Alpha as Ambiguity

Robust Mean-Variance Portfolio Analysis

F. Maccheroni, M. Marinacci, D. Ruffino
U. Bocconi, U. Bocconi, U. Minnesota

LSE

23 May 2012

(Nearly) nothing to fear but fear itself

What is at work is not only objective, but also subjective uncertainty, or what economists, following Chicago economist Frank Knight's early 20th-century work, call "Knightian uncertainty". [...] Subjective uncertainty is about the "unknown unknowns". When, as today, the unknown unknowns dominate, and the economic environment is so complex as to appear nearly incomprehensible, the result is extreme prudence, if not outright paralysis, on the part of investors, consumers and firms. And this behaviour, in turn, feeds the crisis.

Olivier Blanchard, The Economist, January 29, 2009

The celebrated Arrow-Pratt approximation

$$u^{-1}\left(E_Q\left[u\left(w+h\right)\right]\right) \approx w + E_Q\left[h\right] - \frac{\lambda_u\left(w\right)}{2}\sigma_Q^2\left[h\right]$$

has three main merits:

1. Theoretical identification between risk and variance (risk management)
2. Theoretical identification of risk aversion and the proportionality coefficient $\lambda_u\left(w\right)$ (comparative statics)
3. Practical foundation for *the* preference model of investments' finance

$$U\left(X, Q\right) = E_Q\left[X\right] - \frac{\lambda}{2}\sigma_Q^2\left[X\right]$$

(mean-variance utility)

Model Uncertainty a.k.a. Ambiguity

The amount of money $w+h$ is **state contingent** and for each model Q

$$c\left(w+h, Q\right) = u^{-1}\left(E_Q\left[u\left(w+h\right)\right]\right) \tag{1}$$

where u represents the agent's attitude toward **state uncertainty**.
If Q is unknown, then $c\left(w+h, \cdot\right)$ becomes a **model contingent** amount of money itself.
Suppose π to be the agent's prior probability on the possible models and v to be his attitude toward **model uncertainty**. The rationale used to obtain (1) leads to a (second-order) certainty equivalent

$$\begin{aligned} C\left(w+h\right) &= v^{-1}\left(E_\pi\left[v\left(c\left(w+h, \cdot\right)\right)\right]\right) \\ &= v^{-1}\left(E_\pi\left[v\left(u^{-1}\left(E\left[u\left(w+h\right)\right]\right)\right)\right]\right) \end{aligned}$$

see Klibanoff, Marinacci, and Mukerji (2005, henceforth KMM).

Setup

- $L^2(P) = L^2(\Omega, \mathcal{F}, P)$ square integrable random variables w.r.t. a reference model P (e.g., the physical measure)
- $I \subseteq \mathbb{R}$ interval and $w \in \operatorname{int} I$
- $u, v : I \to \mathbb{R}$ twice continuously differentiable with $u', v' > 0$
- π Borel probability measure with bounded support on the **models**

$$\Delta^2(P) = \left\{ Q \ll P : \frac{dQ}{dP} \in L^2(P) \right\}$$

with **barycenter** P, i.e., such that

$$\int_{\Delta^2(P)} Q(A)\, d\pi(Q) = P(A) \quad \forall A \in \mathcal{F}$$

Ambiguous expectations

For all $X \in L^2(P)$

$$E[X]: \begin{array}{ccc} \Delta^2(P) & \rightarrow & \mathbb{R} \\ Q & \mapsto & E_Q[X] \end{array}$$

is a continuous π-a.s. bounded function, with (second order) expectation

$$\int_{\Delta^2(P)} E_Q[X]\, d\pi(Q) = E_P[X]$$

and variance

$$\sigma_\pi^2[E[X]] = \int_{\Delta^2(P)} (E_Q[X] - E_P[X])^2\, d\pi(Q)$$

Arrow-Pratt extended I

Theorem

For all P-a.s. bounded $\mathbf{h} \in L^2(P)^n$ *and* $\mathbf{x} \in \mathbb{R}^n$,

$$
\begin{aligned}
C(w + \mathbf{x} \cdot \mathbf{h}) = w + E_P[\mathbf{x} \cdot \mathbf{h}] - \frac{\lambda_u(w)}{2}\sigma_P^2[\mathbf{x} \cdot \mathbf{h}] \quad & \textit{(Arrow-Pratt)} \\
- \frac{\lambda_v(w) - \lambda_u(w)}{2}\sigma_\pi^2[E[\mathbf{x} \cdot \mathbf{h}]] \quad & \textit{(Ambiguity)} \\
+ o\left(|\mathbf{x}|^2\right) \quad & \textit{(Remainder)}
\end{aligned}
$$

as $\mathbf{x} \rightarrow \mathbf{0}$.

Arrow-Pratt extended II

For $n = 1$

$$C(w+h) \approx w + E_P[h] - \frac{\lambda_u(w)}{2}\sigma_P^2[h] - \frac{\lambda_v(w) - \lambda_u(w)}{2}\sigma_\pi^2[E[h]]$$

- As well known, **risk aversion** corresponds to $\lambda_u(w) > 0$
- **Ceteris paribus**, the greater $\lambda_v(w)$ the greater the ambiguity premium
- KMM show that **ambiguity aversion** corresponds to $\lambda_v(w) > \lambda_u(w)$

The approximation can be rewritten

$$C(w+h) \approx w + E_P(h) - \frac{\lambda_u(w)}{2}E_\pi\left[\sigma^2[h]\right] - \frac{\lambda_v(w)}{2}\sigma_\pi^2[E[h]]$$

Unambiguous prospects

Definition

$X \in L^2(P)$ is *(first moment) unambiguous* iff for all $Q \in \text{supp}\pi$

$$E_Q[X] = E_P[X]$$

It is *(first moment) ambiguous* otherwise.

I.e., X is unambiguous if its expectation is unaffected by model uncertainty. In this case (*and only in this case*)

$$\sigma_\pi^2[E[X]] = 0$$

Classic Arrow-Pratt approximation can thus be viewed as the special case in which all prospects are unambiguous.

Classic risk theory

Theorem

All the components of $\mathbf{h} \in L^2(P)^n$ *are unambiguous iff*

$$\sigma_\pi^2 \left[E \left[\mathbf{x} \cdot \mathbf{h} \right] \right] = o\left(\left| \mathbf{x} \right|^2 \right) \quad \text{as } \mathbf{x} \to 0 \tag{2}$$

In particular, the following facts are equivalent:

- *All elements of* $L^2(P)$ *are unambiguous.*
- $\pi = \delta_P$.

By (2), ambiguity, if present, **does not vanish** in the second order approximation.

Robust Mean-Variance preferences

An agent ranks prospects X in $L^2(P)$ by the following criterion

$$V(X) = E_P[X] - \frac{\lambda}{2}\sigma_P^2[X] - \frac{\theta}{2}\sigma_\pi^2[E[X]]$$

with $\lambda, \theta > 0$, obtained by setting $w + h = X$, $\lambda = \lambda_u$ and $\theta = \lambda_v - \lambda_u$ in the approximation.

The portfolio problem

- A unit of wealth has to be allocated among $n+1$ assets at time 0
- The return on asset i, $i = 1, ..., n$, at time 1, is denoted by $r_i \in L^2(P)$. The $(n \times 1)$ vector of the returns is $\mathbf{r}$ and the $(n \times 1)$ vector of portfolio weights is $\mathbf{w}$
- The return on the $(n+1)$-th asset is risk-free, i.e. equal to a constant r_f
- The end-of-period return $r_{\mathbf{w}}$, induced by a choice $\mathbf{w}$, is

$$r_{\mathbf{w}} = r_f + \mathbf{w} \cdot (\mathbf{r} - r_f)$$

- Markets are frictionless

The optimal portfolio

The vector of portfolio weights **w** can be optimally chosen in $\mathbb{R}^n$ by solving

$$\max_{\mathbf{w}\in\mathbb{R}^n} V\left(r_{\mathbf{w}}\right) = \max_{\mathbf{w}\in\mathbb{R}^n} \left(E_P\left[r_{\mathbf{w}}\right] - \frac{\lambda}{2}\sigma_P^2\left[r_{\mathbf{w}}\right] - \frac{\theta}{2}\sigma_\pi^2\left[E[r_{\mathbf{w}}]\right] \right)$$

Straightforward computation delivers the following optimality condition

$$\left[\lambda \mathrm{Var}_P\left[\mathbf{r}\right] + \theta \mathrm{Var}_\pi\left[\mathrm{E}\left[\mathbf{r}\right]\right]\right]\hat{\mathbf{w}} = \mathrm{E}_P\left[\mathbf{r} - r_f\right] \tag{3}$$

The most attractive feature of (3) is that it allows us to make use of the vast body of research on classic Mean-Variance preferences developed for problems involving only risk to analyze problems involving also ambiguity.

One ambiguous asset

- For $n = 1$, $\mathbf{r} = r$ (non risk-free),

$$\hat{w} = \frac{E_P\left[r - r_f\right]}{\lambda\sigma_P^2\left[r\right] + \theta\sigma_\pi^2\left(E\left[r\right]\right)}$$

- An in $\theta\sigma_\mu^2\left(E\left(r\right)\right)$ – i.e., an increase in either ambiguity aversion θ or ambiguity in expectations $\sigma_\mu^2\left(E\left(r\right)\right)$ – makes the ambiguous asset less desirable and increases the DM's demand for the risk-free asset (a flight-to-quality effect).

One risky and one ambiguous assets

Two (non risk free) assets:

- r_m unambiguous
- r_e ambiguous

Assumptions:

- the portfolio problem admits a unique solution and the ratio of optimal portfolio weights is well-defined
- excess returns on both uncertain assets are strictly positive

Portfolio weights

If $\sigma_P [r_e, r_m] = 0$ then

$$\hat{w}_m = \frac{E_P [r_m] - r_f}{\lambda \sigma_P^2 [r_m]} \quad \text{and} \quad \hat{w}_e = \frac{E_P [r_e] - r_f}{\lambda \sigma_P^2 [r_e] + \theta \sigma_\pi^2 [E [r_e]]}$$

Else if $\sigma_P [r_e, r_m] \neq 0$ then

$$\hat{w}_m = \frac{(E_P [r_m] - r_f) \left(\lambda \sigma_P^2 [r_e] + \theta \sigma_\pi^2 (E [r_e]) \right) - \lambda \sigma_P [r_e, r_m] (E_P [r_e] - r_f)}{\lambda^2 \sigma_P^2 [r_e] \sigma_P^2 [r_m] + \lambda \theta \sigma_\pi^2 (E [r_e]) \sigma_P^2 [r_m] - \lambda^2 \sigma_P [r_e, r_m]^2}$$

and

$$\hat{w}_e = \frac{(E_P [r_e] - r_f) \lambda \sigma_P^2 [r_m] - \lambda \sigma_P [r_e, r_m] (E_P [r_m] - r_f)}{\lambda^2 \sigma_P^2 [r_e] \sigma_P^2 [r_m] + \lambda \theta \sigma_\pi^2 (E [r_e]) \sigma_P^2 [r_m] - \lambda^2 \sigma_P [r_e, r_m]^2}$$

Technical measures: *beta* and *alpha*

- Assume $\sigma_P[r_e, r_m] \neq 0$ and set

$$\beta_{em} = \frac{\sigma_P[r_e, r_m]}{\sigma_P^2[r_m]} \quad \text{and} \quad \alpha_{em} = (E_P[r_e] - r_f) - \beta_{em}(E_P(r_m) - r_f)$$

- β_{em} is a measure of the asset r_e pure risk (in relation to asset m); it is a pure risk adjustment
- $\beta_{em}(E_P(r_m) - r_f)$ is what r_e is expected to earn/lose, net of r_f, given its level of pure risk sensitivity
- α_{em} is the residual component of the expected excess return $E_P(r_e - r_f)$: it is what r_e is expected to earn/lose, net of r_f, given its level of uncertainty uncorrelated with pure risk (such uncertainty is specific to the ambiguous asset r_e)

They solve

$$\min_{\alpha,\beta\in\mathbb{R}} \|(r_e - r_f) - (\alpha + \beta(r_m - r_f))\|$$

Seeking the *alpha*

- Our agent "seeks the *alpha*"

$$\operatorname{sgn} \hat{w}_e = \operatorname{sgn} \alpha_{em}$$

- Agent uses α_{em} as a criterion to decide whether to take a long or short position in the ambiguous asset, i.e., to decide in which side of the market of asset r_e to be

Reducing exposure

- Our agent also reduces exposure to ambiguity as ambiguity aversion increases

$$\operatorname{sgn} \frac{\partial}{\partial \theta} \hat{w}_e = -\operatorname{sgn} \alpha_{em}$$

- Our agent:
  1. goes long on r_e when α is positive and short otherwise
  2. reduces exposure to r_e as ambiguity increases
- E.g., in an international portfolio interpretation of our tripartite analysis, this means that higher ambiguity results in higher home bias.