Optimization with Applications in Portfolio Choice (ME201)

**Course duration:** 54 hours lecture and class time (Over three weeks)

**Summer School Programme Area:** Research Methods, Data Science, and Mathematics

**LSE Teaching Department:** Department of Mathematics

**Lead Faculty:** Dr Albina Danilova (Dept. of Mathematics)

**Pre-requisites:** Multivariate calculus and linear algebra at lower undergraduate level.

**Course Description:**

The purpose of this course is to introduce the mathematical theory of optimization as well as to provide a toolbox for solving problems arising in economics and finance. Although emphasis is placed on problems of optimal investment and consumption, the techniques developed are applicable to problems arising in several other areas.

The course starts with a review of relevant mathematical background. In particular, properties of sets as well as the continuity, differentiability and convexity of functions are studied. Necessary and sufficient conditions for a local minimum or maximum of a smooth function are derived. First and second order conditions are introduced in this context and sufficient conditions for a global minimum or maximum are discussed. Numerical methods for finding a local minimum or maximum, in particular, gradient methods, are also studied.

The course then considers optimization with equality constraints. Lagrange's theorem on necessary conditions for optimality is established. The interpretation of the Lagrange multipliers is discussed and sufficient conditions for optimality are stated. A study of optimization with inequality constraints and the Kuhn-Tucker necessary conditions for optimality follows. Furthermore, the quadratic penalty function method for finding a solution to a constrained optimization problem numerically is developed. To illustrate the theory, the one-period Markowitz portfolio optimization problem is solved. To this end, some elementary probability background is also reviewed.

The course continues with a study of multi-stage optimization problems and Bellman's dynamic programming principle. To apply the theory, the multi-period binomial tree model for a financial market is introduced and the concepts of arbitrage and self-financing portfolios are discussed. In this context, the dynamic programming principle is used to solve optimal investment problems for various agent preferences. In particular, the Merton problem is solved.

**Reading:** The main reading material will be the detailed handouts distributed at the beginning of the course. Optionally, the following books might also be helpful:

Course Structure:
- Lectures: 36 hours
- Classes: 18 hours

Assessment:
The assessment consists of:
Two two-hour written exams. The mid-term exam will take place in week 2 of the programme and the final exam will take place at the end of the programme in week 3. Each exam will count for 50% of the final mark. Both exams will be closed book exams.

Lecture Schedule:

1. Introduction and Overview
- Topics covered:
  - Introduction to the Summer School
  - What is optimization?
  - Course overview
  - Sequences, convergence, and limits
  - Closed, bounded, and compact sets
- Core reading:
  - Sections 1.2.1, 1.2.2, 1.2.7, 1.2.8 of R.K. Sundaram, *A First Course in Optimization Theory*, CUP, 1996.

2. Unconstrained Optimization
- Topics covered:
  - Continuity, differentiability, and convexity of functions
  - Local maximum and minimum of function
  - Taylor’s theorem
  - First order conditions
- Core reading:

3. Unconstrained Optimization (cont’d)
- Topics covered:
  - Second order conditions
  - Global maximum and minimum
  - Sufficient conditions for global extremum
- Core reading:
4. Numerical methods for unconstrained optimization

   Topics covered:
   - Linesearch algorithm
   - Steepest descent algorithm

   Suggested reading:
   - Chapter 1.3.1 and 1.3.2 of D. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods*, Athena Scientific, 1996.

5. Optimization with equality constraints

   Topics covered:
   - Necessary conditions: Lagrange Theorem
   - Application of Lagrange theorem
   - The Lagrange multipliers

   Core reading:

6. Optimization with equality constraints (cont’d)

   Topics covered:
   - Sufficient conditions for optimality: augmented Lagrangian
   - Penalty method for numerical optimization

   Suggested reading:

Tuesday, 7 Aug – Mid-Term Exam

7. Optimization with inequality constraints

   Topics covered:
   - Kuhn-Tucker Theorem
   - Applications of Kuhn-Tucker theorem

   Core reading:

8. Random variables

   Topics covered:
   - Discrete random variables
   - Continuous random variables
   - Expectation
   - Conditional expectation
9. Markowitz optimization problem

Topics covered:
- The general one-period market model
- Formulation and solution of the utility optimization problem

Suggested reading:

10. Bellman Dynamic programming principle

Topics covered:
- Dynamic programming problems
- Dynamic programming principle in discrete time
- Markovian strategies

Core reading:

11. Optimal consumption problem

Topics covered:
- The multi-period binomial model
- Self-financing portfolios
- Utility functions
- An agent’s optimization problem

Suggested reading:

12. Applications and case studies

Topics covered:
- Merton portfolio optimization problem
- Optimal consumption and investment

Suggested reading:

Friday, 17 Aug – Final Exam

Content of Seminars:
The seminars will consist of exercises on the mathematical tools developed in the course.
Credit Transfer: If you are hoping to earn credit by taking this course, please ensure that you confirm it is eligible for credit transfer well in advance of the start date. Please discuss this directly with your home institution or Study Abroad Advisor.

As a guide, our LSE Summer School courses are typically eligible for three or four credits within the US system and 7.5 ECTS in Europe. Different institutions and countries can, and will, vary. You will receive a digital transcript and a printed certificate following your successful completion of the course in order to make arrangements for transfer of credit.

If you have any queries, please direct them to summer.school@lse.ac.uk