/m but below a threshold q^*. Those invest in capital according to its Sharpe ratio: $(k/a)^q = (mq - r)/\delta(q)^2$. For those with lower quality, this is less than their assets, so they invest the remainder in the public bonds or lending. For those with higher

[17] The graphs would suggest that $r \in [\gamma m/(1-\alpha), m]$ and that $\kappa \in [0, (1-\gamma)/\alpha]$. However, it must be that $s/b \geq 0$ or that $g \geq r$. This puts an upper bound on the interest rate $\hat{r} < m$, and therefore an upper bound on κ as well that is below $(1-\gamma)/\alpha$.

quality, they start borrowing in private credit markets, but their borrowing constraint is still slack. Finally, those with $q > q^*$, do not buy any public bonds, borrow up to the limit and, and invest everything in their superior capital: $(k/a)^q = r/(r - \gamma m q)$. The appendix shows that a sufficient condition for $q^* > r/m$ is that there is a $q > 0$ such that $r\delta(q)^2 > 0$. That $q^* < 1$ is guaranteed by the fact that there is a positive mass of agents with $q = 1, \delta(1) = 0$. Combining these investment choices with the previous equation gives the general model's equilibrium *bubble-premium condition*:

$$\rho + \frac{s}{b} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)}\right)^2 dQ(q) + \int_{q^*}^{1} \left(\frac{mq - r}{1 - \frac{\gamma m q}{r}}\right) dQ(q). \tag{19}$$

As in the simple economy, the market clearing conditions imply that: $1/(1+\kappa) = k/a$. But, aggregating over the different types: $k/a = \int \int (k/a)^q (a_t^{qi}/a) dG(q) di$. Combining these two equations with the investment choices discussed above gives the second equilibrium *debt-size condition* in the general model:

$$\frac{1}{1+\kappa} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)^2}\right) dQ(q) + \int_{q^*}^{1} \left(\frac{r}{r - \gamma m q}\right) dQ(q). \tag{20}$$

The equilibrium is the joint solution for r, κ over the two equations as a function of the exogenous s/b. From here it follows that:

Proposition 2. *In the general economy there is an equilibrium where government can run a permanent deficit paid for by the bubble premium and:*

- *More spending (s/b) requires a higher bubble premium $m - r$.*
- *More spending (s/b) lowers the ratio of public debt to private capital κ.*
- *More spending (s/b) increases inequality of consumption and asset growth between those at the top of the income distribution (with $q > q^*$) and those at the bottom (with $q < r/m$).*
- *There is a finite fiscal capacity S, which is smaller if the marginal product of capital is lower (low m), the economy is more financially developed (high γ), if there are more high productivity types (lower $G(q^*)$), or if there is less idiosyncratic risk in the economy (weakly lower $\delta(q)$ for all q)*

All of these properties mirror those in the simple model, with one addition: the consideration of idiosyncratic risk. In the simple model, there were only high and low quality types. In the general model, there is also an intermediate type, which finds refuge in the public debt because it provides some safety against the risk of capital investment. The bubble premium now includes also a safety premium because of the demand for public debt from these agents.

As the proposition shows, both the liquidity and the safety premium work in the same direction. More spending still requires a higher bubble premium because of the debt arithmetics. This still comes with less public debt relative to private capital, as the public debt is less attractive, and it still hurts the bottom of the income distribution because they disproportionately hold the public debt. There is still a finite fiscal capacity, which is smaller if private credit markets work better.

The novelty is that less idiosyncratic risk now directly reduces the demand for safety. Therefore, it lowers the safety premium, and so it reduces fiscal capacity.[18]

5 Other considerations

This section considers extensions of the model covering many of the other considerations considered in the literature.

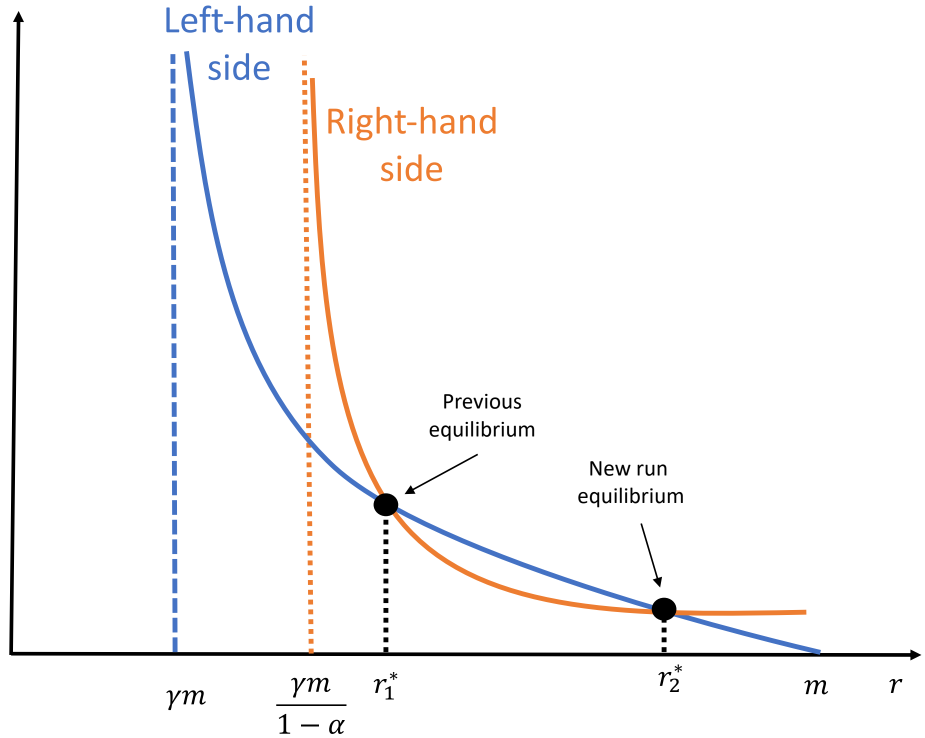
5.1 Net spending as a ratio of private assets

If policy chose a stationary exogenous amount for s, then spending would become an irrelevant fraction of income as time goes by and the economy grows. If spending grows at the rate at which debt, capital, or output grow, the policy choice is just at what level to set s_0. In the model, this choice was made relative to the public debt at that period, since this followed naturally from the debt arithmetics in section 2. At the same time, the propositions showed that when s/b rises, then b/k falls, leaving open whether spending was actually higher or not.

Since initial assets are also exogenous, a natural alternative is to have spending set as a ratio of assets. So, now s/a is exogenous. In the simple model, the two equilibrium conditions in equations (14)-(15) now have to be solved simultaneously for the interest rate and bond holdings. Combining the two, in terms of the new exogenous variable, the

[18] For readers interested in isolating the safety premium, the appendix solves the case where there is a single type q, so there is no liquidity premium in the debt.

Figure 3: Equilibrium with a fiscal rule for s/a



equilibrium interest rate now is:

$$\frac{\alpha(m-r)}{1-\frac{\gamma m}{r}} = \rho + \frac{s}{a}\left(1 - \frac{\alpha}{1-\gamma m/r}\right)^{-1}. \tag{21}$$

As before, the left-hand side falls monotonically with r starting at infinity when r is close to γm, and falling to 0 when $r = m$. Now, the right hand side also falls with r, starting at infinity when r is close to $\gamma m/(1-\alpha)$, the lowest level it can be that is consistent with non-negative bond holdings, and falls to ρ. This is displayed in figure 3.

If s/a is too high, then there is no intersection between the two curves, as higher s/a shifts the right-hand side upwards. In other words, there is still a finite fiscal capacity S. However, now if spending is low, there are two possible equilibria. In one of them, interest rates are low, the bubble premium is high, but bond holdings are small. In this equilibrium, the same implications stated in proposition 1 hold.

Yet now, it is also possible that a change in spending (or even a sunspot) leads to a sudden "run on the debt", where interest rises to a higher level. The bubble premium is then lower, but bond holdings are higher, so the implicit bubble revenues for the government are the same. Empirically, sovereign debt crises indeed feature sudden increases in the debt and sharp rises in interest rates and clear falls in $m - r$. The model suggests that a country may have supplied safe debt for decades, and used the premia on it to sustain permanent spending, but if it is tempted to issue more debt it will find the premium falls and no extra revenue is generated.

5.2 Aggregate uncertainty

Next, consider the case where the shocks that hit the economy have an aggregate component, since: $dz_t^{qi} = \zeta dz_t + d\hat{z}_t^{qi}$. The shock dz_t hits all, so ζ is the covariance of shocks across agents, while $d\hat{z}_t^{qi}$ is idiosyncratic and integrates to zero across households.

Of the two equations determining equilibrium, the debt-size condition in equation (20), is clearly unchanged since it was not affected by the shocks. The bubble-premium condition in equation (19), is now different because it has a new term in the second line of the following equation:

$$\begin{aligned}\rho + \frac{s}{b} = & \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)(1+\zeta)} \right)^2 dQ(q) + \int_{q^*}^{1} \left(\frac{mq - r}{1 - \frac{\gamma m q}{r}} \right) dQ(q) \\ & - \left[\int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)(1+\zeta)} \right) dQ(q) + \int_{q^*}^{1} \left(\frac{\delta(q)(mq - r)}{1 - \frac{\gamma m q}{r}} \right) dQ(q) \right] \zeta dz_t. \end{aligned} \tag{22}$$

An aggregate shock that raises depreciation now lowers the right-hand side. This lowers interest rates, just as raising spending did, since the economy has less output now.

For there to be a BGP, then s/b can no longer be constant. Rather, spending would have to fall whenever there is a bad depreciation shock; by how much, is given by the expression in square brackets. Conditional on doing so, then a change in average spending as a ratio of bonds would have the exact same effects as described in the propositions. However, because the aggregate shocks raise the risk of investing, the first term on the right hand side is also now lower: agents want to hold less capital. This raises the safety premium on the debt, and so it raises the bubble premium and increases fiscal capacity.[19]

[19] A different, quantitative, question is whether discounting by a now-stochastic discount rate, as opposed to the average MPK, raises or not the size of the bubble premium term. This is left for future work.

5.3 Foreign demand for public bonds

Imagine now that, on top of the demand from households with lower-quality capital, there is also a foreign demand for domestic government bonds. Therefore, total government bonds b_t, also include an amount $B(r - r^f)$, growing with the economy at rate g, so it does not become negligent. The foreign demand falls with the gap between the domestic interest rate and a foreign counterpart r^f, as the returns offered to foreigners are smaller.[20] Of the two equilibrium equations of the model, only the debt-size condition in equation (20) changes, as the left-hand side has a new term:

$$\frac{1}{1+\kappa} + \frac{B(r)}{a_0} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)^2} \right) dQ(q) + \int_{q^*}^{1} \left(\frac{r}{r - \gamma mq} \right) dQ(q). \tag{23}$$

At one extreme, imagine that there is perfect capital mobility, so unless $r = r^f$ there is infinite demand or short selling by the foreigners of the public debt. Given the fixed r at its international level, then the other equilibrium bubble-premium condition, equation (19), shows that there is a unique s/b consistent with that equilibrium. The reason is that for a given r, the bubble premium $m - r$ is now exogenous in this economy. The government budget constraint imposes that s/b can no longer be freely chosen, but there is a unique level of it consistent with a positive value of government debt. The fiscal capacity S is now equal to this level. For the fixed interest rate, more financial development (γ) still raises the fiscal capacity as before.

At the other extreme, imagine that $B(r - r^f)$ is inelastic, or a constant. Then, nothing changes in the analysis of the model. There is a constant in the left-hand side of equation (23), and if foreigners demand more bonds, this obviously raises the share of public debt to private capital in the economy, and raises the fiscal capacity S.

In between these two cases, if $B(r - r^f)$ has a finite negative derivative, the model works in the same way and proposition 2 still holds. Now, more spending, lowers domestic interest rates, which lowers demand for public bonds from domestic households and now also from foreigners. Therefore the bubble revenue increases by less, and the fiscal capacity, though higher, is reached faster.

[20] Since this paper is about public debt, I focus on the government borrowing from abroad. If instead households can also borrow from abroad at a fixed r^f, then the model becomes less interesting, since domestic private lending no longer actively constrains the allocation of capital across types.

5.4 Diminishing returns and transition dynamics

Assume now that output is given by $y_t = A_t K_t^{\theta}$, where A_t grows deterministically at the rate $g/(1-\theta)$, and K_t is quality adjusted capital defined just as before: $K_t = \int \int q_t k_t^{qi} dG(q) d(i)$. The marginal product in the first best economy would now be:

$$m_t = \theta \left(\frac{Y_t}{K_t} \right), \tag{24}$$

which continues to be deterministic.

Replacing this expression for where the exogenous m_t appeared in the model, the other parts of the model are unchanged. Since this is a neoclassical model in terms of aggregates, the economy converges to a balanced-growth path where output and the capital stock all grow at rate g. Therefore, $m_t \to m = \theta Y/K$ a constant. The two equilibrium conditions along the balanced growth path are just the same as in the general model already studied.

There are however two important differences relative to the previous analysis. First, the steady state m is now endogenous. More spending lowers the interest rate R as before. This lowers the growth of income for the low-types, while raising it for the high-types, who can now increase their leverage. In equilibrium, as agents substitute from bonds to capital, the steady state capital stock rises and the marginal product of capital m fall. In turn this makes the interest rate fall by less than it did before. Because the steady state capital-output ratio changes in response to more spending, the impact of spending on the bubble premium is smaller, and so the fiscal capacity is higher.

Second, there is now a transition to that steady state. During that transition, as the economy accumulates capital, the marginal product of capital m falls. During that transition whether r rises or not will depend on the fiscal rule for spending and how it changes with respect to capital and public bonds over time. Therefore, the bubble premium may converge from above or below to its steady state giving an illusion that the debt is safer and more liquid, or vice-versa, then before. In fact, it is the relative value of that safety and liquidity, that is changing, and the bubble premium measures it.

6 Monetary-fiscal policy trade-offs

Monetary, regulatory, transfer, and taxation policies affect the equilibrium growth rate and interest rate in the economy, interacting with the amount of government spending.

Therefore, they affect the bubble premium on the government debt, and so the ability to run perpetual deficits and their size. Insofar as the policies are chosen by a different policymaker than the one choosing public spending, conflicts will arise. This section studies these effects, and their trade-offs.

I investigate this in two complementary ways. First, by asking whether the policy lowers or raises the fiscal capacity S. I answer this by changing the policy, while keeping spending s/b fixed (at a level that is below the capacity both before and after the change). Second, I instead let government spending respond to the policy change in order to keep the interest rate fixed, and ask whether the policy leads permanent spending s/b to rise or fall. In this second question, by keeping r fixed, the bubble premium $m - r$ is unchanged with the policy, so the effect of the policies on spending will come from whether they increase the demand for government bonds. If s/b is higher after the policy, I say that *fiscal space* has increased (as opposed to capacity, S).[21]

6.1 Monetary policy: inflating the debt or deflating the bubble?

Assume that inflation is positive and stochastic:

$$dp_t/p_t = \pi_t dt + \sigma_\pi dz_t^\pi, \tag{25}$$

where π_t is the expected inflation rate, and dz_t^π are aggregate shocks to inflation, uncorrelated with the idiosyncratic shocks to the depreciation of the capital stock. I take these as given, implicitly assuming that the classical dichotomy holds. It would be standard to assume there is a central bank that chooses a nominal interest rate according to a Taylor rule, in which case π_t could be its inflation target and dz_t^π the monetary shocks.[22]

The debt dynamics are now given by:

$$db_t = s_t dt + r_t b_t dt - b_t \sigma_\pi dz_t^\pi, \tag{26}$$

since r_t is an ex ante real return, but ex post an inflation shock lowers the real value of the debt.[23] If $\sigma_\pi = 0$, then nothing of substance changes in the analysis. Because r_t is the real

[21] One potential use of this second question would be in applying the ideas in this paper in a fiscal theory of the price level. The adjustment in s, a real variable, could come about by an adjustment in the price level.

[22] Introducing nominal rigidities, and studying how they affect the both inflation and the equilibrium real interest rate is left for future work.

[23] The assumption that inflation is iid simplifies the analysis because this expression holds regardless of the maturity of the debt. In general, the maturity of debt held by the public has a large impact on how

return on the bonds, a higher expected inflation π_t simply leads the nominal price of the bond to rise faster over time leaving r_t unchanged.[24]

Ex post, shocks to inflation make a difference. A large positive shock lowers the real value of the debt, which would loosen the government's budget constraint and directly allow for s_t to rise that period. At the same time, in the other direction, unexpectedly lower inflation raises the real value of the debt. On average the ex post effects cancel out.

Ex ante, inflation uncertainty matters. Assume that the fiscal rule is $(s_t/b_t)dt = \sigma dt + \sigma_\pi dz_t^\pi$ neutralizing the ex post effect so the economy is still in a deterministic balanced growth path. Then, all that remains is the uncertainty from investing in the government bonds. All else equal, this uncertainty lowers bond holdings, as public debt is less safe. The interest rate rises, and this shrinks the fiscal capacity. Also, adjusting σ to keep r fixed, then this lowers bond-holdings κ. Therefore, the implicit revenues from the bubble are smaller, and permanent spending has to fall.

Collecting all of these results:

Proposition 3. *Changes in expected inflation π_t have no consequence on government spending. A higher variance of inflation σ_π lowers the fiscal capacity S and the fiscal space s/b.*

During times of fiscal trouble, discussions about inflation tend to focus on the ex post benefits from inflating the debt. However, if the spending was being financed using the bubble premium that results from $r < g < m$, then the desire to inflate leads in equilibrium to an inflation risk premium in bonds. This reduces the safety and store of value premium on those bonds. Therefore, the bubble is smaller. As a result, attempts to inflate away the debt when bond holders are forward-looking reduces the bubble value of the debt, and tightens the budget constraint of the government.

To loosen the debt burden on the fiscal authority, the best action for monetary policy in this economy is to stabilize inflation as much as possible. This has a footprint on the government's budget, because it permanently lowers the inflation risk premia that must be paid on the debt, creating fiscal capacity. Price stability generates fiscal resources, while a switch to monetary instability can trigger a rise r and cause a sovereign debt crisis.[25]

much of the debt can be inflated away (Hilscher, Raviv and Reis, 2014).

[24] Using the case of the bond as a nominal annuity introduced earlier, now the nominal price of the bonds v_t is simply given by $\dot{v}_t/v_t = (r_t - \xi + \pi)/(1 - \xi)$, but the real value b_t is unchanged.

[25] Consistent with the model, Galli (2020) finds empirically that higher σ_π is positively correlated with a higher r and the incidence of fiscal crises when the country hits its fiscal capacity.

6.2 Financial repression: sacrificing private credit

A common form of financial repression is to force the financial system to hold under-priced government bonds. This is sometimes done by central banks through reserve requirements that do not pay interest. Other times, it is done by financial regulators that require financial institutions' assets to be held in safe investments for macro-prudential reasons, when in many countries the only safe asset is a liability from the government. In more extreme times, of war or after large expenses, the government may legally or through strongly-stated moral suasion force financial markets to lend funds to support public programs at a fixed discounted rate.

In the model, separate the public debt into a coerced and a voluntary amount:

$$b_t = b_t^c + b_t^v. \tag{27}$$

The voluntary debt is freely chosen by agents given a return r_t just as before. The coerced debt is mandatory and pays a below-market return. For simplicity, I assume that every agent holds the same amount of coerced debt, and that the forced return is zero. The debt dynamics are therefore now given by:

$$db_t = s_t dt + r_t b_t^v dt. \tag{28}$$

The household choices on consumption and investment do not change with respect to the their voluntarily-disposed assets $a_t - b_t^c$. The bubble-premium condition in equation 19 that determines the interest rate is now:

$$\rho + \frac{s}{b} - r\left(\frac{b^c}{b}\right) = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)}\right)^2 dQ(q) + \int_{q^*}^{1} \left(\frac{mq - r}{1 - \frac{\gamma mq}{r}}\right) dQ(q). \tag{29}$$

The only difference is the new term on the left-hand side. It implies that the left-hand side now also falls with r, just as the right-hand does.

All else equal, an increase in the share of debt that is coerced will raise interest rates. The implicit revenue from financial repression lowers the need to have the bubble premium to finance the spending. At the same time, the voluntary bond-holding can be smaller. Both effects combined mean that the fiscal capacity S rises. Alternatively, keeping the bubble premium fixed, if r is unchanged, the right-hand side of the equation above is unchanged. Then, a higher b^c/b must allow for a higher s/b and the fiscal space rises.

Collecting these results:

Proposition 4. *An increase in the share of public debt that is coerced raises the fiscal capacity S and the fiscal space s/b.*

Financial repression imposes a repression premium on the coerced debt. This unambiguously loosens the government budget constraint, which perhaps explains why this policy is so often used when governments get in fiscal trouble. At the same time, it also lowers growth because it worsens the allocation of capital. If tax revenues depended on economic activity, the decline caused by repression would automatically raise net spending. Moreover, repression affects the chances that there are financial crises, and these can require very large increases in spending s.[26]

6.3 Redistributive policy: lowering inequality versus raising public spending

Households with access to high-quality capital earn higher returns and so have a higher income than those who are unfortunate to have low-quality capital. Since all households are ex ante identical, a utilitarian social planner would be tempted to address this inequality by taxing the former and redistributing to the latter. This may even raise welfare by providing social insurance.

Usually, redistribution comes with distortions to incentives. Since these will be studied in the next case, here I consider a best-case scenario where the redistribution is done through a tax-and-transfer on the initial assets of households. The government taxes a share ψ of the assets of the high-quality types (those with $q > q^*$) and transfers its proceeds directly to the lower types that only earn the safe rate r on their income (those with $q < r/m$). The debt dynamics are unchanged since the program does not generate net revenues for the government. However, now the key equilibrium condition that determines interest rates becomes:

$$\rho + \frac{s}{b} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)} \right)^2 dQ(q) + (1 - \psi) \int_{q^*}^{1} \left(\frac{mq - r}{1 - \frac{\gamma mq}{r}} \right) dQ(q). \tag{30}$$

An increase in ψ lowers the left-hand side. It therefore raises the equilibrium interest rate on government debt and lowers the public bonds as a ratio of private capital that is

[26] See Reis (2020), Acharya, Rajan and Shim (2020) for the interaction of these fiscal footprints of macroprudential policy.

held in equilibrium. This brings the economy closer to the point where agents to not want to hold debt. Therefore, it lowers the fiscal capacity S. In turn, it follows immediately from the equation above that, keeping r fixed, a higher ψ lowers the right-hand side, and so leads to a fall in fiscal space s/b.

Proposition 5. *A larger redistribution program lowers the fiscal capacity S and the fiscal space s/b.*

The intuition is that more redistribution gives the low-quality capital types more assets that they want to lend out as they await for a good entrepreneurial opportunity. The financial friction prevents this lending to go to higher-quality capital, trapping assets with their poor owner, as those who want to borrow cannot do so as much, since they have fewer assets. The growth rate of the economy falls, shrinking the ability to sustain government spending. Proposition 2 already showed that more government spending increases inequality. Conversely, reducing inequality through redistribution lowers the spending the government can do.

This gives rise to an interesting policy trade-off. A policymaker that is focused on inequality and approves a large transfer program will constrain the ability of a different policymaker that is focused instead on spending, say to provide public services or infrastructure.[27] A conflict will arise between the two. Consider a political system where parties alternate in power and have different preferences for inequality vis-a-vis public spending. It is well known, empirically and theoretically, that polarization of this type may lead to over-spending. The result above suggests that, because polarization also lowers fiscal space and fiscal capacity, it will heighten the risk that the public debt bubble pops.

6.4 Fiscal policy: tax cuts that pay for themselves?

Consider now the effects of distortionary taxation. Income is taxed at the rate τ and its proceeds are rebated to the households through lump-sum transfers T_t^q. I assume that the transfers depend on the quality type so that the tax cut involves no redistribution, already studied in the previous case. Likewise, I assume that the scheme's budget is balanced so

[27]More formally, assume both policymakers maximize the ex ante expected utility, integrating over all types, but that while the policymaker puts a very large weight on welfare on the additive term in s, the other one gives it a zero weight. The former will then focus on maximizing s, while the latter will focus on social insurance and redistributing to complete markets.

to leave out the direct (and uninteresting) effect of tax revenues in creating fiscal space.[28]

The budget constraint of the household in equation (9) changes to:

$$\begin{aligned} da_t^{qi} &= (1-\tau)(r_t b_t^{qi} + r_t^l l_t^{qi} + r_t^{qi} k_t^{qi})dt + (T_t^q - c_t^{qi})dt - \delta(q) dz_t^{qi}, \\ -r_t^l l_t^{qi} &\leq (1-\tau)\gamma m_t q_t k_t^{qi}. \end{aligned} \tag{31}$$

The tax lowers the returns on investment, as shown in the first equation. I assume that depreciation is not deductible but that interest payments on the debt are deductible. More novel, and interesting, taxes also tighten the borrowing constraint. This is because the entrepreneurs cannot pledge the tax bill for repayment of their debts. In most countries, taxes are collected throughout the year, and the legal penalties from not paying taxes are much higher than those from not paying back private debts.

This form of taxation changes the model in two ways. First, the borrowing constraint is now tighter since taxes are always paid and deduct from pledgeable income. This raises the demand for government bonds, since private lending is restricted. In the two-type simple model, the debt-size condition in equation (15) just becomes:

$$\kappa = \frac{1-\alpha}{\alpha} - \frac{(1-\tau)\gamma m}{\alpha r}. \tag{32}$$

The fall in the right-hand side with higher taxes shifts this condition to the left in figure 2. Thus, fiscal capacity S rises.

At the same time, by reducing leverage, taxes lower the return from investing in capital. Moreover, because taxes lower the returns to saving, the after-tax marginal product of capital is also lower. The bubble-premium condition in equation (14) becomes:

$$\frac{\alpha(1-\tau)(m-r)}{1-\frac{(1-\tau)\gamma m}{r}} - \alpha\tau r = \rho + \frac{s}{b}. \tag{33}$$

A higher τ lowers the left-hand side, so it shifts the condition downwards in figure 2. Therfore, for a fixed r, the feasible spending is lower. Therefore, fiscal space falls.

Collecting these results:

Proposition 6. *A marginal increase in the tax rate on all income that is fully rebated to each type of household, raises fiscal capacity S but lowers fiscal space s/b.*

[28] Since in the model all income is from capital investments, another way to think of the policy change is as a capital income tax financing a cut in labor income taxes.

The direct effect of an increase in a tax rate is to increase fiscal revenues, which by itself of course raises fiscal space and capacity. A well-known counter effect is that taxes reduce the incentives for saving and capital. The proposition points to a new effect: with misallocation, the tax also lowers the ability of high-quality types to borrow. This directly lowers private investment and growth, and with it the bubble premium. If the increase in tax revenues is more than offset by the fall in bubble revenues, fiscal space falls.

As for the fiscal capacity, the tax was akin to a fall in development of the financial market (a lower γ) which by itself raises capacity (proposition 2). Taxes in this case would not just bring revenue, but also raise the demand for government bonds by reducing activity in private credit markets.

The question of whether tax cuts ever pay for themselves is a classic one in economics. The empirical debate revolves around measuring whether the tax base rises after a fall in the tax rate, because it incentivizes investment or labor supply. The perspective offered in the result above is different. First, because it suggests that deficit-financed tax cuts, by increasing the public debt, raise a source of revenue for the government. Second, because it suggests measuring how $m - r$ responds to a tax cut. This would combine estimates of multipliers with estimates of direct crowding-in effects of tax changes on interest rates, but where the elasticity of investment to interest rates is irrelevant. These are intriguing cues for future research to pursue.

7 Conclusion

Public debt is expected to exceed 120% of GDP by 2021 on average across advanced economies, matching or exceeding the previous peak (in 1945) in the last 140 years. At the same time, interest rates relative to the growth rate of the economy are low in most advanced economies, even relative to a history where $r < g$ quite frequently.[29] This paper asked what is the constraint on public debt when there is a bubble ($r < g$) but the economy is dynamically efficient ($g < m$). Unlike previous results in dynamically inefficient economies, it found that there was a well-defined government budget constraint and a limit to spending. The constraint is relaxed by a bubble premium on the debt, which is the difference between the return that private agents can earn on the marginal unit of capital as opposed to lending to the government. This makes permanent deficits feasible even if the public debt is positive, but their size may be quantitatively small, and spend-

[29] Sources: IMF Fiscal Monitor of October 2020, and Mauro and Zhou (2020).

ing more will affect the bubble premium changing debt arithmetics. Monetary and fiscal policies may unexpectedly bring the existing public debt closer (or further away) from the fiscal capacity limit. In sum, by focussing on the bubble premium $m - r$, and what determines it, this paper showed that new lessons arise on the feasibility as well as limits of government spending.

Working through debt arithmetics led to four questions, stated in the introduction. First, why is $r < g < m$? The paper provided a model that matches the empirical measures that public debt provides safety and liquidity creating a bubble premium / convenience yield / seignorage that can pay for some spending. Second, how does spending affect the equilibrium bubble premium and bond holdings, as well as other variables? The model predicts that more spending raises the bubble premium but lowers bond-holdings, increasing inequality along the way. Third, is the fiscal capacity of the economy—the largest spending feasible without driving the value of debt to zero—finite, and what does it depend on? There is a finite limit to spending, and it is smaller if the economy is less productive, has more developed private financial markets, and less idiosyncratic risk. How do policies affect fiscal capacity and fiscal space? Inflation backfires, repression works but is costly, redistribution results in a tighter government budget constraint, and distortionary income taxation changes the size of the public debt bubble by changing the allocation of capital, on top of its usual revenue and Laffer-curve effects.

Some of the results were surprising, while others perhaps less so, but all together they lay out clear policy trade-offs. The novelty is to think of debt limits as partly driven by the bubble premium, the gap between the marginal product of capital and the interest rate on government bonds. This raises some new empirical challenges: How much does the bubble premium respond to shocks to government spending? How much does it change with other policies? How do different compositions of the public debt affect the overall bubble premium? Can the three-way interaction between the stochastic discount factor, the bubble premium, and the amount of debt make quantitative progress on solving the debt valuation puzzle? Do the considerations of $r < g < m$ quantitatively change the strategic incentive for countries to default well before they reach their fiscal capacity? What is clear for now is that while there are no fiscal free lunches, there are more ways to pay for the public debt.

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Appendix – For Online Publication

This appendix contains further derivations not in the main manuscript.

A General model: the household problem and q^*

The household problem in (9) can be written as (dropping superscripts):

$$\max_{\{c_t/a_t, k_t/a_t\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_t dt \right]$$

$$\text{subject to:} \quad da_t = \left[r_t + (m_t q_t - r_t)(k_t/a_t) - c_t/a_t \right] a_t dt - \delta(k_t/a_t) a_t dz_t \tag{A1}$$

$$0 \leq k_t/a_t \leq \frac{r_t}{r_t - \gamma m_t q_t}$$

For generality, let the conditional distribution across types be $Q(q'|q)$, and say that changes in type arrive as a Poisson process with rate v. Then, the state variables for the household are her assets-type pair. Ignoring the constraints on capital, the associated Hamilton-Jacobi-Bellman equation is:

$$\rho V(a,q) = \max\{ \log(c) + V'(a,q) \left[r + (mq - r) \left(\frac{k}{a} \right) - \frac{c}{a} \right] a + \tag{A2}$$

$$\left(\frac{V''(a,q)}{2} \right) \left(\frac{k}{a} \right)^2 \delta^2 a^2 + v \int (V(a,q') - V(a,q)) dQ(q'|q) \} \tag{A3}$$

where $V'(.) \equiv \partial V(.)/\partial a$. It is standard to derive that at an optimum:

$$c = \rho a \tag{A4}$$

$$\frac{k}{a} = \frac{mq - r}{\delta(q)^2} \tag{A5}$$

$$V'(.) = \frac{1}{\rho a} \tag{A6}$$

$$\lim_{t \to \infty} e^{-\rho t} V'(.) a_t = 0 \tag{A7}$$

Combining A6 and A7, it is clear that the transversality condition is always respected as long as $\rho > 0$.

Recall that $\delta(q)$ decreases monotonically with q. Therefore, the optimal k/a above is monotonically increasing in q. It then follows that imposing the constraints leads to those with $q < r/m$ choosing $k/a = 0$, while those with a high q will be at the borrowing constraint and so have $k/a = r/(r - \gamma mq)$. This finishes the solution to the household problem. Finally, note that for the iid case, I take $Q(q'|q) = Q(q')$, so independent of the current q, and $v \to \infty$ so that every instant the agents draw a new type.

Next comes showing that $r/m < q^* < 1$. Recall that q^* is the quality threshold after which agents exhaust their borrowing constraint and hold all assets in capital. The q^* is defined by:

$$\frac{mq^* - r}{\delta(q^*)^2} = \frac{r}{r - \gamma mq^*} \quad \Leftrightarrow \tag{A8}$$

$$\gamma m^2 q^{*2} - (1+\gamma)mrq + r(r + \delta(q^*)^2) = 0. \tag{A9}$$

Start with the case where $\delta(q^*)$ is constant, so this is a quadratic equation of the form $f(q^*) = 0$ with a single root in $[0, 1]$. Next, it is easy to see that $f(0) = r\delta(0)$ and $f'(0) = -(1+\gamma)mr$. Therefore, as long as $r\delta(0)^2 > 0$ we will have $q^* > 0$. Next, note that $f(1) = (m - r)(\gamma m - r) < 0$ since there is a positive mass of agents with $q = 1, \delta(1) = 0$. Therefore, $q^* < 1$.

B Liquidity-only model: an upper bound on the interest rate

Recall from equation (2) that $s/b = g - r$,so the condition $g \geq r$ reduces to $s/b \geq 0$ or that permanent net spending (which is exogenous) is positive. Using equation (14), this condition becomes:

$$r^2 - (m - \rho)r - \gamma m\rho \geq 0. \tag{A10}$$

It is easy to show that the quadratic has a single root in $[\gamma m/(1-\alpha), m]$, call it $\hat{r}$, and that this root is interior, proving that there is an upper bound on an upper bound on the interest rate $\hat{r} < m$. Since κ increases with r, this also puts an upper bound on κ that is below $(1-\gamma)/\alpha$.

C A safety-only economy

In this case, there is a single type, say $q = \eta, \delta(q) = \delta$, and no private credit is possible, so $\gamma = 0$. In this case, there is no role for the public debt in allowing low-types to transfer value into the future when they might become high-types. Only the role for public debt as providing safety remains, so this special case allows us to isolate and study it.

In this case, the growth rate of assets is

$$\frac{\dot{a}_t}{a_t} = r - \rho + \left(\frac{m\eta - r}{\delta}\right)^2. \tag{A11}$$

Since the left-hand side is equal to g, which in turn from the government budget constraint is equal to $s/b + r$, then the equilibrium interest rate solves

$$r = \eta m - \delta\sqrt{\frac{s}{b} + \rho}. \tag{A12}$$

Next, since all households choose to hold the same share of assets in bonds (equation A5), the equilibrium bonds held as a ratio of assets is:

$$\frac{b}{a} = 1 - \frac{m\eta - r}{\delta^2}. \tag{A13}$$

Recalling that $1 - b/a = 1/(1+\kappa)$ and using the solution for r form above this becomes:

$$\kappa = \frac{\delta}{\sqrt{\frac{s}{b} + \rho}} - 1. \tag{A14}$$

Note right away that an increase in uncertainty (δ) raises the desire for precautionary savings. So, it raises the holdings of government debt κ, while pushing down the interest rate r.

An increase in permanent spending as a ratio of debt lowers both r and κ fall, just as happened in the simple model with only a demand for liquidity, and in the general model with a demand for safety and liquidity. The intuition is that more spending requires a larger bubble premium. But, since the bubble premium is solely a safety premium, it must be that households are investing more in the risky technology. Because this increases overall risk, then the safety of government debt is more valuable, its bubble premium rises, and so more persistent spending is possible. Higher spending also raises inequality since the lower safe interest rates induce households to invest more in their risky technologies, which have dispersed ex post returns. Finally, the upper bound on spending so that the government can spend forever is now:

$$S = \delta^2 - \rho. \tag{A15}$$

There has to be enough risk in the economy to drive r sufficiently down and create a bubble premium in the public debt. As in the general model, if the economy becomes less risky with financial development, then the upper bound S falls

In short, the two motives for why $r < g$ in the model— the uses of public debt as a store of value and as a safe harbor—complement each other.