Dissipative and Non-dissipative Models for Correlated Wild Fluctuations in Complex Geosystems

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Thanks:

1. Many collaborators over the years, particularly Sandra.

2. Meeting organisers Valerio & Tobias. Martin Rypdal and Ola Lovsletten for our ongoing collaboration on multifractals **[Fall AGU, 2012]**

3. **Many historical sources, including:**

Statistics for Long Memory Processes, Beran, CRC, 1994

Selfsimilar processes, Embrechts & Maejima, Princeton, 2002

Fractals, Feder, Plenum 1988.

Mandelbrot, Selecta Volumes, Springer 1997-2002.

Obituary of H E Hurst, Hydrological Sciences Bulletin, 24, 4, 12/79

Abstract:

I will discuss the hierarchy of approaches to complex systems, focusing particularly on stochastic equations. I discuss how the main families of model advocated by the late Benoit Mandelbrot fit into this classification, and how they continue to contribute to cross-disciplinary approaches to the increasingly important problems of correlated extreme "bursts", and unresolved scales. The burst models complement the fluctuation theorems under discussion at NeSEE. Their applications have ranged across science areas as diverse as the heavy tailed distributions of intense rainfall in hydrology, after which Mandelbrot named the "Noah effect"; the problem of correlated runs of dry summers in climate, after which the "Joseph effect" was named; and the intermittent, bursty, volatility seen in finance and fluid turbulence.

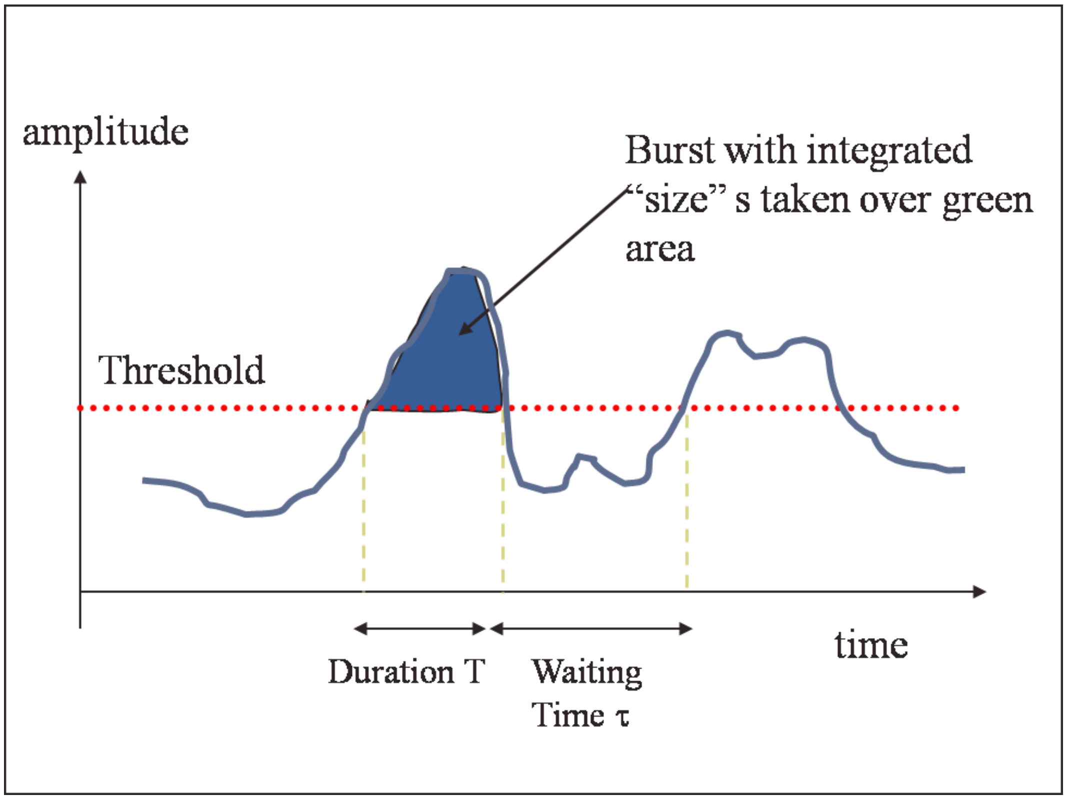
Based on: **Watkins et al, PRE, 2009**

Watkins et al, in "Extreme Events & Natural hazards: the complexity perspective" , AGU Monograph, 2012

Watkins, GRL Frontiers, 2013, doi:10.1002/grl.50103

What problem am I considering ?

Integrated bursts in complex time series



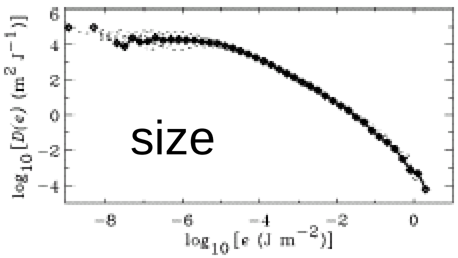
$$s = \int_{t_i}^{t_f} y(t)dt$$

$$T = t_{d_i} - t_{u_i}$$

$$\tau = t_{u_{i+1}} - t_{d\,i}$$

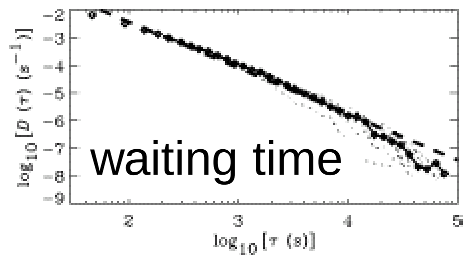
Why do we want to study it ?

log P(s)



log s

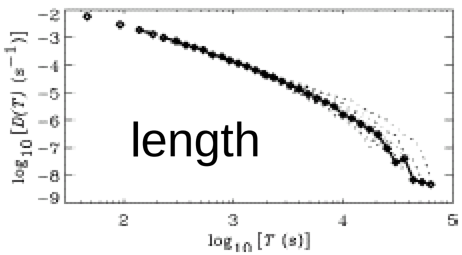
log P(τ)



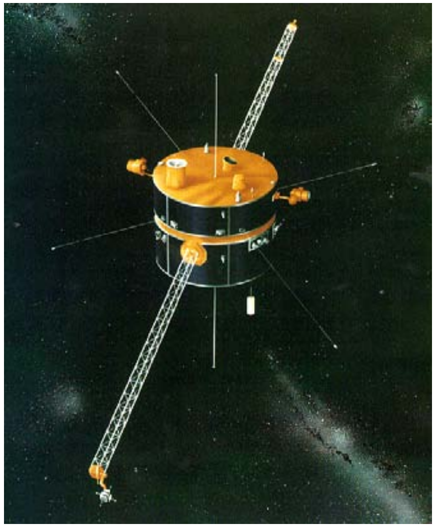
log τ

Data

log P(T)



logT



Many reasons, one example:

Poynting flux in solar wind plasma from NASA Wind Spacecraft at Earth-Sun L1 point
Freeman, Watkins & Riley [PRE, 2000].

What do we already know?

Quite a lot in the Gaussian, white noise-based cases

• Times for white noise level crossings governed by Rice formulas [e.g. Kratz, Probab. Surveys, 2006] .
Amplitudes for exceedances above (high) thresholds governed by Pickands Balkema de Haan theorem [e.g. Coles, 2001]

•Random walks are H-selfsimilar, so can use scaling arguments on burst problem [e.g. Kearney and Majumdar, J Phys A 2005] (or something very close to it)

•Langevin equation seems trickiest case, as has intrinsic (damping) scale, and sacrifices H-selfsimilarity for long time stationarity.

Much less is known in general case

Talk first briefly recaps some textbook equations of fluctuations

- Cold linear (& nonlinear) deterministic dynamics.
- Cold deterministic dissipative dynamics vs warm stochastic dissipative **Langevin**
- Hot, **Smoluchowski** limit, diffusion equation

Then considers 2 of the anomalies highlighted by Mandelbrot and his modern successors [c.f. Watkins, GRL Frontiers, 2013]

- First anomaly identified in finance in 1963, heavy tails, Noah effect.
- Second anomaly was in hydrology 65-68s, long range dependence, Joseph effect
- United in a fractional hyperbolic noise model in 1969, forerunner of LFSM.

Then show work in progress [also Watkins et al, PRE, 2009] on bursts in heavy tailed, LRD walks and, time permitting, multifractals.

Cold deterministic dynamics

Linear: e.g. Newton's falling apple

$$F = ma = m\ddot{x} = -V'(x)$$

Nonlinear: e.g. Henon-Heiles perturbed oscillator

$$V(x,y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Dissipative dynamics: cold & warm

Cold (x is deterministic): Ohmic

$$m\ddot{x} = -\eta\dot{x} - V'(x)$$

Warm (x is now stochastic): Ohmic Langevin

$$m\ddot{x} = -\eta\dot{x} - V'(x) + \xi$$

Hot diffusive dynamics

$$\frac{\partial}{\partial t} p(x,t) = D_0 \nabla^2 p(x,t)$$

Results in Smoluchowski limit, rather than the trajectories followed by the Langevin equation, evolves the pdf.

Mandelbrot's "anomalies" [1963 – 1974]

- Heavy tailed fluctuations in 1963 in cotton prices---advocates α-stable model (J. Business, 1963). Then abstracts out H-selfsimilarity idea.

- Hears about River Nile and "Hurst effect". Initially (Selecta) believes will be explained by heavy tails, but sees that fluctuations are light tailed & applies self-similarity [Comptes Rendus1965] in a long range dependent (lrd) model, source of fractional Brownian motion. Classic papers on fBm in SIAM Review and Water Resources Research, with Van Ness, and Wallis [1968-1969].

- Demonstrates new self-similar model, fractional hyperbolic noise, in 1969 paper with Wallis on "robustness" of R/S. Combines heavy tails & lrd.

- Becomes dissatisfied with purely self-similar models, & advocates multifractal cascades, initially in context of turbulence, JFM 1974. Subsequent applications include finance [above story told in his Selecta, and in Mandelbrot and Hudson, "The (mis)behaviour of markets"]

“anomalously”wild symptoms …

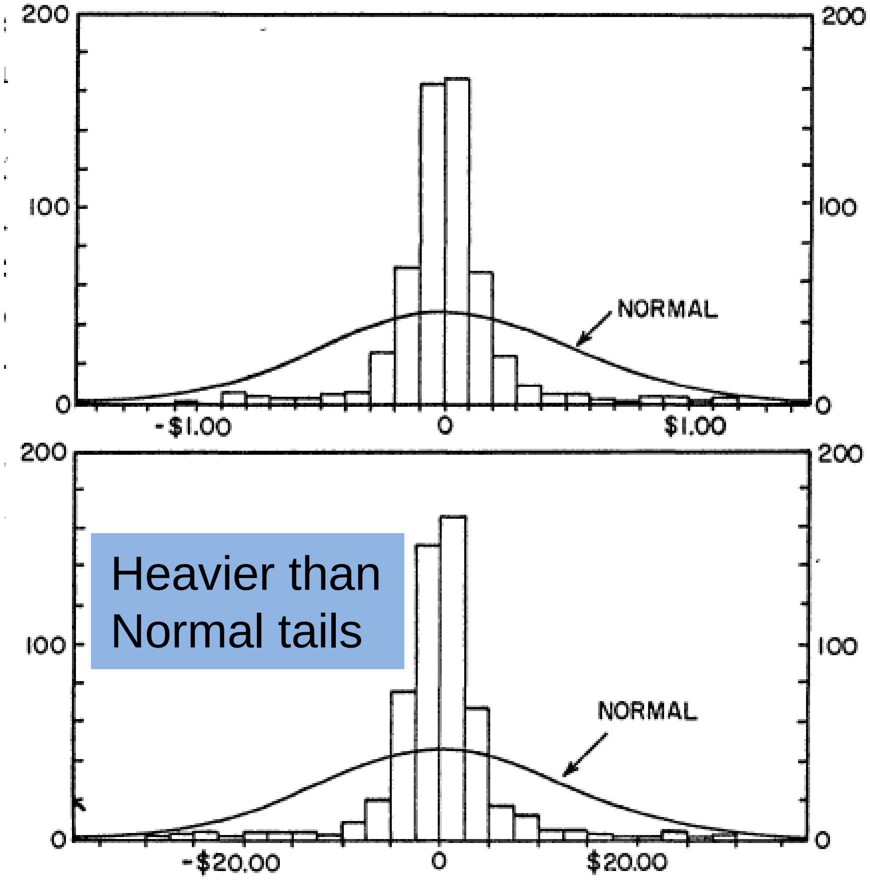


FIG. 1.—Two histograms illustrating departure from normality of the fifth and tenth difference of monthly wool prices, 1890–1937. In each case, the continuous bell-shaped curve represents the Gaussian “interpolate” based upon the sample variance. Source: Gerhard Tintner, *The Variate-Difference Method* (Bloomington, Ind., 1940).

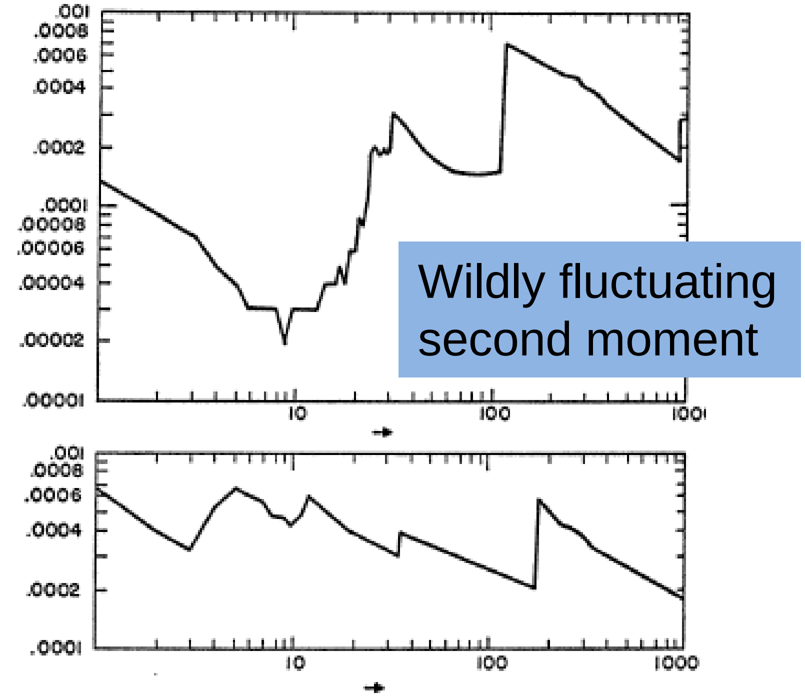


FIG. 2.—Both graphs are relative to the sequential sample second moment of cotton price changes. Horizontal scale represents time in days, with two different origins T°: on the upper graph, T° was September 21, 1900; on the lower graph T° was August 1, 1900. Vertical lines represent the value of the function

$$(T-T^\circ)^{-1}\sum_{t=T^\circ}^{t=T}[L(t,1)]^2\,,$$

where $L(t,1)=\log_e Z(t+1)-\log_e Z(t)$ and $Z(t)$ is the closing spot price of cotton on day t, as privately reported by the United States Department of Agriculture.

Mandelbrot, J. Business, 1963

… and α-stable cure ?

Tails of ccdf for α 1 to 2

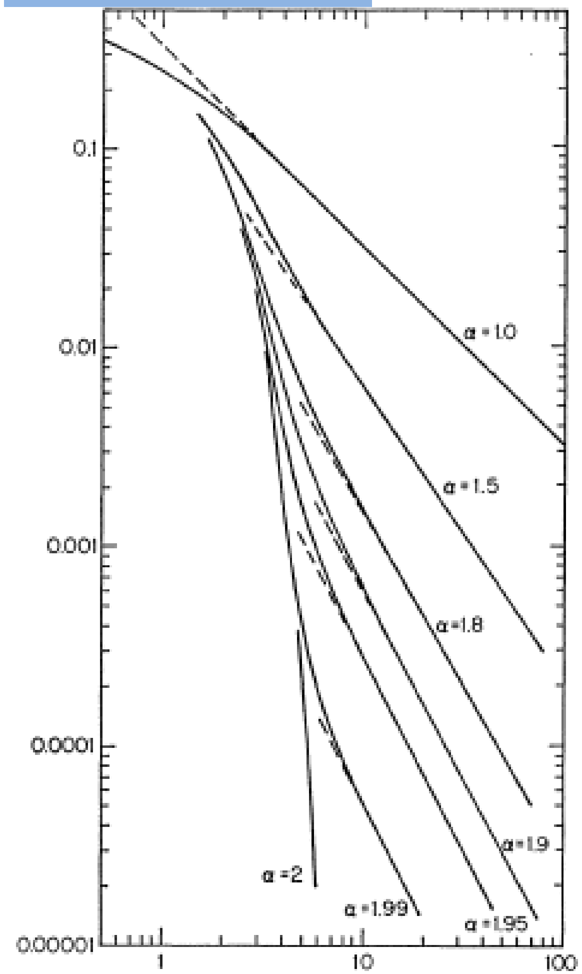


FIG. 3.—The various lines are doubly logarithmic plots of the symmetric stable Paretian probability distributions with $\delta = 0$, $\gamma = 1$, $\beta = 0$ and various values of α. Horizontally, $\log_e u$; vertically, $\log_e Pr(U > u) = \log_e Pr(U < -u)$. Sources: unpublished tables based upon numerical computations performed at the author's request by the I.B.M. Research Center.

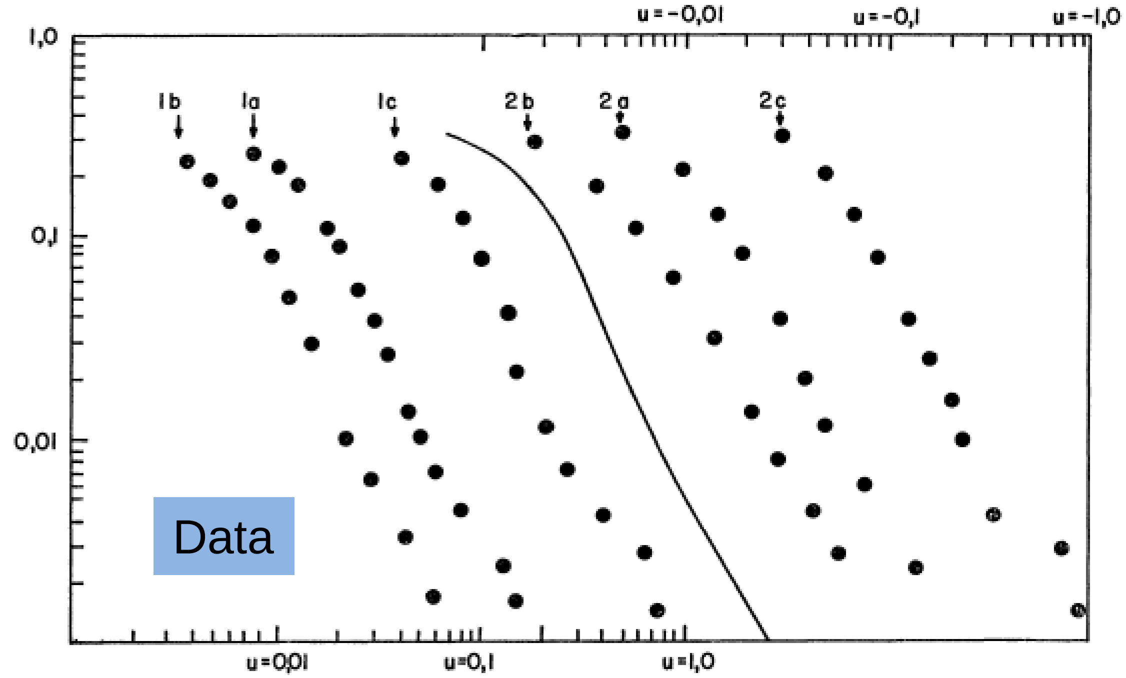


FIG. 5.—Composite of doubly logarithmic graphs of positive and negative tails for three kinds of cotton price relatives, together with cumulated density function of a stable distribution. Horizontal scale u of lines 1a, 1b, and 1c is marked only on lower edge, and horizontal scale u of lines 2a, 2b, and 2c is marked along upper edge. Vertical scale gives the following relative frequencies: (1a) $Fr[\log_e Z(t + \text{one day}) - \log_e Z(t) > u]$, (2a) $Fr[\log_e Z(t + \text{one day}) - \log_e Z(t) < -u]$, both for the daily closing prices of cotton in New York, 1900–1905 (source: private communication from the United States Department of Agriculture).

The reader is advised to copy on a transparency the horizontal axis and the theoretical distribution and to move both horizontally until the theoretical curve is superimposed on either of the empirical graphs; the only discrepancy is observed for line 2b; it is slight and would imply an even greater departure from normality.

Mandelbrot, J. Business, 1963

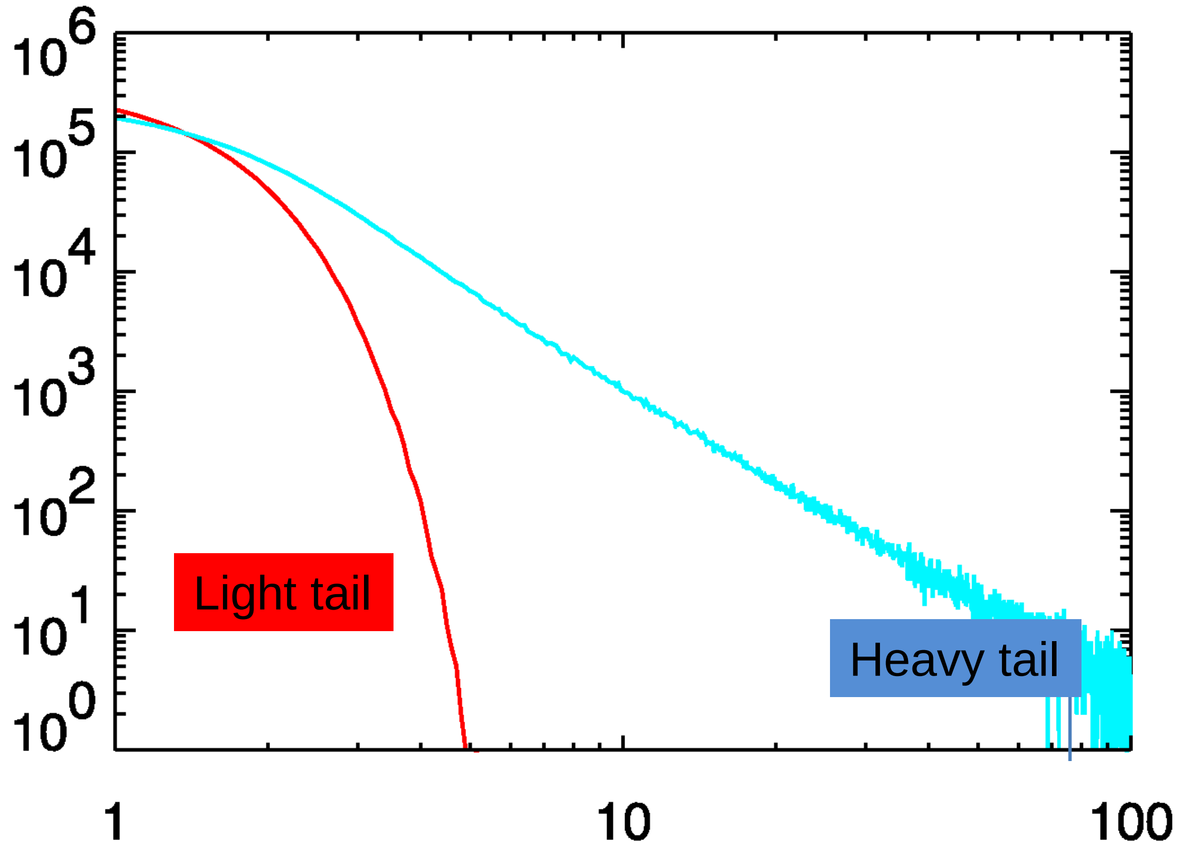
The α-stable laws

- Pdfs that have

$$p(x) \sim x^{-(1+\alpha)}$$

tail exponent in range

$$0 < \alpha < 2$$

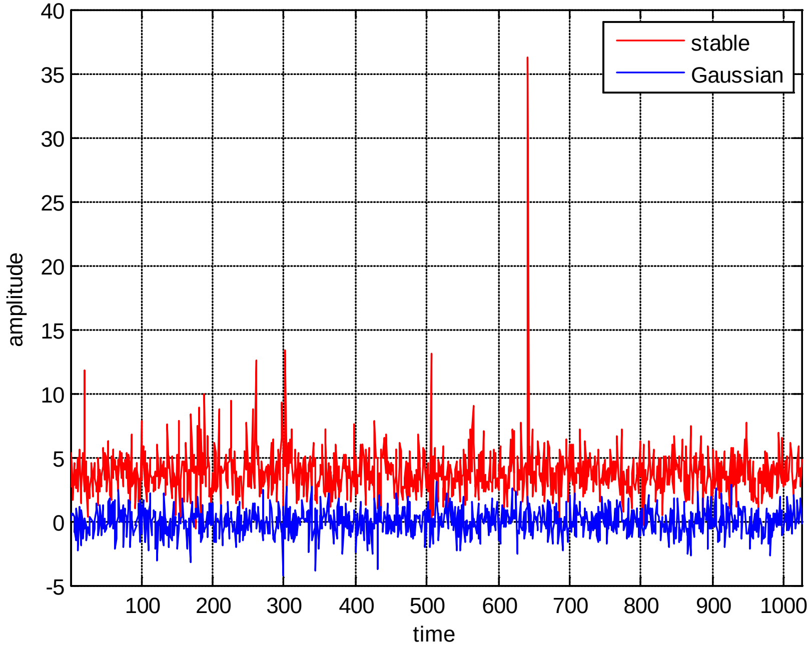


obey the ECLT due to Levy, and are attracted to an α-stable law. Infinite variance !

H-selfsimilarity & alpha-stability

- For walks formed from α-stable noises

$$H = \frac{1}{\alpha}$$



H is selfsimilarity exponent, measures roughness. Smaller α, shallower pdf, wilder jumps, more roughness.

Stable, alpha=1.8
Gaussian alpha=2,
noises

Fractional diffusion equation

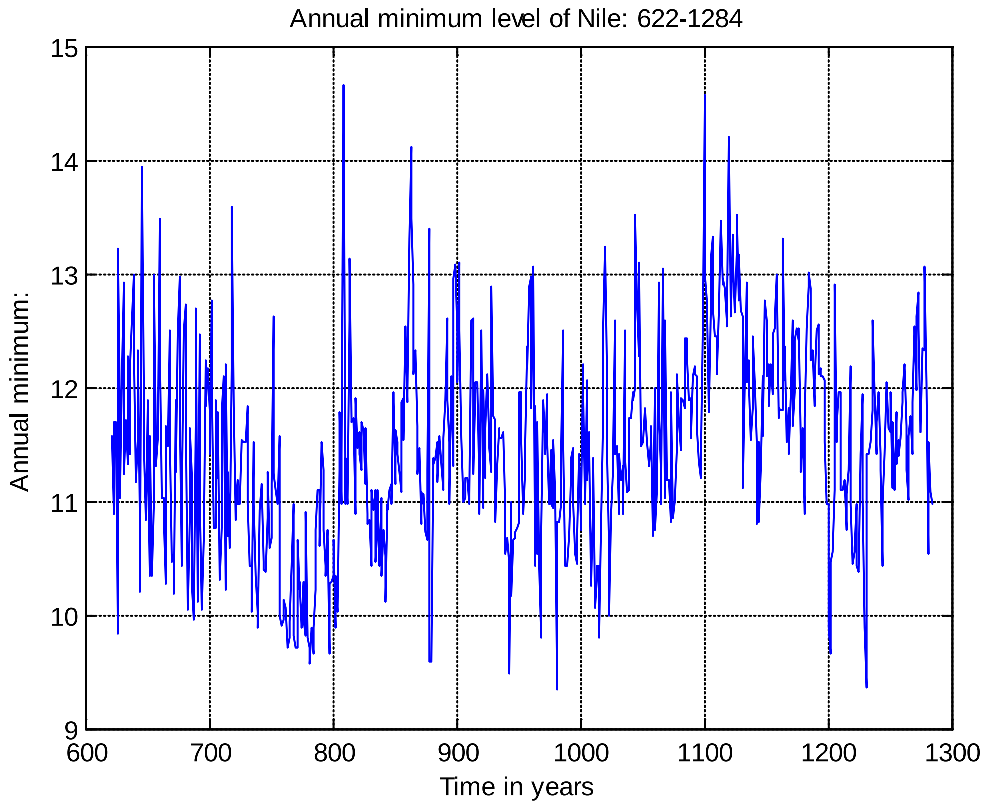
- Capture by modifying diffusion equation **[e.g. Klafter and Sokolov, OUP, 2012]**

$$\frac{\partial}{\partial t} p(x,t) = D_0 \nabla^{\alpha} p(x,t)$$

Using Reisz fractional derivative of order alpha.

$$D_0 \frac{\partial^{\alpha}}{\partial x^{\alpha}} p(x,t) = -\frac{\bar{\sigma}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \, |k|^{\alpha} \exp(-\bar{\sigma} \, |k|^{\alpha} \, t)$$

A new puzzle: the Nile …



Hurst (1880-1978) studied this dataset from point of view of design of "large and long term over-year storage … 'century storage'"

Ideal reservoir

- Average influx over

years, need to ensure annual released volume equals mean influx:

$$< \xi >_\tau = \frac{1}{\tau}\sum_{t=1}^{\tau}\xi(t)$$

Accumulated deviation of the influx from the mean:

$$X(t,\tau) = \sum_{u=1}^{t}\{\xi(u) - < \xi >_\tau\}$$

(Rescaled) range

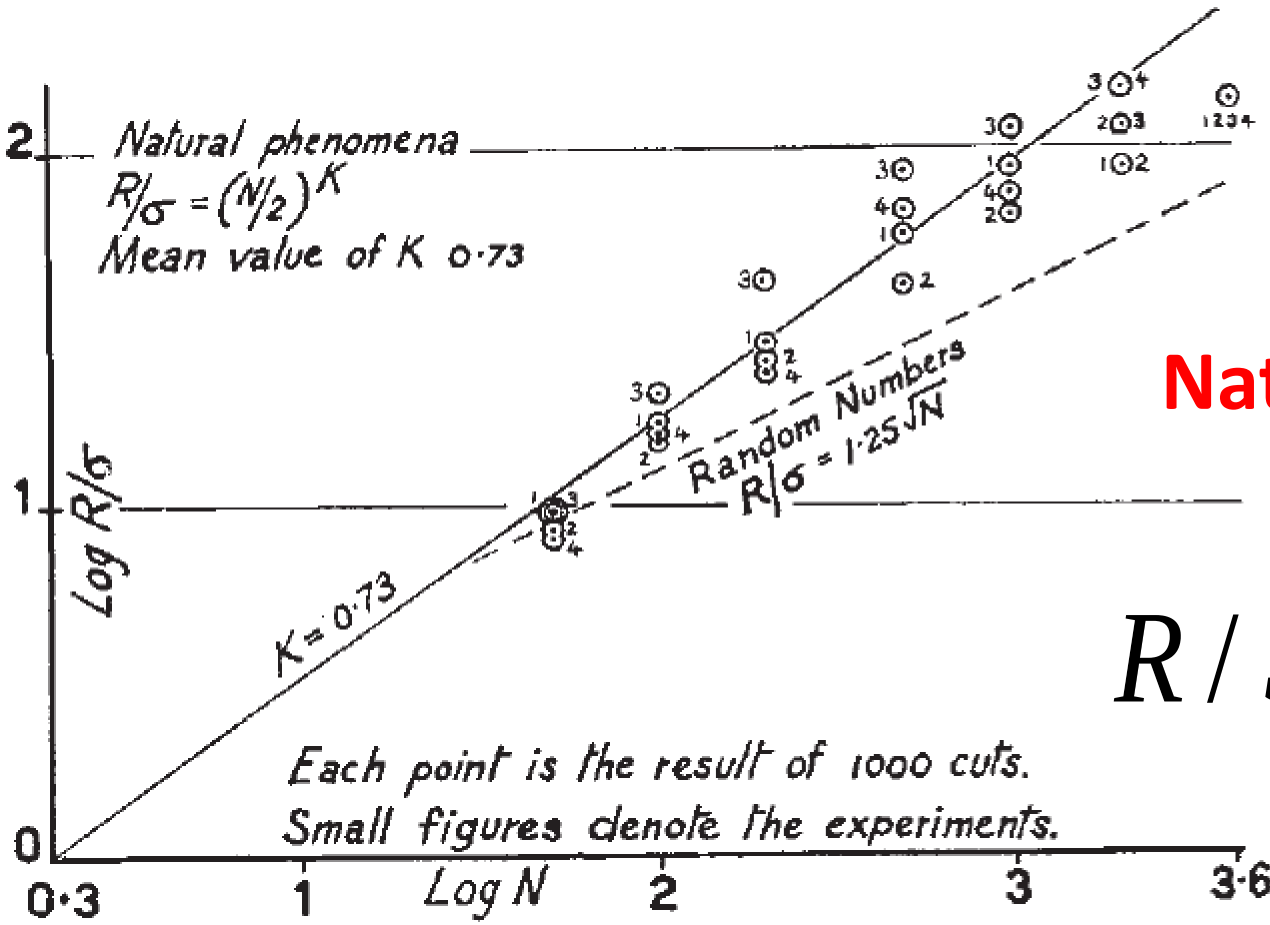
Range: $\max_{0<t<\tau} x(t,\tau) - \min_{0<t<\tau} x(t,\tau)$

Standard deviation: $S = (\frac{1}{\tau}\sum_{t=1}^{\tau}\{\xi(t) - <\xi_\tau>\}^2)^{1/2}$

Form $\frac{R}{S}$ against interval τ Plot loglog.

White Gaussian noise prediction $R/S \sim \tau^{1/2}$

Hurst’s effect



Nature, 1957

$$R / S \sim \tau^{J}$$

Hurst found anomalous growth of rescaled range, J about 0.7

Joseph Effect:

... there came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine ...

Genesis: 41, 29-30.

Pharoah's dream of 7 years of plenty and 7 years of drought. Now shuffle

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Point is that marginal distribution, of sample at least, unaffected by shuffling, but that the two series represent very different worlds for insurers, or Pharoahs. Former unlikely to happen without LRD.

fBm, an LRD and H-selfsimilar model for the Hurst effect

H = d+1/2

$$X_{H,2}(t) \sim \int_R \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dL_2(s)$$

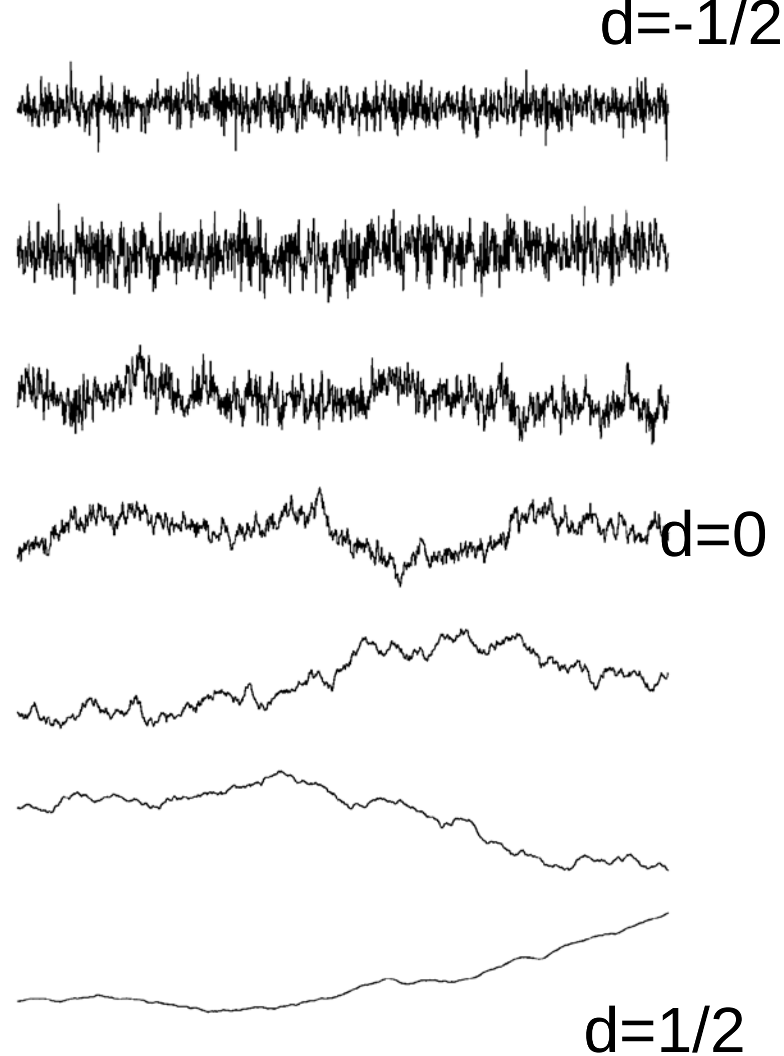
Memory kernel:
Joseph effect

Gaussian jump

- Mandelbrot **Comptes Rendus, 1965**; **Mandelbrot & van Ness, SIAM Review 1968**; **Kolmogorov, 1940** "Wiener Spiral"

Strength of LRD

- Measured by dependence parameter *d*
- d of LRD noise, e.g. fGn is related to H-selfsimilarity exponent of walk, e.g. fBm by H=d+1/2.



For white noise d=0 and J=1/2.

Diffusion equation for fBm ?

- "Fix up" diffusion eqn: correct pdf for fBm

[e.g. Wang & Lung, 1990; Lutz, 2001]

$$\frac{\partial}{\partial t} p(x,t) = D(t)\nabla^2 p(x,t)$$

$$D(t) = 2D_0 H t^{2(H-1/2)}$$

H = d+1/2

But still Markovian and so can't reproduce correct trajectory-based information e.g. FPT. NB, fractional time diffusion equation not same as fBm [Lutz, PRE,2001]

Langevin equation for fBm?

- If particle moves in a correlated heat bath we need full form of Langevin equation:

$$m\ddot{x} = -m\int_0^t dt'\,\rho(t-t')\dot{x}(t') + \xi - V'(x)$$

- Which can be simplified in LRD special case (d≠0)

$$\rho(\tau) \sim \tau^{-(1+2d)}$$

H = d+1/2

Fractional Langevin equation [Lutz, 2001]

- Use Riemann-Liouville fractional derivative

$$\frac{\partial^{\lambda}}{\partial t^{\lambda}} f(t) = \frac{1}{\Gamma(-\lambda)} \int_0^t d\tau \, (t-\tau)^{-(1+\lambda)} f(\tau)$$

- To re-express the memory kernel as fractional derivative

$$m\ddot{x} = -m\rho_{1+2d} \frac{\partial^{2d}}{\partial t^{2d}} \dot{x}(t') + \xi - V'(x)$$

H = d+1/2

LRD and H selfsimilarity, common history ...

Recap **Mandelbrot (& Wallis), mid 1960s:** 2 departures from AR(1), "Biblical geoscience" illustrations, selfsimilarity exponent H

- heavy tails in amplitude, cotton prices.

"Noah" effect, 40 days and 40 nights of rain.

- long range dependence in Nile level.

7 lean & 7 fat years: "Joseph effect".

… not necessarily common <u>origin</u>

- <u>Both</u> models fBm & Levy flights are H-selfsimilar, only fBm is LRD.

- Fractional Brownian motion: H = d + ½
- Levy flights: $H = 1/\alpha$, where α *is* exponent of pdf heavy tail.
- Both are special cases of $H = 1/\alpha + d$
- To Mandelbrot's surprise, R/S turned out to measure d+1/2(=J), NOT H ! [Mandelbrot & Wallis, 1969].

Linear Fractional Stable Motion, a model with heavy tails & Joseph effect

$$X_{H,\alpha}(t) = C_{H,\alpha}^{-1} \int_R \left((t-s)_+^{H-\frac{1}{\alpha}} - (-s)_+^{H-\frac{1}{\alpha}} \right) dL_{\alpha}(s)$$

H = d+1/α

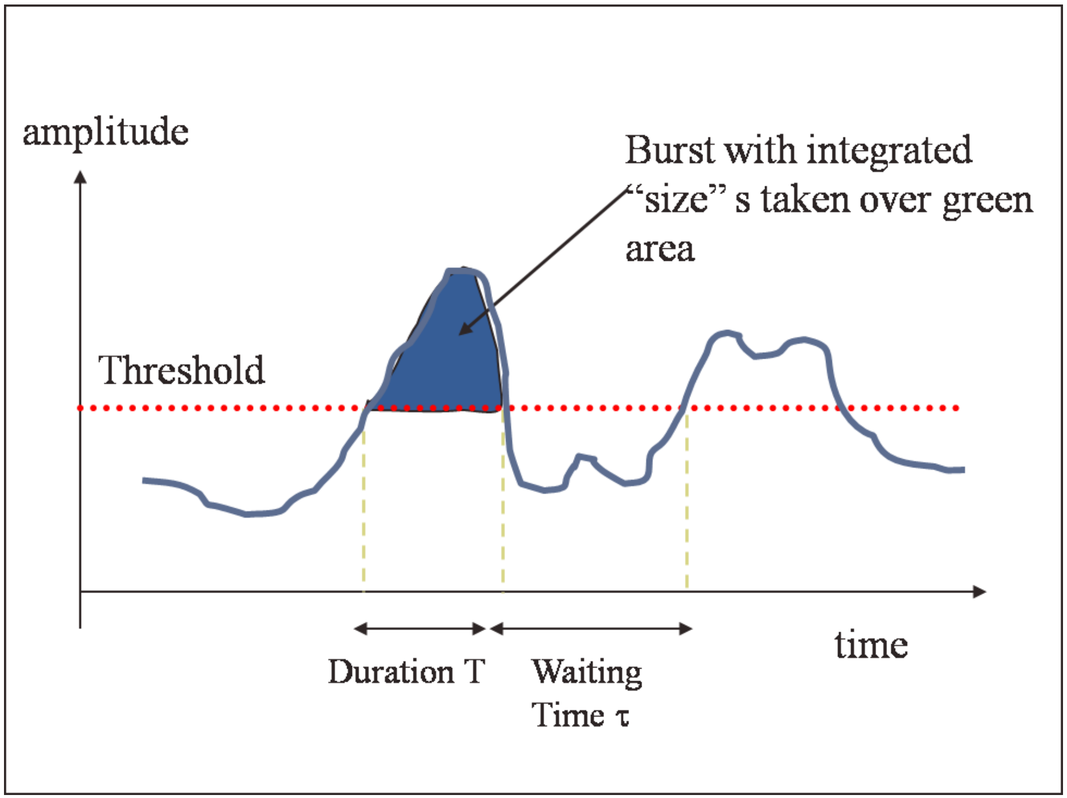
Memory kernel: Joseph effect

α-stable jump: heavy tails

- fBm kernel, but α-stable rather than Gaussian jumps (e.g. **Samarodnitsky & Taqqu book**).

Burst scaling in LFSM?

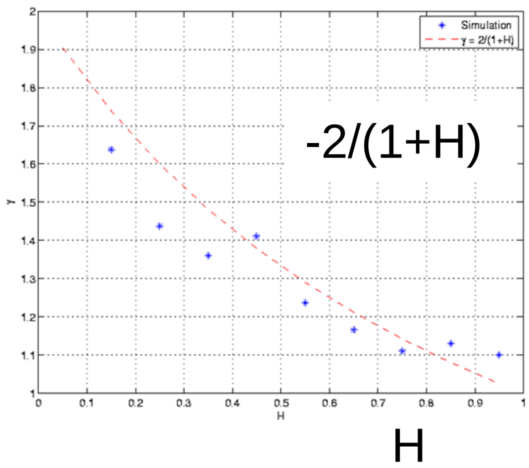
Scaling arguments used in **Watkins et al, PRE, 2009**

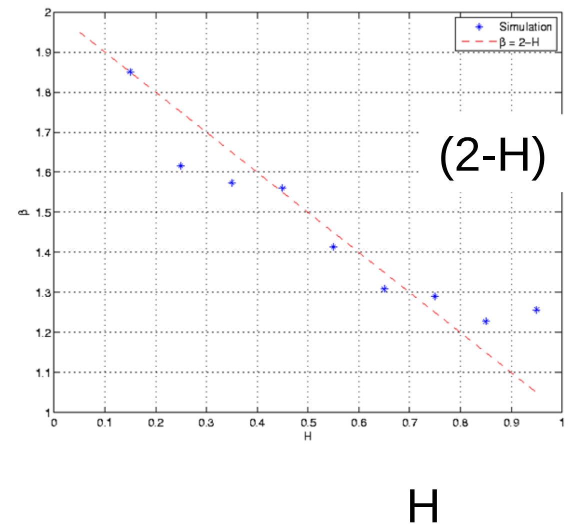


$$p(s) \sim s^{-\frac{2}{1+H}}$$

$$p(T) \sim T^{-(2-H)}$$

Simulations of light-tailed bursts





$$p(s) \sim s^{-\frac{2}{1+H}}$$

$$p(T) \sim T^{-(2-H)}$$

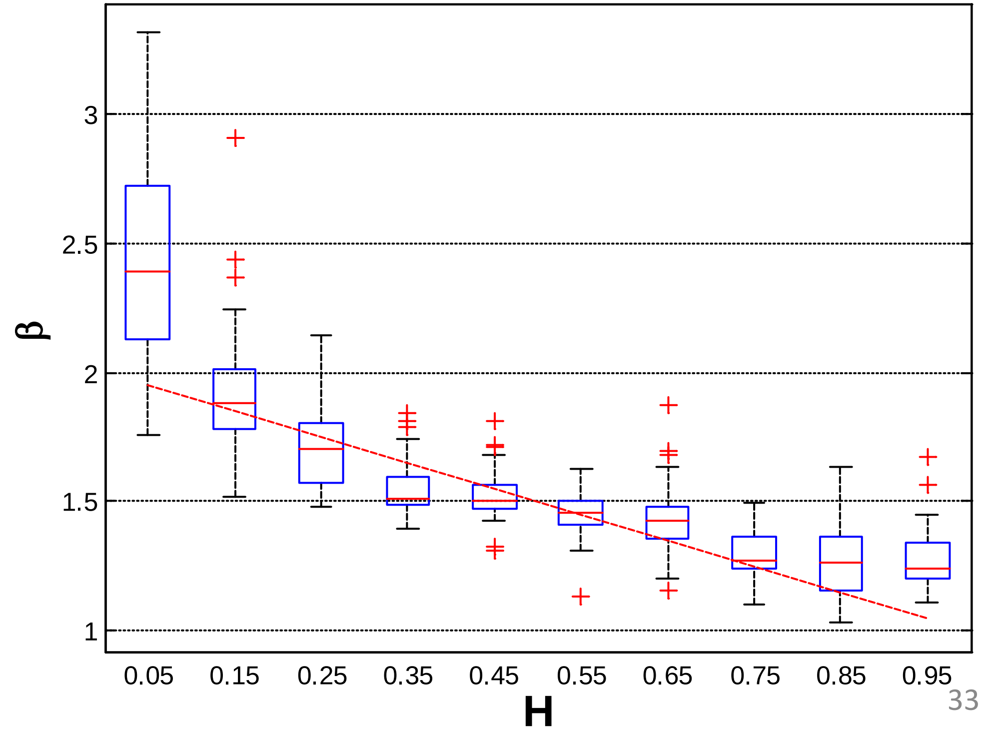
- **Reasonable agreement in Gaussian (fBm) limit: Confirmed findings of Carbone *et al* [PRE, 2004] who used a DFA inspired variable threshold.**

Simulations of heavy-tailed bursts

- **Watkins et al, PRE, 2009** found expressions also reasonable down to $\alpha \sim 1.6$, but to fail completely by $\alpha = 1$.

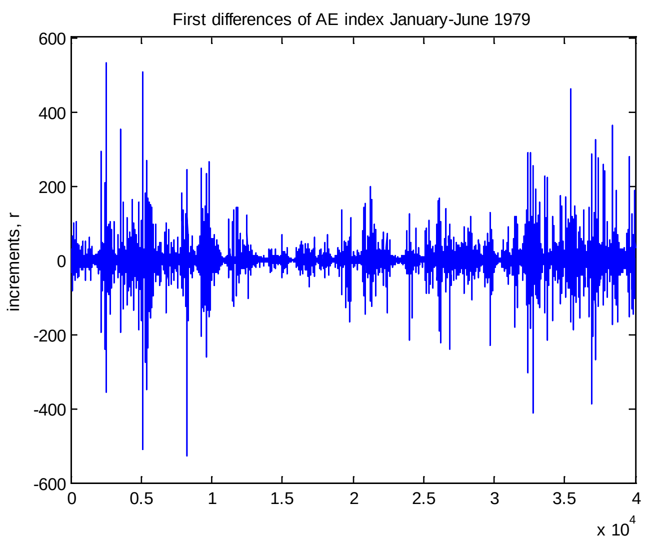
Nonstationarity of walk is a conceptual isssue, also technical ones, work in progress.

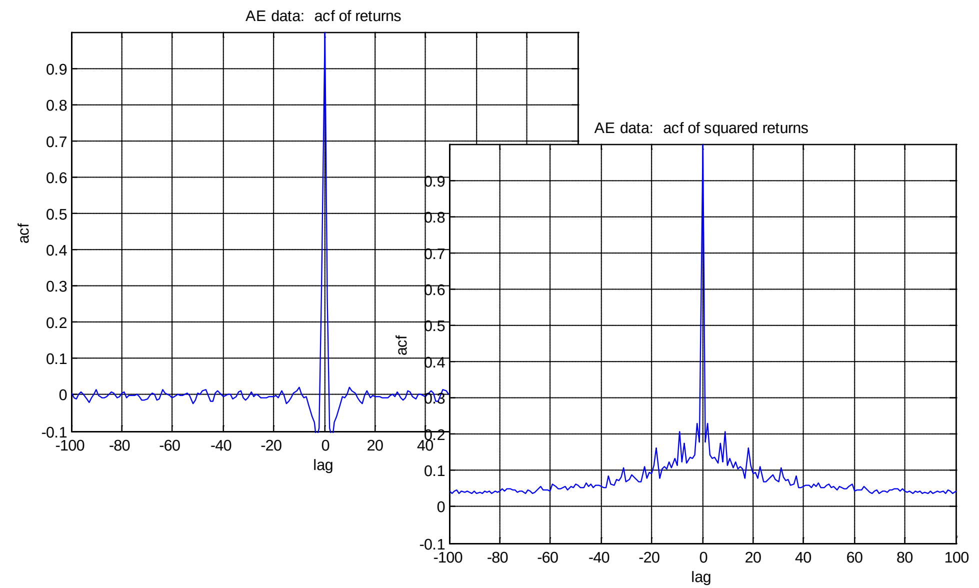
Burst length exponent, β, vs. H for α=1.6, &40 trials / exponent



Alternative burst model ?

- Dependence sometimes seen not in amplitude but squared amplitude-essentially in the unsigned magnitudes. Natural counterpart of the “volatility bunching” seen in finance---multifractals capture this effect.





Scaling of multifractal bursts

For fractional Brownian motion we had burst duration exponent

$$\beta = 2 - H$$

Replace time by a multifractal time with a scaling function zeta then we have found

$$\beta = 1 - \zeta(1 - 1/H)$$

Effect of multifractality here is to increase effective burst duration. Work in progress **[Rypdal, Watkins and Lovesletten, 2013]**

Can model LRD (d) without assuming complete H-selfsimilarity

Don't actually need completely H-selfsimilar models to exhibit LRD (just asymptotic)

In 1980s **Granger and Joyeux** modified SRD **A**uto **R**egressive **M**oving **A**verage [ARMA(p,q)] models to allow LRD via **F**ractional **I**ntegration of order d [ARFIMA(p,d,q)].

Physically interesting: High frequency p term(s) that turns nonstationary, H-ss random walk into weakly stationary AR(p) i.e. dissipation

Next steps:

- Check reasons for discrepancy between scaling arguments and numerics in LFSM ("fractional Levy" case).

- Develop analytical and numerical understanding of bursts in multifractal random walks.

- Study bursts in FARIMA/fractional Langevin problems.

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