SECTION D

Answer **FIVE** of the six questions from this section.

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1. (a) For real numbers $x \neq 2, 3$, the function, $f(x)$, is defined by

$$f(x) = \frac{x+1}{x^2 - 5x + 6}.$$

Find the values of x for which $f(x) < 0$.

(b) Find all the solutions to the equation $3^{x^2} = \dfrac{27}{9^x}$.

(c) Find all the solutions to the equation $x^3 + 6 = 4x^2 - x$.

2. (a) Mr. Smith takes out a mortgage for £20,000 at the beginning of 2011 where compound interest is charged at a rate of 5% at the end of each month. He makes his first repayment of £P at the end of January 2011 and, by making a repayment of £P at the end of each subsequent month, he intends to repay the mortgage completely in 20 payments. Clearly explain why his repayments will satisfy the equation

$$20,000(1.05)^{20} = P + P(1.05) + P(1.05)^2 + \cdots + P(1.05)^{19},$$

and hence find the value of P to two decimal places.

(b) Due to unforeseen circumstances, Mr. Smith can not make the 10th payment. If he agrees with the bank that he will compensate for this by making a repayment of £P for payments 11 to 19 and a repayment of £R for payment 20, what is the value of R to two decimal places if you use your answer to (a) for P?

(c) If, instead, he agrees with the bank that he will compensate for this missed payment by making a repayment of £S for payments 11 to 20, what is the value of S to two decimal places if you use your answer to (a) for P?

3. Consider the function $f(x) = 3x^5 - 5x^4$.

(a) Find and classify any stationary points of $f(x)$.

(b) Find any points of inflection of $f(x)$, clearly justifying your answer.

(c) Sketch the curve $y = f(x)$.

(d) Find the equation of the tangent to the curve $y = f(x)$ when $x = 1$.

4. (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2},$$

and evaluate the integral

$$\int_{-1}^{1} \frac{x^2}{1+x^2} \, \mathrm{d}x,$$

simplifying your answer as far as possible.

(b) Suppose that $m, n \geq 1$ are integers. If

$$I_{m,n} = \int_0^1 x^m (1-x)^n \, \mathrm{d}x,$$

show that

$$I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}.$$

Hence find $I_{m,n}$, simplifying your answer as far as possible.

5. A quantity, x, of some product generates a revenue given by $R(x) = px$ where p is a constant and costs an amount $C(x) = x^3 - 3x^2 + 3x$ to produce. Find the profit function, $\pi(x) = R(x) - C(x)$, and clearly explain why it is the case that:

(a) If $p > \frac{3}{4}$, then the maximum value of $\pi(x)$ occurs when $x = 1 + \sqrt{p/3}$.

(b) If $0 \leq p < \frac{3}{4}$, then the maximum value of $\pi(x)$ is zero.

If $p = \frac{3}{4}$, find the value of x that maximises $\pi(x)$ and give your reason for choosing this value of x.

6. A firm manufactures two products, A and B, and both of these need to be processed on three machines, M1, M2 and M3. M1 is available for a total of 24 hours a week, M2 for 9 hours, and M3 for 24 hours. The production of a unit of A requires a processing time of 1 hour on M1, 1 hour on M2, and 3 hours on M3. The production of a unit of B requires a processing time of 4 hours on M1, 1 hour on M2, and 2 hours on M3.

(a) Explain, *very briefly*, why the production of x units of A and y units of B will require the following inequalities to hold:

$$x + 4y \leq 24, \quad x + y \leq 9, \quad 3x + 2y \leq 24, \quad x \geq 0, \quad y \geq 0.$$

(b) Indicate on a diagram the region, $\mathcal{R}$, of all points (x, y) in the plane that satisfy these inequalities. (You should clearly label any relevant points of intersection.)

(c) The profit on each unit of A is \$2 and the profit on each unit of B is \$1. By considering the lines $2x + y = c$ where c is a constant in relation to your diagram, or otherwise, find the values of x and y which maximise the firm's weekly profit. (You should clearly explain how you reached your answer.)

(d) Suppose, instead, that the profit on a unit of A is \$1 and the profit on a unit of B is \$2. Find the values of x and y which maximise the weekly profit in this case. (You should clearly explain how you reached your answer.)