Enlightenment in Shadows

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Abstract. Numerical weather forecasting has functioned both as one of the major inspirations for the development of the theory of nonlinear dynamical systems, and as one of its leading applications. While ensemble forecasts used by operational forecast centres both in the US and the EC provide the best operational estimates of the reliability of a given day's forecast, many open questions regarding the construction and evaluation of the ensembles remain. The concepts of shadowing are illustrated and applied to evaluate ensembles for the thermally driven rotating fluid annulus. Low-dimensional dynamical systems are obvious test-beds for proposed improvements, yet the question arises of whether the simplicity that one often observes in very high-dimensional weather models (with millions of apparent degrees of freedom) fails 'even in or only in' low-dimensional chaotic systems; this is addressed and initial results on the uniformity of 'the linear range' presented for the annulus.

Forecasting nonlinear phenomena is a driving force of the applications of nonlinear dynamical systems as we head into the next millennium; the forecasting of physical phenomena in general, and the Earth's atmosphere in particular, has been perhaps the major single force in the formation and perpetuation of nonlinear dynamics this millennium. The role of atmospheric dynamics at the turn of this century has many parallels with that of celestial mechanics at the turn of the last: providing a slightly too difficult problem.

We consider the prediction problem given an imperfect model, an uncertain initial condition and an incomplete understanding of observational noise, as opposed to assuming a perfect model with uncertain parameters or inexact arithmetic. Two physical systems are considered, the thermally driven rotating fluid annulus and the Earth's atmosphere [3]. The annulus has the advantage of being somewhat simpler physically than the atmosphere, admitting rather simpler forecast models, and enabling analysis of data sets with a longer duration (measured in characteristic times of the system); nevertheless both are infinite dimensional (fluid) systems for which no perfect model exists.

Implications of imperfect models and uncertain observations are discussed, illustrating the concepts of shadowing for the annulus using a radial basis

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function (RBF) model [5]. Moving from $\mathbf{R}^5$ to $\mathbf{R}^{10^6}$, the question arises of whether the simplicity observed in high dimensional systems fails 'even in or only in' (EIOOI) low dimensional chaotic systems. A viable test of internal consistency is proposed, given the operational constraints in the weather forecasting scenario, and results presented for the annulus.

Dynamics of Uncertainty

Given a series of observations and some knowledge of the observational uncertainty, we may define a perfect model as one which is able to generate a solution which differs from the observations in a manner consistent with the uncertainty. In general, imperfect models will differ from the perfect model scenario in more fundamental ways than simply having inexact (estimated) parameters[1]. Often the structure of the model is wrong in the sense that the true dynamics cannot be represented over 'long' periods of time by *any* set of model parameters (*e.g.* the model consists of a truncated Taylor expansion of the perfect model), also the state-space of the model may differ from that of the system. How should the 'best' parameters in such models be defined?

In practice, the lack of sufficient observations to completely specify a model state, the knowledge that the available observations are uncertain, and discrepancies between the model state-space and that of the physical system make it inadvisable to initiate a forecast from an initial condition based solely on current observations. An *analysis*, $A(t, \tau)$, for time t is formed by combining observations until time τ using the model dynamics; thus the initial analysis $A(0,0)$ serves as the best estimate of the state corresponding to the initial system state given past observations for this model. Knowledge of the uncertainty distribution is represented by a 'ball' around the analysis (an isopleth of the corresponding probability distribution function). If the model is perfect then (i) there exists a model state within the t=0 ball which represents the system state and (ii) evolution of this model state under the model will yield a trajectory which passes through the uncertainty balls at t=1, 2, 3, ... This model state need not be unique (fig. 1a), and will not be if the model is also hyperbolic. An alternative schematic, more familiar to low dimensional dynamical systems, is given in fig. 1b; the subset of consistent initial conditions is determined by considering the intersection of forward and backward projections (under the model) of past and future uncertainty bounds respectively, with that at $t = 0$.[2]

The model is never perfect. If the system is uniformly hyperbolic and the model is good then there will exist a system trajectory which stays close to, or

[1] The underlying physical system may have dynamics which can not be represented by a system of ODEs (or PDEs), or may not be closed in the thermodynamic sense.

[2] The consistent subsets of fig. 1a are the intersection of the states within the uncertainty at $t = 0$ and the backward projections (onto $t = 0$) of uncertainties at times $t = 1, t = 1, 2, \ldots$

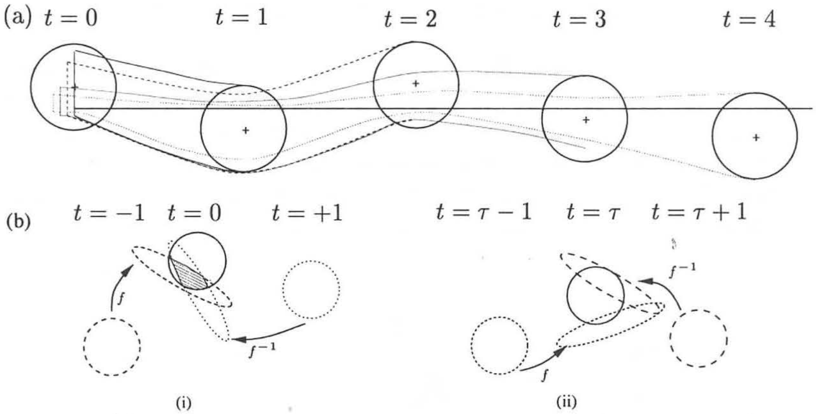

FIGURE 1. (a) Perfect model scenario: the 'true' trajectory (horizontal line), analyses (+) and corresponding uncertainty (circles). Subsets of initial conditions consistent with observations at $t = 0, 1$ (solid line), $t = 0, 1, 2$ (long dashes),... demonstrate the collapse of the subset of consistent initial conditions to a non-empty subset containing the 'true' initial state as more future observations are obtained. (b) Schematic reflecting the existence of a shadowing trajectory from projection of uncertainty distributions. (i) Imperfect model; an ι-shadowing trajectory exists, passing through the shaded region. (ii) No ι-shadowing trajectory exists as there is no intersection between projections of past and future uncertainties and the current uncertainty.

ϵ-shadows the model trajectory. (The Anosov-Bowen lemma states that every 'good' model can be ϵ-shadowed [1].) But the question of interest here is finding model trajectories sufficiently near the observed system trajectory to be consistent with the observational uncertainty distribution. In general, we can make precise statements about the dynamics of the model (we know its functional form), but not about the system itself (we know only the observations). And inasmuch as the Hénon map is not hyperbolic, it is doubtful that current weather models are. In reality, there may be a time τ_ι, the ι-shadowing time, at which *all* initial conditions consistent with the observational uncertainty at $t = 0$ are inconsistent with some observation at time $\tau \in (0, \tau_\iota]$ (see fig. 1bii). Note that this time, τ_ι, is a function of the initial state of the system, the initial analysis and associated uncertainty, and the particular model.

Variational data assimilation schemes may needlessly degrade the analysis at $t < \tau$ *and* $t > \tau$ due to the model error. By explicit use of the distribution of measurement uncertainty and the acceptance of systematic (if unknown) macroscopic model error, ι-shadowing is able to cut a trajectory at a location like fig. 1bii. This suggests regions for model improvement and provides a measure of optimality, the distribution of τ_ι, which may be used to contrast models with similar average forecast errors. Unlike a root mean square error cost function, ι-shadowing times will not penalise realistic model sensitivity.

Low–dimensional dynamics in the Annulus

The thermally driven rotating annulus is an infinite dimensional laboratory analogy of the mid–latitude circulation systems in the Earth's atmosphere. Fluid is held between concentric cylinders each held at constant temperature (the inner one being cooler), and the entire apparatus (including the temperature probe) rotates at a fixed rate Ω. For large Ω, the flow appears spatially irregular and local co-rotating temperature measurements are chaotic [4,5,7].

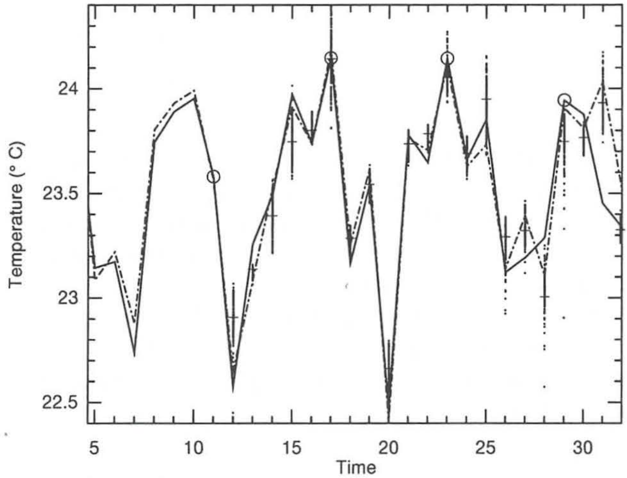

FIGURE 2. An observed temperature time series (solid) from the annulus and an ι-shadowing trajectory (dot-dashed) from a RBF model which stays 'close' to the observed trajectory for 30 time steps. Four 6–step ensemble predictions are shown: an ensemble of 128 points, normally distributed within 0.075 degrees of (each component of) the initial observation, is initiated at times 11, 17, 23 and 29 (circles) and iterated under the model to give a distribution after 1,2,3,4,5 and 6 steps (dots), the mean of which is denoted (+).

Figure 2 shows ensemble predictions for a RBF model of the annulus; in $\mathbf{R}^5$ the distributions of the ensembles show regions of rapid error growth, return of skill, and model error (*e.g.* at times 12, 13 and 31 respectively). At time 31 the scenario illustrated in fig. 1bii occurs; no shadowing trajectory exists. Even in low dimensional systems, models are imperfect.

High–dimensional Atmospheric Dynamics

The computational complexity of high-dimensional numerical weather prediction (NWP) models poses an operational restriction limiting real time ensembles to about 2^6 members. This suggests the use of constraints so that the "most relevant" perturbations are investigated. A topic of much discussion is the optimal choice of constraining sub-space, which is often defined using singular vectors of the linear propagator, $\mathcal{M}$, over an optimisation time t_{opt}. *Singular vectors* (SV) [2], employed by ECMWF, are the right singular vectors of the linear propagator over $[0, t_{\text{opt}}]$; they represent the directions of the linearised system which grow most rapidly over $[0, t_{\text{opt}}]$. The *Lyapunov vectors* (LV) are the left singular vectors of the linear propagator in the limit $t_{\text{opt}} \to \infty$, over time $[-t_{\text{opt}}, 0]$ and represent the directions of the linearised system which have grown the most. If the model is perfect and the perturbation infinitesimal the *breeding vectors* (BV) [8], employed by NCEP, evolve toward the local orientation of the first global LV. Given an imperfect model and finite perturbation the BV contains information on model error; in this case there may be no LV, as no trajectory need pass through the analysis.

Ideally, the best (possible) representation of the system initial condition for a given model is included in the ensemble; such a current initial condition may be defined by trajectory which shadows farthest both from the past and into the future. Defining the perturbation from the current analysis to such an initial condition as the 'dream perturbation', the projection of the constrained vectors onto this dream perturbation provides a method for (*a posteriori*) evaluation of ensemble construction.

The treatment of nonlinearity in NWP is based upon several assumptions which may be shown to fail *even in* low ($m \sim 2^2$) dimensional dynamical systems. A prime example is the assumption of uniformity in the time scale over which the linear propagator yields a reasonable approximation to the dynamics for different initial conditions. One familiar with high ($m \sim 2^{20}$) dimensional nonlinear dynamical systems, and the Earth's atmosphere, might respond that the huge fluctuations observed in low dimensional systems happen *only in* low dimensional systems: the 'even in or only in' (EIOOI) dilemma.

The use of SV subspaces assumes that the linearised model remains a good approximation for the optimisation time, about 2 days operationally. We plan to evaluate this assumption for NWP models by exploiting the common practice of running twin perturbations in operational ensembles in order to avoid the computational cost of additional model runs. Specifically, for every initial condition in the ensemble with perturbation $\boldsymbol{\delta}$, $\boldsymbol{\delta} \in \mathbf{R}^m$, from the control, a twin initial condition with perturbation -$\boldsymbol{\delta}$ is included in the ensemble. These twin trajectories reflect the time scale over which the linear propagator might provide a reasonable approximation of the dynamics of finite perturbations, thereby providing both a direct measure of the uniformity, or otherwise, of this time scale with initial condition and a test of the internal consistency of SV ensembles. Taking the control run as the origin, consider the dynamics of each pair of twin perturbations, $\boldsymbol{\delta}^+(t)$ and $\boldsymbol{\delta}^-(t)$. In the linear approximation $\boldsymbol{\delta}^+(t)+\boldsymbol{\delta}^-(t) = \mathbf{0}$, thus we monitor the time at which the quantity $\theta(t)$ exceeds a given threshold, where

$$\theta(t) = \frac{||\boldsymbol{\delta}^+(t) + \boldsymbol{\delta}^-(t)||}{||\boldsymbol{\delta}^+(t)|| + ||\boldsymbol{\delta}^-(t)||} \tag{1}$$

and $||\cdot||$ is one of several possible metrics. Results from twin experiments run under a RBF model for the annulus show that, for uncertainties comparable to the analysis error and a 20% threshold (*i.e.* $\theta < 0.2$), linearity breaks down rapidly and non-uniformly. Choosing an optimisation time of 4 time steps, SV perturbations to 90% of analysis values show a break down of linearity within within 6 time steps; only $\sim$20% are linear until optimisation time. Linearity breaks down non-uniformly and slightly later in the BV subspace (90% within 7 time steps).

CONCLUSIONS

While Lyapunov exponents place no *a priori* limit on the predictability of a dynamical system [6], the ι-shadowing time does place an upper bound on the predictability given the combination of a specific model of a given system and some particular (distribution of) uncertainty. Even this bound is of limited utility in practice, since it assumes infinitely large ensembles. However, ι-shadowing offers an alternative to the usual prediction error cost functions in defining the 'best' model of a system; it penalises model error while utilising realistic model sensitivity, rather than penalising both model error and sensitivity. Shadowing failure at an anomalous result yields a trajectory containing useful information over its duration and indicating regions for model improvement. An assumption of NWP ensemble formation was tested for a low-dimensional system and the applicability of these results to high-dimensional systems queried due to the EIOOI dilemma. Twin experiments are currently being run for an operational NWP model, the results of which will address an aspect of this dilemma.

REFERENCES

1. Farmer J.D., and Sidorowich J.J., *Physica D* **47**, 373–392 (1991).
2. Mureau R., Molteni F., and Palmer T.N., *Q. J. R. Meteorol. Soc.* **119**, 299–323 (1993).
3. *Predictability*, ECMWF, Shinfield Park, Reading, U.K., 1996.
4. Read P.L. *et al.* , *J. Fluid Mech.* **238**, 599–632 (1992).
5. Smith L. A., *Physica D.* **58**, 50–76 (1992).
6. Smith L. A., *Phil. Trans. R. Soc. Lond. A* **348**, 371–381 (1994).
7. Smith R.L., *J. R. Statist. Soc. B* **54(2)**, 329–352 (1992).
8. Toth Z., and Kalnay E., *Bull. Am. Meteorol. Soc.* **74(12)**, 2317–2330 (1993).

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