APPLIED CHAOS: QUANTIFYING COMPLEX SYSTEMS

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Abstract

This contribution discusses dynamic reconstructions and their application to the identification of, quantification of and discrimination between complex systems. Phase space reconstructions which reproduce the flow of dynamical systems in time are constructed from (multiple probe) time series data. These dynamic reconstructions are used to quantify the unknown system. In the case of chaotic systems, this is accomplished through the isolation of unstable periodic orbits. The method is applied to data from the Ikeda map, where it is shown that the data requirements of this approach are modest relative to those required for other types of analysis. The application of this approach to systems where the underlying dynamics are stochastic is also discussed.

1. Introduction

In recent years there has been a rapid increase in the application of nonlinear dynamical systems ideas to the analysis of complex time series. These ideas have provided a new paradigm which has proven useful in many fields of study. Originally, reconstructions were "static" in that they attempted to reconstruct the geometry from a time series [1]. Unfortunately, methods of analysis developed for the study of low dimensional low noise systems have sometimes been applied to data sets with which they cannot produce meaningful results. This is particularly true of dimension calculations [2] where many results may be disqualified on simple geometrical constraints [3]. While much work has been done on determining the reliability of such algorithms [4], a major draw back of these approaches is that they fail in subtle ways. More recently, there have been propositions to build dynamic reconstructions for predicting chaotic systems [5, 6] (additional references are given in Ref. [7]). One advantage

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Information Dynamics, Edited by H. Atmanspacher and
H. Scheingraber, Plenum Press, New York, 1991

of these new approaches is that they may be used to quantify the dynamics of the system instead of attempting to analyze the geometry. A second property is that they fail robustly; the value of which cannot be overstressed. In this paper, I develop a reconstruction and apply it not for predicting, but rather to quantify the chaotic system underlying the observations. The reconstructed dynamical system will then be used to locate unstable periodic orbits and quantify their stability. This information may then be used, for example, to determine the origin of a time series known to have come from one of several chaotic systems.

It will be shown that this approach yields excellent results with reasonable data requirements. Note however that the particular predictor constructed here is not claimed to be the best; the fundamental point of this paper is to demonstrate that reconstructing the dynamics from a time series yields a testable and, in some cases, reliable model upon which analysis and system identification techniques can be usefully applied.

2. The Ikeda Map

Before describing the reconstruction, I introduce the system which will be used to generate most of the time series described herein, namely the Ikeda map [8]. (Initial studies with the Henon map showed it to be unsuitable as described in Ref. [7].) The Ikeda map is

$$\begin{aligned} x' &= 1 - \mu(x\cos(t) - y\sin(t)) \\ y' &= \mu(x\sin(t) + y\cos(t)) \end{aligned} \tag{1}$$

where $t = 1/(1 + x^2 + y^2)$. This system displays chaotic behavior over a range of values for the parameter μ including the values chosen here.

In order to quantify the similarity of a dynamic reconstruction with the original system, we compute the spectrum of unstable periodic orbits of each. This spectrum provides information on the global properties of the system as well [9, 10]. Two approaches for determining these orbits were employed. First, a trajectory was followed for n iterations; if the n^{th} iteration of the map was within a distance δ from the initial point, a gradient flow/Newton-Raphson scheme was implemented to search for a periodic orbit. Typically, δ was about one tenth the diameter of the attractor; large values of δ resulted in more computational time per iteration of the map, but tended to find different orbits in fewer iterations while small δ lead to the testing of only near returns. Alternatively, initial points may be taken on a fixed grid. This approach has the advantage of "immediately" sampling regions of the attractor which are visited only rarely, but at the cost of searching regions far from the attractor. As it is very difficult to ascertain whether all the periodic orbits have been found, a combination of these approaches is recommended. In addition, searching from small, random displacements from known periodic orbits eases the identification of nearby orbits in the case where one of them dominates the global search.

The results of these searches ($\mu = 0.83$) yield the following numbers of periodic orbits of period 1 through 12 of (1, 1, 2, 3, 2, 3, 4, 6, 14, 13, 22, 40) . We have searched for orbits up to period 23, at this time well over 4000 orbits have been found. This number is still increasing slowly. Obviously, it is quite difficult to determine if all the periodic orbits of given length have been located. The number of periodic points of any given period provides an estimate of the entropy of the map [10]. In this case we have the estimates near 0.5. Equations (1) also have a fixed point near (1.532, -2.071); as this point appears far from the attractor of interest, we have omitted it from the listing above.

3. The Reconstructed Flow

The basic question in reconstructing a flow or map is one of interpolation in a high dimensional space with irregularly spaced data points. As noted below, the data may be combined from several different probes. To each observation, $\mathbf{x}_i$, there will correspond an image or future vector, $\mathbf{y}_i$. Following Casdagli [11], I shall use a radial basis function as the basis for interpolation.

Consider a deterministic system with phase space dimension M_s. Given n distinct base points $(\mathbf{x}_i, i = 1, 2, ..., n)$, consider a single component of each image vector $\mathbf{y}_i$. Let this component be labeled s_i $(i = 1, 2, ..., n)$. The goal is to determine a predictor $f(\mathbf{x}_i) : R^{M_s} \rightarrow R$ such that:

$$f(\mathbf{x}_i) = s_i \tag{2}$$

Following Powell [12], consider $f(x)$ of the form

$$f(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \phi(||\mathbf{x} - \mathbf{x}_i||) \tag{3}$$

where $\phi(r)$ are radial basis functions and the λ_i are constants which are uniquely determined by Equations (1) provided the matrix

$$A_{ij} = \phi(||\mathbf{x}_i - \mathbf{x}_j||) \tag{4}$$

is nonsingular. This is always the case when the $\mathbf{x}_i$ are distinct and the $\phi(r)$ are radial basis functions (see Refs. [11, 12]). We will restrict attention to the particular cases $\phi(r) = r$ and $\phi(r) = (r^2 + c^2)^{-1/2}$. Reconstructions with the former are more robust in the presence of noise, while those with the latter have lower data requirements (in the low noise case).

In this way, a prediction for one component, s_i, of the image vector $\mathbf{y}_i$ has been constructed. This procedure must then be repeated for each component of the image vector. The resulting set of functions will be referred to collectively as the rbf-map. Note that, with experimental data, the requirement that the base points be distinct may not be satisfied when the data are digitized at low resolution. Even with 64 bit accuracy, it is found that the reconstruction is improved if a minimum distance between the $\mathbf{x}_i$ is maintained.

In the past the $\mathbf{x}_i$ have often been constructed by the method of delays from a single time series. An alternative approach is to use different measurement functions of the system for each element of $\mathbf{x}_i$ (potentially combined with time delays). Here the measurement functions may be distinct probes or a projection of the signal onto a singular value decomposition or Fourier basis. The introduction of measurements in different physical units requires careful thought in determining the scaling of distances in different coordinate directions of the reconstructed phase space. One approach is to choose these scales by optimizing the prediction error profile described in Ref. [7]. Note that the question of rescaling arises in the simple example of a set of coupled o.d.e.'s where the measurement functions may be taken as x, x', x'', and so on, or as nonlinear combinations of these values.

Once an rbf-map has been generated, characteristics of the reconstructed map are accessible. For example, a gradient flow/Newton Raphson style technique may be developed (using finite difference estimates of the local derivatives) to locate the periodic orbits of the rbf map and determine their stability. Note that the stable eigenvalues are best estimated by a reconstruction of the series under time reversal. A comparison of the periodic orbits found in a rbf-map with those found with a knowledge of the Ikeda map is given in the next section.

4. Discussion and Conclusions

Several rbf maps were constructed based on a 2^7 observations taken from 2^8 iteration trajectories from the Ikeda map. In a $\phi(r) = (r^2+1)^{-1/2}$ reconstruction, all known periodic orbits on the attractor with period less than 8 were present with their locations accurate to 3 decimal digits and their unstable eigenvalue to within 2%. Estimates of the unstable eigenvalues of orbits up to period 10 were within 5% of their true values. (In some reconstructions erroneous orbits were also found, these however were always far from the attractor.) Note that this accuracy is more than enough to distinguish series originating from the $\mu = 0.83$ case from those from the $\mu = 0.85$ case.

It is desirable to compare the data requirements of this approach to that of other methods for retrieving similar data, for example, by comparing the amount of data required to quantify the unstable periodic orbits by direct observation. This is particularly interesting due to the independent usefulness of this data as noted above.

The most straightforward method for detecting periodic orbits is simply to scan the data stream for near returns; if the $(n+1)^{th}$ iteration of a point is sufficiently near that point, that location is a candidate for a periodic orbit. Clearly, this approach requires a careful definition of "sufficiently near" which should depend on the particular orbit.

Consider the dynamics within a radius ℓ of a period n point. In the limit $\ell \to 0$, the system's dynamics are governed by the linearization of the map about the periodic point; the linear dynamics may be characterized through the stable and unstable eigenvalues, λ_s and λ_u. As a trajectory moves away from a periodic point, each n^{th} iteration will be approximately a factor λ_u farther away from the point. Call this (location dependent) ratio of distances the effective value of λ_u. We estimate the size of the linear region by determining the distance along the unstable manifold at which this effective value differs by more than some threshold from the true value, say 5%. This then defines a visitation radius for each periodic orbit: if the properties of the orbit are to be deduced directly from near visitations, a passage within this radius must be observed. Thus the visitation time provides an estimate of the amount of data required to quantify a periodic orbit to desired accuracy. Note that this is only a lower bound on the length of trajectory required. If the difficulty of finding period n orbits is denoted by the average visitation time of the least visited period n orbit, then the visitation times for orbits of period 1 to 8 ($\mu = 0.83$) are approximately $(2^7, 2^{8.5}, 2^{12}, 2^{11.5}, 2^{12.8}, 2^{15.7}, 2^{16}, 2^{18})$. These results are based on an average over 20 initial conditions, each of which is iterated up to 2^{22} times. This was not long enough for every initial condition to visit some orbits of period 8 and greater; orbits which were accurately located by the 2^7 point rbf-map described above. This demonstrates that the construction of dynamic flows is a vastly more efficient method to determine the spectrum of periodic orbits.

Rbf-maps from experimental data of a nonlinear electronic oscillator [13] have also been constructed. In this case it is not possible to compare the periods and eigenvalues with the "true" values. Examination of the time series near the predicted periodic orbits in addition to the comparison of results from maps constructed from different surfaces of section show that consistent estimates are obtained. A detailed description of these results is presented elsewhere [14].

Finally we note that this approach is applicable to systems where multiple measurement functions are available simultaneously (e.g., where several sensors are in use). Even in the stochastic case, different measurement functions will be related through their dependence on the current state of the system in such a way as to create forbidden regions in the reconstruction space. An application of these ideas to multiple sensor systems has been proposed [15].

In conclusion, we have seen that a system's dynamics, as characterized by the unstable periodic orbits, are quantitatively reproduced in a dynamic reconstruction using radial basis functions. The data requirements of this method are modest in comparison with other approaches and data from systems with similar dynamics may be distinguished using this approach. Applications to experimental data and higher dimensional systems appear to support the belief that these methods shall provide a powerful new tool for the analysis of nonlinear dynamical systems.

Acknowledgments

I am happy to acknowledge discussions with P. Cvitanovic, M. Muldoon, I. McIntosh, M. Powell, and C. Sparrow. This work was supported by the Science and Engineering Research Council and the U.S. Navy under Grant # N00014-88-J1197.

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