Section D

This section is marked out of 100. The marks achieved account for 50% of your final exam result.

Instructions to candidates: This section contains five questions.
Answer FOUR questions as follows.

- You MUST answer Question 1.

- Answer any THREE of the remaining four questions.

All questions carry equal marks. For full credit, you must show all of your working and clearly explain your reasoning. Calculators are permitted.

1. (a) Solve the equation $x^3 - 4x^2 - 7x + 10 = 0$. **(6)**

(b) Solve the equation $2\ln(x) = \ln(x + 6)$. **(6)**

(c) Solve the equation $\dfrac{5^{x^2}}{25} = 5^{-x}$. **(5)**

(d) Find the values of x that satisfy the inequality

$$\frac{x-1}{x+1} + \frac{x}{x-2} < 1 + \frac{10}{(x-2)(x+1)}.$$

What values of x satisfy this inequality if '$<$' is replaced by '$\leq$'? **(8)**

2. Orange Ltd. can produce a quantity q of its product (where $q \geq 0$ and measured in millions of units) at a cost given by $C(q) = 2 + 2q + q^2$ (measured in millions of dollars). Its profit is given by

$$\pi(q) = R(q) - C(q),$$

where $R(q) = pq$ is its revenue and p is the price (for a million units measured in millions of dollars).

(a) Suppose that Orange Ltd. supplies its good to a market which will pay a price given by $p = 8 - 2q$ if q is the total quantity that is being supplied. Assuming that everything they supply sells at this price and that they are the only supplier to this market, explain why Orange Ltd.'s profit is given by

$$\pi(q) = -3q^2 + 6q - 2.$$

Hence find the number of units that they should be selling if they want to maximise their profits. What price per unit does this correspond to? **(8)**

(b) A competitor enters this market and, in order to establish itself, supplies 1.5 million units. Explain why Orange Ltd.'s profit is now given by

$$\pi(q) = -3q^2 + 3q - 2.$$

Find the number of units that they should be selling if they want to maximise their profits. Should Orange Ltd. continue to sell in this market? **(6)**

(c) Orange Ltd. seeks a new market and wants to maximise its profits by selling 4 million units at a price of 14 dollars per unit. If a market will buy at a price given by $p = a - bq$ where a and b are positive constants, what values of a and b are needed if they want to achieve this goal? **(11)**

3. Consider the curve with equation $y = \dfrac{1}{x - \sqrt{x}}$.

(a) Find and classify the stationary point of the curve. **(6)**

(b) Sketch the curve $y = f(x)$ clearly labelling any important features. **(8)**

(c) Find the area bounded by the curve, the x-axis and the vertical lines $x = 4$ and $x = 9$. **(5)**

(d) Find the equation of the tangent line to the curve when $x = 4$.

Using your graph, determine the values of x for which the tangent line to the curve will have two distinct points of intersection with the curve. **(6)**

4. (a) Find $\dfrac{\mathrm{d}}{\mathrm{d}x} \tan x$. **(4)**

(b) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \dfrac{1}{1 + x^2}$. **(4)**

(c) If $y = \left(\tan^{-1} x\right)^2$, show that $(1 + x^2)^2 y'' + 2x(1 + x^2)y' = 2$. **(4)**

(d) If $g(x) = (1 + x^2)^2 f(x)$, find $g'(x)$ in terms of $f(x)$ and $f'(x)$.

Hence solve the differential equation

$$(1 + x^2)^2 z''(x) + 4x(1 + x^2)z'(x) = 2,$$

where $z(0) = 1$ and $z'(0) = 0$. **(13)**

5. (a) Mark wants to buy a house for $200,000 and decides to take out a twenty five year mortgage to pay for it. Given that the interest on the mortgage is 2% per month, explain why his monthly repayments of X will be given by the equation

$$200,000(1.02)^{300} = X + X(1.02) + X(1.02)^2 + \cdots + X(1.02)^{299},$$

assuming that he makes his first payment one month after taking out the mortgage. Hence find X. **(8)**

(b) After he has made 100 repayments of X, Mark decides to move his mortgage to another lender. How much does he owe to his original lender at this point? **(6)**

(c) If the new lender charges interest at a monthly rate of 1% and assuming that he wants to repay what remains of the loan in what remains of the original 25 year period, what will his new monthly repayment be? **(5)**

(d) Suppose that, in addition to the monthly repayment found in (c), he can also repay an extra $200 every month. How long will it take him to repay the mortgage? **(6)**