Pattern Formation by Particles Settling in Viscous Flows

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1. Introduction

The discovery of an increasing number of fractal objects in the natural environment prompts the present study of the settling of particles in a moving fluid. We describe here simulations of the motion of small objects in two-dimensional cellular flows. One of our aims is to see how an initial distribution of such particles deforms in time. We assume throughout that the particles do not interact with each other or affect the fluid, whose motion is two-dimensional. Even without sedimentation effects, and without many other complications that real fluids provide, intricate structures in the swarming dust arise through their chaotic motions.

It is not hard to construct incompressible flows in three dimensions with chaotic Lagrangian orbits (Arter, 1983). In two-dimensional time-dependent flows with open steamlines, the fluid particles can have chaotic orbits, as in Aref's (1984) explorations. Even when the fluid streamlines are closed, the motion of particles moving in the fluid may be chaotic. Our preliminary results indicate that chaotic motion of swarms of noninteracting particles in cellular flows may produce fractal structures whose dimension seems insensitive to the control parameters of the flow.

We consider small bodies immersed in a fluid with stream function $\psi(x,y,t)$. With the neglect of hydrodynamic mass, the nondimensional equations are

$$\ddot{x} = -\mu(\dot{x} - \psi_y) \qquad (1.1a)$$

$$\ddot{y} = -\gamma - \mu(\dot{y} + \psi_x) \qquad (1.1b)$$

where μ and γ are constants. The structures seen in this problem are interesting, as a preprint of Maxey and Corrsin shows. We here consider the reduced problem of very viscous flows and neglect inertial terms. This limit has geological interest (Huppert, 1984).

If particle acceleration is negligible,

$$\dot{x} = \phi_y \ ; \qquad \dot{y} = -\phi_x \qquad (1.2a,b)$$

where the stream function for particle motion is

$$\phi = \gamma x + \psi. \qquad (1.3)$$

These reduced equations are equivalent to a Hamiltonian system with one degree of freedom and a time dependent Hamiltonian equal to ϕ.

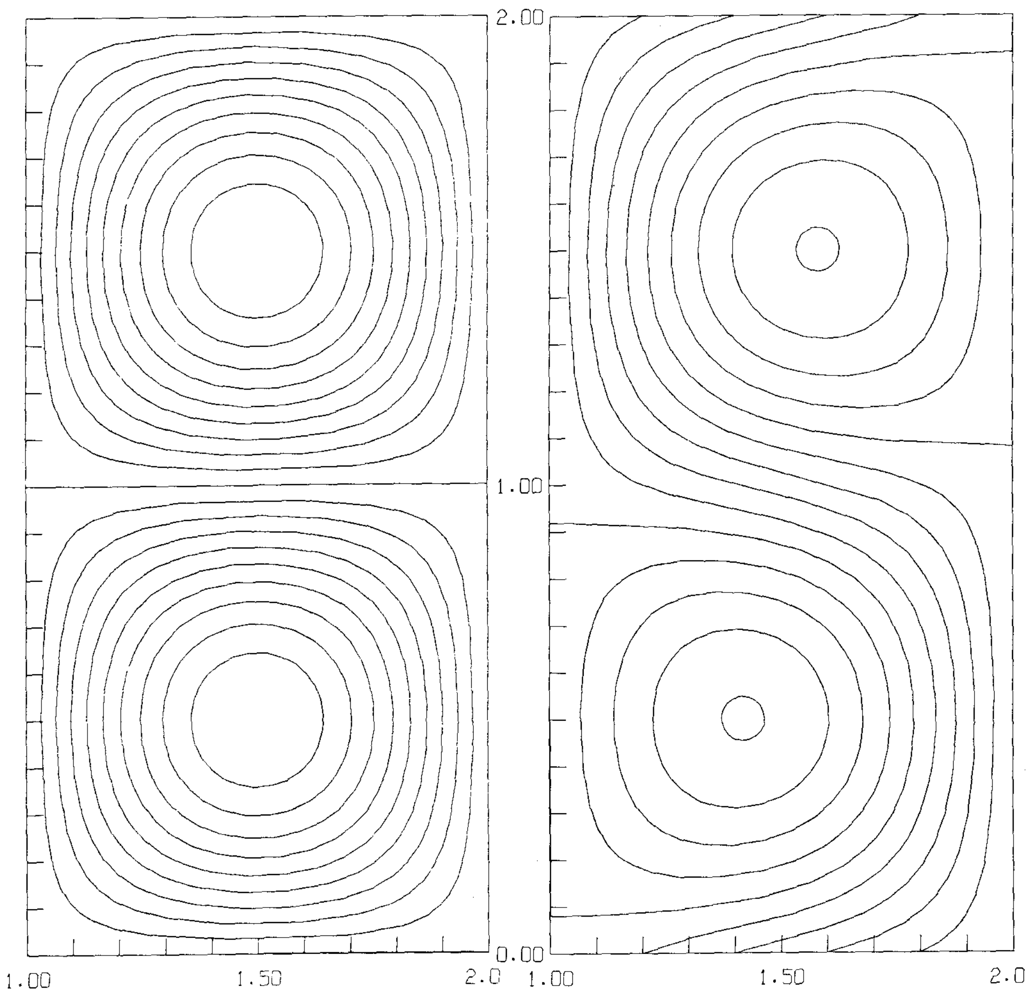


Fig. 2.1. (a) Fluid streamlines (ψ) according to (2.1). (b) Particle streamlines (ϕ) according to (1.3) and (2.1). For $A=1$ and $\gamma=0.25$.

2. Steady Flow

Stommel (1949) considered the motion of particles for steady ψ with closed streamlines. His results are illustrated in Fig. 2.1 which shows the situation for the case

$$\psi = \frac{A}{\pi} \sin \pi x \ \sin \pi y. \qquad (2.1)$$

where A is a constant. In Fig. 2.1a we show the streamlines of ψ and in Fig. 2.1b the streamlines of ϕ. Experiments suggested by Stommel's results have been reported by Toobey, Wick and Isaacs (1977).

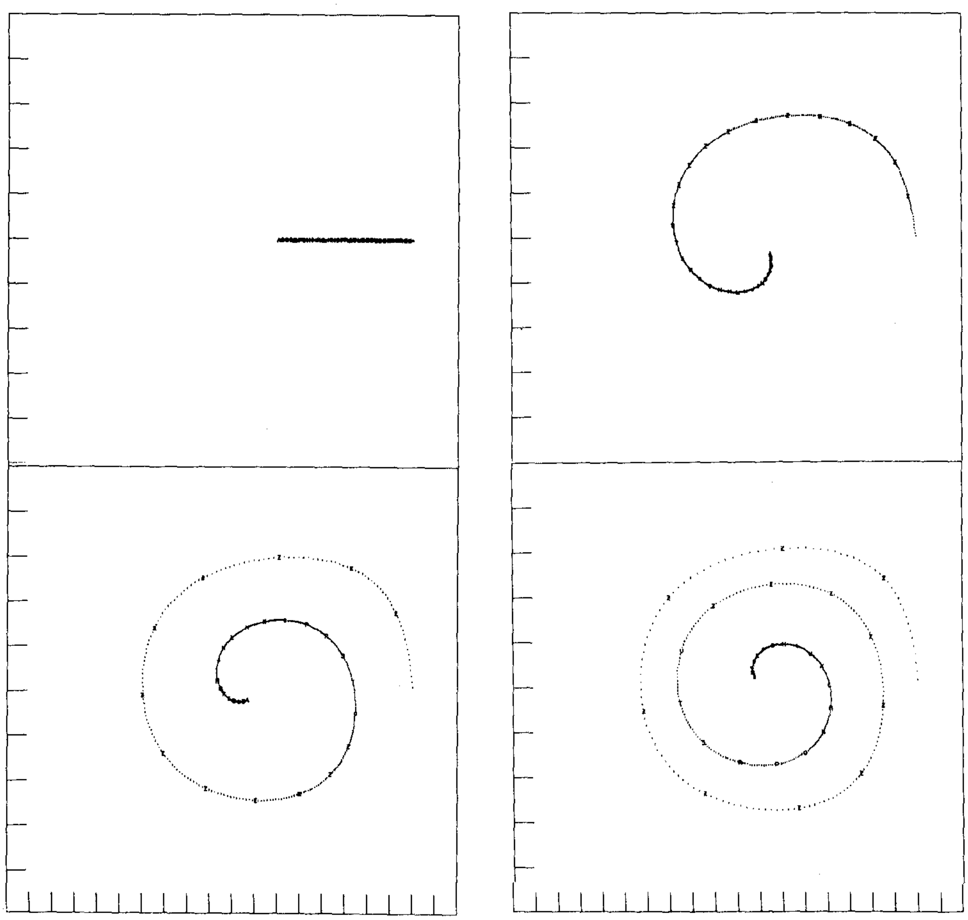


Fig. 2.2. The evolution of a line of particles with the stream function of Fig. 2.1. The distributions are for t = (a) 0 (b) 18 (c) 35 (d) 54 and the unit of length marked off is 0.1 in the vertical.

The problem with steady ψ has interesting consequences which may be taken over from a stellar dynamical study by Quinn (1984). A sheet of particles introduced into a fluid layer one cell deep is rolled up by the flow (2.1) as in Fig. 2.2. The density of the particles, when projected onto a horizontal plane, will evolve as in the upper panels of Fig. 2.3. If the particles are visible in the fluid, they may look like the lower panels of this figure. The well-defined structures in the projected density of the dust is a familiar phenomenon to those who have watched vortex rollup in suitably dyed fluids. Quinn calls the process phase wrapping (since y is momentum in his case). Stommel motivated his original study with a discussion of patterns formed in the sea. The results of Quinn may bear on such questions as dune formation in shallow water along beaches, though the inertial effects may be signifigant in such cases.

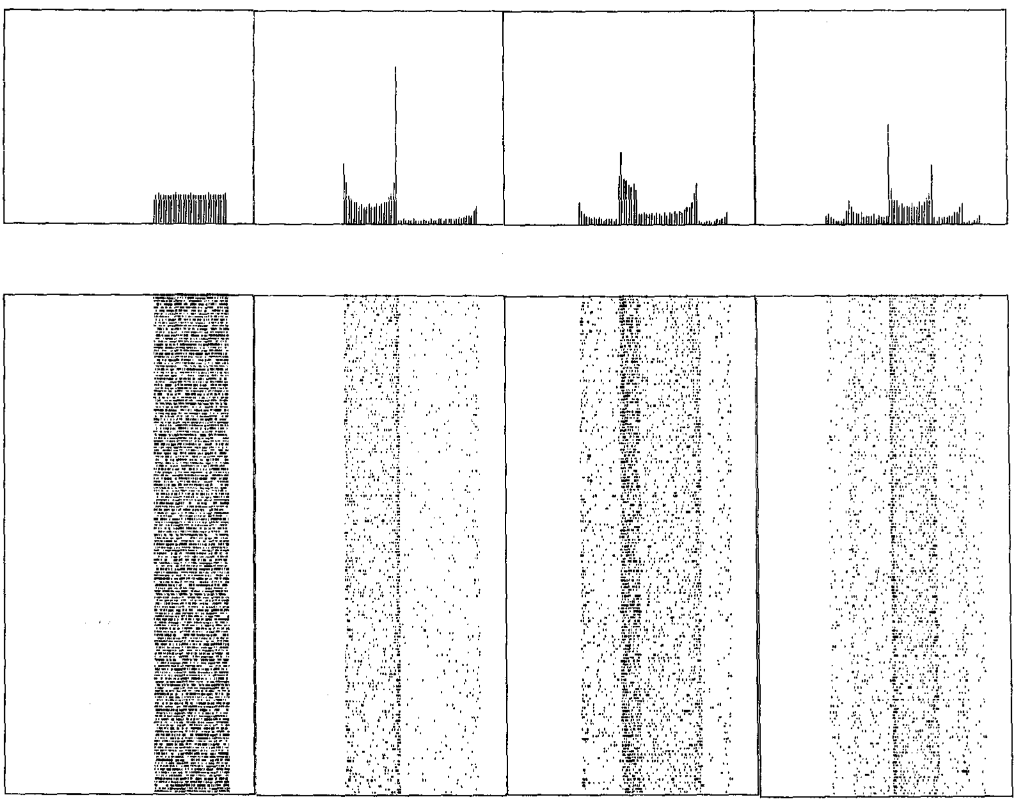


Fig. 2.3. The particle densities integrated (in y) for the four states of Fig. 2.2. The upper panels show the particle number explicitly and the lower panels are 'dust plots' simulating the appearance of the convected dust.

3. Dusty Chaos

Stommel (1949) studied particle trajectories in the steady cellular flow of Fig. 2.1a. He found regions of retention where particles are trapped indefinitely. Particles are trapped when the stream function of the fluid has an oscillatory dependence on time as well. Let

$$A = 1 + \varepsilon \cos(\omega t). \qquad (3.1)$$

We find that particles remain suspended in the fluid even when $\varepsilon \sim 10$, for $\omega \sim 1$. Results of this kind are easy to obtain numerically and surfaces of section of spatial orbits are easily drawn.

In Fig. 3.1 we illustrate the motion of a particle for the parameters indicated in the caption. In the first panel we show an orbit in the x-y plane. In 3.1b we show a stroboscopic view of the same orbit, that is, the x and y coordinates of the particle at the succession of times $t=0,P,2P,\ldots$, where $P=2\pi/\omega$.

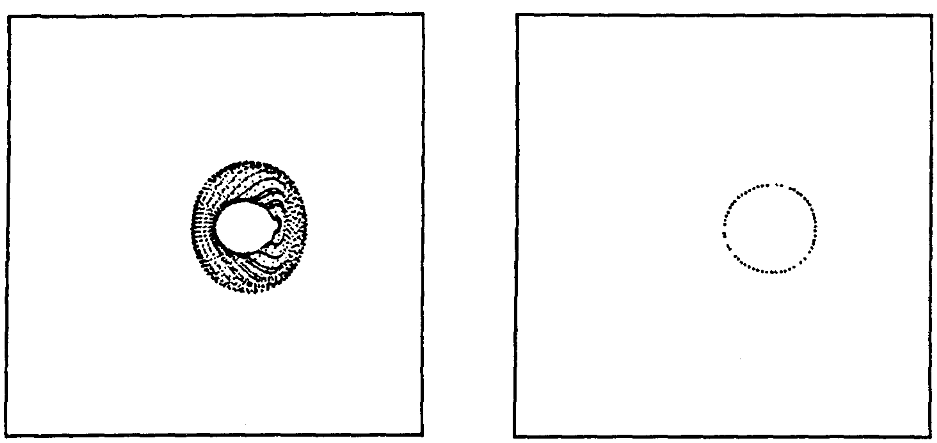


Fig. 3.1. (a) A trapped orbit in the time-dependent case for $\varepsilon=0.5$, $\gamma=0.25$, $\omega=\pi/2.2$. (b) The orbit seen stroboscopically.

Fig. 3.2 gives a corresponding pair of plots, for different values of the parameters, showing islands. The system displays the textbook behavior of Hamiltonian chaos (Lichtenberg and Lieberman, 1983) but we are more interested in the behavior of swarms of particles.

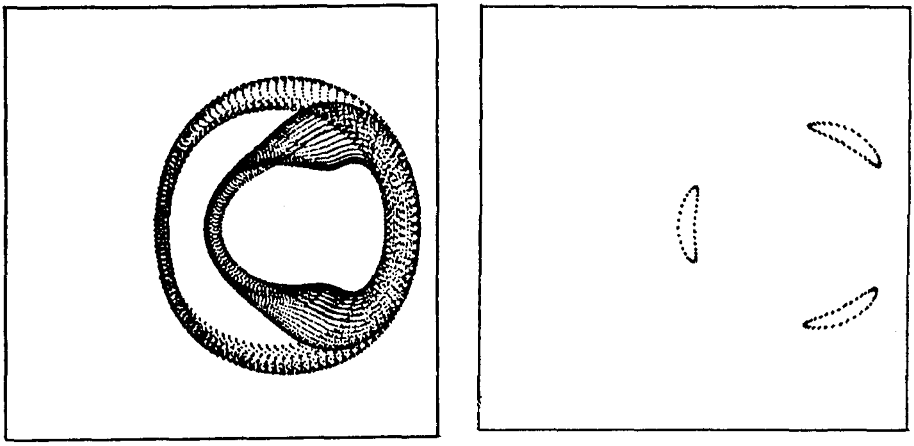


Fig. 3.2. Same as 3.1 but with $\omega = \pi/2.25$.

Fig. 3.3a shows stroboscopic views of the orbits of 32 particles that were uniformly distributed on the line $y = 3/2$ at $t = 0$. The largest of the concentric closed tori in the upper right outlines a region of particle retention. The particles shown outside this region fall to the bottom boundary of the cell where they are dealt with by the periodic boundary conditions. In 3.3a, the blank region surrounding the region of retention corresponds to chaotic motion. The region of chaotic motion is shown in Fig. 3.3b for a long run following four particles. The chaotically falling particles spend some time entrained in individual cells. The large lacuna on the lower left in 3.3a is another region of particle trapping, into which no particles can enter from the outside. Fig. 3.4 is similar to 3.3 but with an increase in ε to show the development of the region of chaotic fallout; the intial conditions have been changed to emphasize the islands.

The two large islands embedded in the chaotic region of Fig. 3.4a contain particles that remain in the initial cell. Their orbits resonate with the fluid oscillations. The fluid velocity is maximum whenever they are at their extreme y-values. A higher order resonance produces the island chain forming within these large islands.

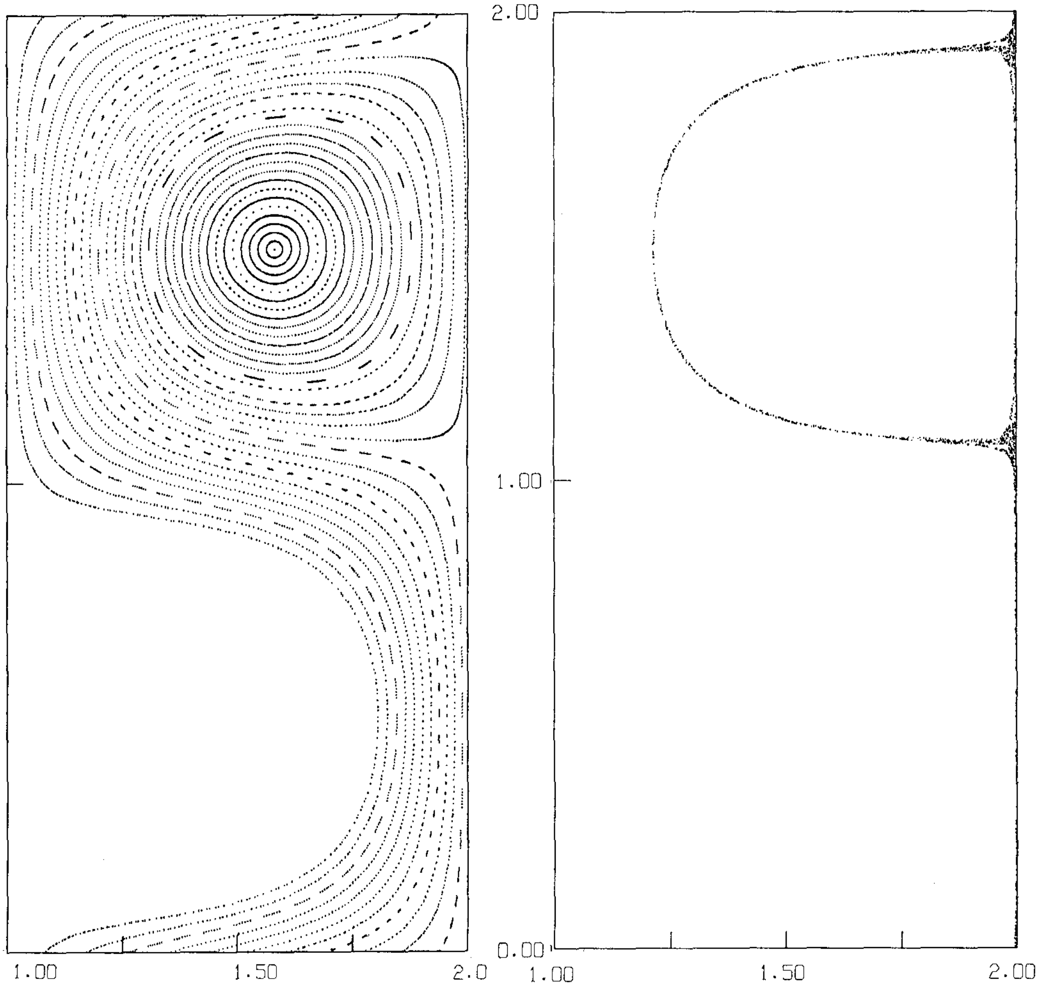


Fig. 3.3. Orbits with falling particles subjected to periodic boundary conditions and shown stroboscopically for $\varepsilon = 0.01$, $\gamma = 0.25$, $\omega = \pi/2.25$. (a) For thirty-two particles starting out uniformly on $y = 3/2$ and followed for 300 oscillation periods of the fluid flow. (b) For four particles starting out on $y = 2$, so as to be in the chaotic region, and followed for 500 periods.

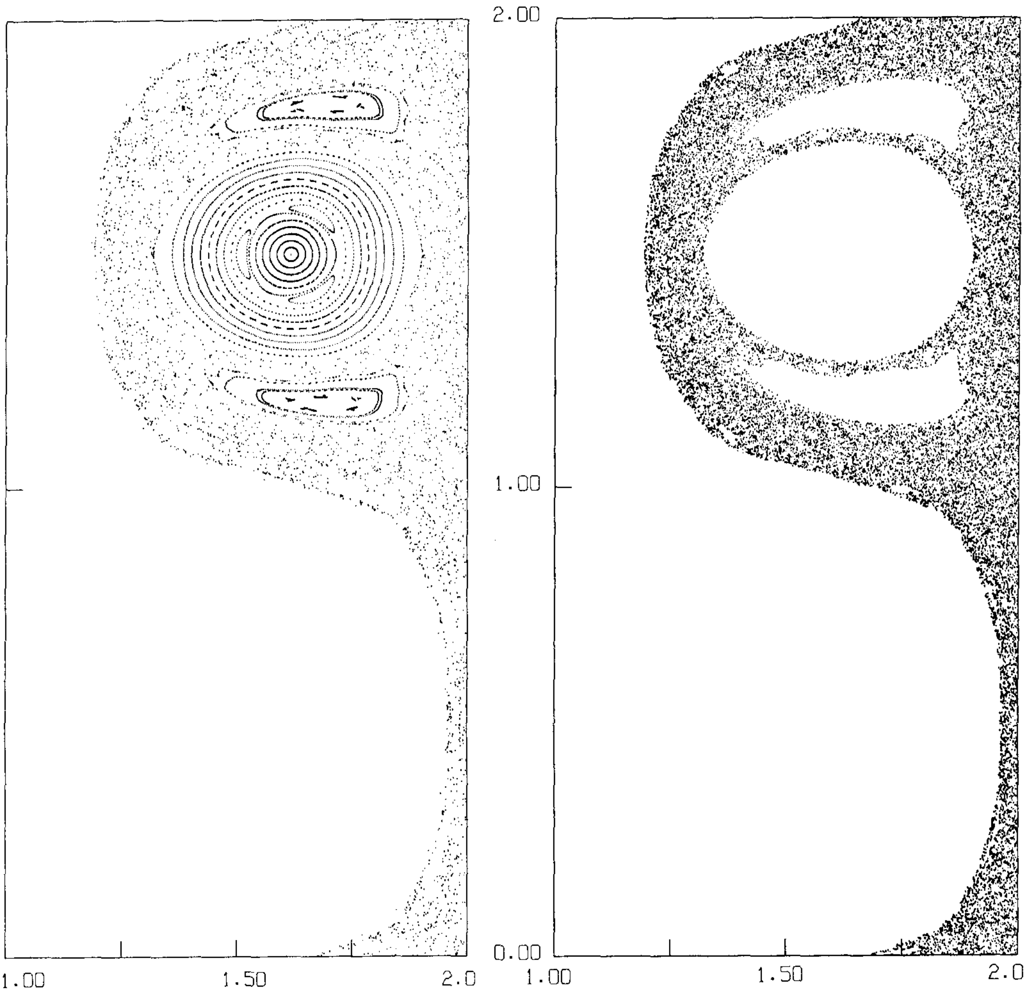
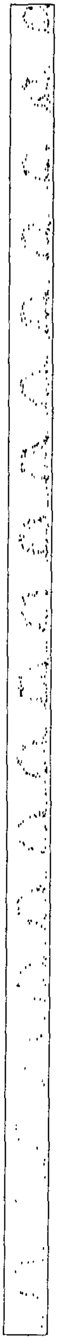


Fig. 3.4. Same as in Fig. 3.3 but with $\varepsilon = 0.05$. This time, the orbits in (a) have been followed for 400 oscillation periods and those in (b) for 3000. To the left edge of the chaotic region integrable orbits are found, though they are not shown in the figure.



Strung out along the left of this page is Fig. 3.5 showing the locations of 512 particles that were initially spread along the line $y = 2.0$ in the x interval [1.85,1.90]. Particles from this interval fall through chaotically. The particles spread in vertical extent, but stay within the original cell width. In this case, the abcissa is the value of y (not mod 2). We wish to see whether a fractal structure develops. However, there are not enough points in Fig. 3.5 to do this. In Fig. 3.6 we show the results of a calculation designed to suggest the detail in the loops of Fig. 3.5. In this calculation we applied the periodic boundary counditions in y and Fig. 3.6 shows only the upper half of a cell. There are long intertwined filaments extending into the lower half of the cell where the segment was stretched exponentially. This poses a resolution problem that we circumvented in two ways. First, we projected the points in Fig. 3.5 onto a vertical line and used the Grassberger-Procaccia algorithm (1983a,b) to calculate a correlation exponent of 0.78.

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Fig. 3.5. The locations after 50 periods of particles that started in the interval $1.85 \le x \le 1.90$ on $y = 2.0$. For $\varepsilon = 0.5$, $\gamma = 0.25$, $\omega = \pi/2.25$.

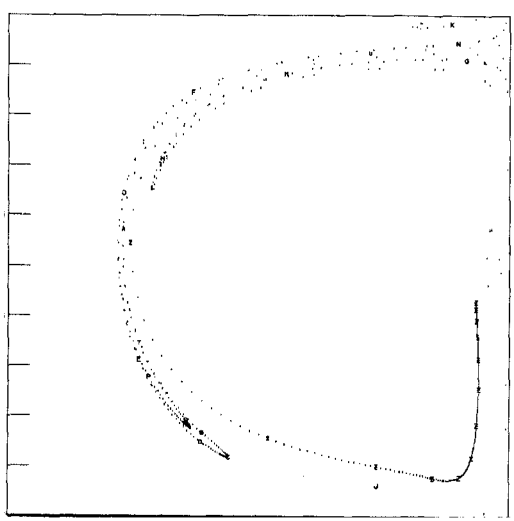

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Fig. 3.6. A closeup of one of the loop-like features in Fig. 3.5 but calculated as explained in the text.

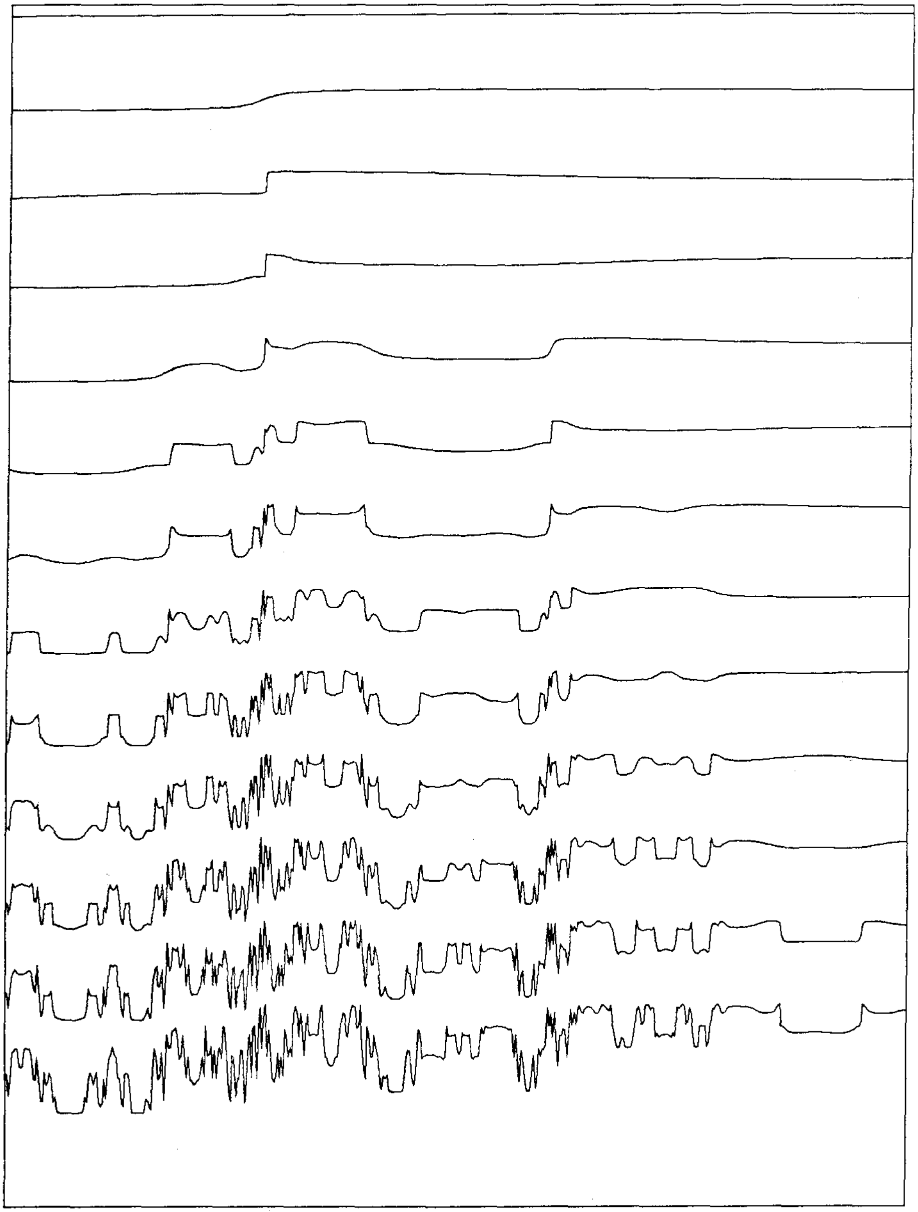


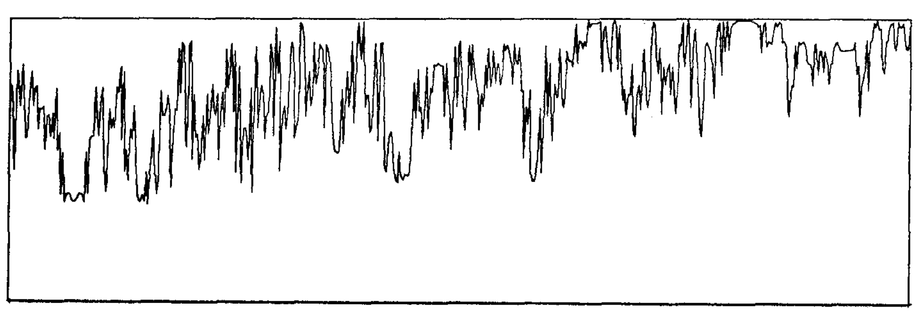
Fig. 3.7. The evolution of an initially simple particle distribution. Shown for a sequence of times separated by one period of oscillation of the fluid. The range in x is 0.0 to 0.1. The height of the box is 110 cell heights. The vertical offset of each curve is linear with time.

The second method focused on the vertical separation of initially nearby particles. In Fig. 3.7 we show the (scaled) y-coordinate of each particle as a function of x_0, the initial x-coordinate, for a sequence of times. Portions of these curves with very large slopes correspond to initial line segments of particles that have been stretched over many cell heights. The stretching and folding in x-y space of the original line of particles produces the self-similar structure seen in Fig. 3.8. We broke the x_0 coordinate into steps of size Δx_0 and found the length, L, of each curve as a function of Δx_0. Then we fit L to the formula

$$L \propto (\Delta x_0)^{-\sigma}. \qquad (3.2)$$

As a function of time, σ behaves as shown in Fig. 3.9. The precise form of this evolution depends on how we perform the calculations; the plateau at $\sigma = 0.76$ is characteristic and is independent of the initial particle density and extent. This plateau does not last indefinitely at fixed resolution. As expected, it persists longer if the initial density is higher. We therefore assign the value $\sigma = 0.76$. We do not have a good way to determine the precision of this result but, on the basis of many such calculations, we would estimate that the internal errors are less than 10%.

We found that σ was not sensitive to ε and γ. We have results when these parameters are in the range 0.1 to 0.8. For ε less than about 0.1, the effect of time dependence is so weak, that the particle spreading is very slow, while for large ε the time steps required become prohibitively small.



The increase of L with decreasing Δx_0 suggests that a fractal object is being formed. We expect its dimension to be approximately $1+\sigma$ (Mandelbrot, 1975). The agreement of σ with the Grassberger-Procaccia correlation exponent reinforces this belief. Like others before us, we have learned that it is easier to find a process that makes fractal objects than to understand its dynamics.

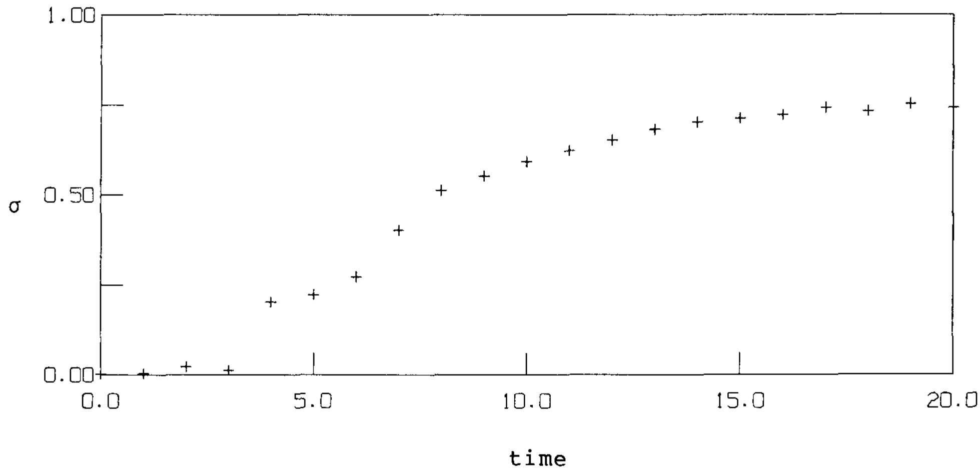


Fig. 3.9. The evolution of σ of (3.2) in time.

4. Conclusion

When studying sedimentation, one often neglects the fluid motion. Yet, as we have seen, the effect of fluid motion on the trajectories of settling particles can organize particle motion. When particle effects on the fluid are allowed, we have a self-consistency problem that may be relevant to the formation of structures in fluid dynamics. While their delicate features are sensitive to noise, diffusion and Brownian motion, mean effects may survive to feed back on the diffusive process. Our calculations already reveal features that hint at such macroscopic implications.

The problem that has most interested us in this study is the formation of fractal swarms. This work raises the problem of finding a dynamical argument to postdict the dimension of the swarms. This question of the dynamics of deformation of a line of passive particles is a first step in the understanding of the stretching of an active vortex pointing out that simple numerical simulations provide a hint about the former problem.

This study is a direct outcome of work (Smith, 1984) begun as a result of a lecture by Herbert Huppert in the GFD Summer Program at the Woods Hole Oceanographic Institution. We are indebted to Hassan Aref and Walter Robinson for helpful discussions. The work has received support from the NSF under grant PHY 80-2371 to Columbia University and from the NASA Cooperative Agreement NCC 5-29 through GISS.

5. References

Aref, H., 1984, Stirring by chaotic advection, J.Fluid Mech., **143**, 1-21.

Arter, W., 1983, ERGODIC STREAM-LINES IN STEADY CONVECTION, Phys. Lett., **97A**, 171-174.

Cocke, W.J., 1969, Turbulent Hydrodynamic Line Stretching: Consequences of Isotropy, Phys. Fluids, **12**, 2488-2492.

Grassberger, P. and Procaccia, I., (1983a) Characterization of Strange Attractors, Phys. Rev. Lett., **50**, 346-349.

Grassberger, P. and Procaccia, I., (1983b) MEASURING THE STRANGENESS OF STRANGE ATTRACTORS, Physica, **9D**, 189-208.

Huppert, H.E., 1984, Lectures on Geological Fluid Dynamics, Woods Hole Ocenaographic Institution, G.F.D. course, to be expected.

Lichtenberg, A.J. and Lieberman, M.A., 1983, Regular and Stochastic Motion, Springer-Verlag.

Mandelbrot, B.B., 1975, Les objets fractal: forme, hasard et dimension. Paris: Flammarion.

Neuringer, J.L., 1968, GREEN'S FUNCTION FOR AN INSTANTANEOUS LINE PARTICLE SOURCE DIFFUSING IN A GRAVITATIONAL FIELD AND UNDER THE INFLUENCE OF A LINEAR SHEAR WIND, Siam J. Appl. Math., **16**, 834-841.

Quinn, P.J., 1984, ON THE FORMATION DYNAMICS OF SHELLS AROUND ELLIPTICAL GALAXIES, Astrophys. J., **279**, 596-609.

Smith, L.A., 1984, PARTICULATE DISPERSAL IN A TIME-DEPENDENT FLOW, Fellow's lecture in G.F.D. Course of W.H.O.I., with several figures.

Stommel, H., 1949, Trajectories of small bodies sinking slowly through convection cells, J. Mar. Res., **8**, 24-29.

Toobey, P.F., Wick, G.L. and Isaacs, J.D., 1977, The Motion of a Small Sphere in a Rotating Velocity Field: A Possible Mechanism for Suspending Particles in Turbulence, J. Geophys. Res., **82**, 2096-2100.