Efficient Estimation of Risk Measures in a semiparametric GARCH model

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Abstract

This paper proposes efficient estimators of risk measures in a semiparametric GARCH model defined through moment constraints. Moment constraints are often used to identify and estimate the mean and variance parameters and are however discarded when estimating error quantiles. In order to prevent this efficiency loss in quantile estimation, we propose a quantile estimator based on inverting an empirical likelihood weighted distribution estimator. It is found that the new quantile estimator is uniformly more efficient than the simple empirical quantile and a quantile estimator based on normalized residuals. At the same time, the efficiency gain in error quantile estimation hinges on the efficiency of estimators of the variance parameters.

1 Modeling Framework

We consider the following popular AR(p)-GARCH(1,1) model

$$\begin{aligned} y_t &= \sum_{j=1}^{p} \rho_j y_{t-j} + h_t^{1/2}\varepsilon_t \\ h_t &= \omega + \beta h_{t-1} + \gamma u_{t-1}^2, \end{aligned}$$

where $u_t = h_t^{1/2}\varepsilon_t$, and $\{\varepsilon_t\}$ is an i.i.d sequence of innovations with mean zero and variance one and p is a finite and known integer. We suppose that ε_t has a density function $f(\cdot)$, which is unknown apart from the two moment conditions:

$$\int x f(x)dx = 0; \int x^2 f(x)dx = 1.$$

The conditional Value-at-Risk of y_t given $\mathcal{F}_{t-1}$ is:

$$\xi_t(\alpha) = \sum_{j=1}^{p} \rho_j y_{t-j} + h_t^{1/2} q_\alpha$$

The conditional expected shortfall of y_t given $\mathcal{F}_{t-1}$ is

$$\begin{aligned} \chi_t(\alpha) &= E[y_t | 1(y_t \le \xi_t(\alpha)), \mathcal{F}_{t-1}] \\ &= \sum_{j=1}^{p} \rho_j y_{t-j} + h_t^{1/2} E[\varepsilon_t | 1(\varepsilon_t \le q_\alpha)] \\ &= \sum_{j=1}^{p} \rho_j y_{t-j} + h_t^{1/2} ES_\alpha \end{aligned}$$

Let $\theta = (\omega, \beta, \gamma)$. The goal of this paper is to estimate the parameters $(\theta, q_\alpha, ES_\alpha)$ efficiently and plug in these efficient estimators to obtain the conditional quantile $\widehat{\xi}_{n,t} = h_t^{1/2}(\widehat{\theta})\widehat{q}_\alpha$ and the conditional expected shortfall $\widehat{\chi}_t(\alpha) = h_t^{1/2}(\widehat{\theta})\widehat{ES}_\alpha$.

2 Efficient Estimation

Efficient Estimaton of θ :

We rewrite the volatility model of GARCH(1,1) using adaptive methods, which is

$$h_t = c^2 + ac^2 y_{t-1}^2 + b h_{t-1}$$

The finite dimensional parameter in this model $\theta = (c, a, b)^\top \in \Theta \subset \mathbb{R}^3$ is to be partitioned into two parts: $(c, \beta^\top)$ where $\beta = (a, b)^\top \in B$ for the reason that only β is adaptively estimable.As a result, we can rewrite the volatility as $h_t(\theta) = c^2 g_t(a, b)$, where $g_t(\beta) = 1 + a u_{t-1}^2 + b g_{t-1}(\beta)$.

We construct the efficient estimator for θ in 3 steps:

1. Let $\widehat{\theta}_1 = (\widehat{\beta}_1^\top, \widehat{c}_1)^\top$ be an initial $\sqrt{T}$-consistent estimator, for example the QMLE, and compute the residuals $\widehat{\varepsilon}_{1t} = y_t / h_t^{1/2}(\widehat{\theta}_1)$.
2. Update the estimator of β by using the Newton–Raphson method:

$$\begin{aligned} \widehat{\beta} = \widehat{\beta}_1 + & \left[\frac{1}{n}\sum_{t=1}^n \widehat{l}^*_{1t}(\widehat{\beta}_1)\widehat{l}^*_{1t}(\widehat{\beta}_1)^\top\right]^{-1} \\ & \times \frac{1}{n}\sum_{t=1}^n \widehat{l}^*_{1t}(\widehat{\beta}_1) \end{aligned}$$

$$\widehat{l}^*_{1t}(\widehat{\beta}_1) = -\frac{1}{2}\left[G_t(\widehat{\beta}_1) - \frac{1}{n}\sum_{s=1}^n G_s(\widehat{\beta}_1)\right]\widehat{R}_3(\widehat{\varepsilon}_{1t})$$

3. Denote $\widehat{e}_t = y_t g_t^{-1/2}(\widehat{\beta})$ and the efficient estimator for c is

$$\widehat{c} = \sqrt{\frac{1}{n}\sum_{t=1}^n \widehat{e}_t^2 - \frac{1}{n}\frac{\sum_{t=1}^n \widehat{e}_t^3}{\sum_{t=1}^n \widehat{e}_t^2}\sum_{t=1}^n \widehat{e}_t}.$$

Theorem 1 *Suppose that assumptions A hold. Then, then there exists an efficient estimator $\widehat{\theta}$ that has the following expanded*

$$\sqrt{n}(\widehat{\theta} - \theta_0) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \psi_t(\theta_0) + o_P(1),$$

$$\psi_t(\theta_0) = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} R_3(\varepsilon_t) \\ R_2(\varepsilon_t) \end{pmatrix}.$$

where

$$\begin{aligned} A &= -\frac{1}{2}E[l_{1t}^* l_{1t}^{*\top}]^{-1}\{G_t - G\} \\ B &= \frac{c_0}{4}G^\top E[l_{1t}^* l_{1t}^{*\top}]^{-1}\{G_t - G\} \\ C &= \frac{c_0}{2}(-E\varepsilon^3, 1) \end{aligned}$$

Efficient Estimation of quantile:

Case 1: Quantile Estimation with true error available: We compare three estimators:
a) The empirical distribution function $\widehat{F}(x) = n^{-1}\sum_{t=1}^n 1(\varepsilon_t \le x)$ is commonly used but it does not impose these moment constraints.
b) A modified empirical distribution, $\widehat{F}_N(x) = n^{-1}\sum_{t=1}^n 1((\varepsilon_t - \widehat{\mu}_\varepsilon)/\widehat{\sigma}_\varepsilon \le x)$.
c) In this paper, we consider a new weighted empirical distribution estimator $\widehat{F}_w(x) = \sum_{t=1}^n \widehat{w}_t 1(\varepsilon_t \le x)$, where the empirical likelihood weights $\{\widehat{w}_t\}$ come from the following:

$$\begin{aligned} & \max_{\{w_t\}} \Pi_{t=1}^n w_t \\ \text{s.t.} \quad & \sum_{t=1}^n w_t = 1; \sum_{t=1}^n w_t \varepsilon_t = 0; \quad \textit{and} \\ & \sum_{t=1}^n w_t(\varepsilon_t^2 - 1) = 0. \end{aligned}$$

It is clear that by construction $\widehat{F}_w$ satisfies the moment restrictions.

Theorem 2 *Suppose that assumptions A.2-A.5 hold. The quantile and expected shortfall estimators are asymptotically normal:*

$$\begin{aligned} \sqrt{n}(\widehat{q}_\alpha - q_\alpha) &\Longrightarrow N(0, V_1) \\ \sqrt{n}(\widehat{q}_{N\alpha} - q_\alpha) &\Longrightarrow N(0, V_2) \\ \sqrt{n}(\widehat{q}_{w\alpha} - q_\alpha) &\Longrightarrow N(0, V_3) \\ \sqrt{n}(\widehat{ES}_\alpha - ES_\alpha) &\Longrightarrow N(0, V_4) \\ \sqrt{n}(\widehat{ES}_{N\alpha} - ES_\alpha) &\Longrightarrow N(0, V_5) \\ \sqrt{n}(\widehat{ES}_{w\alpha} - ES_\alpha) &\Longrightarrow N(0, V_6). \end{aligned}$$

Case 2: Quantile Estimation with true error available. The result is similar as Case 1, but we use the polluted error instead.

3 Simulation and Empirical Study

Simulation: We follow Drost and Klaassen (1997) to simulate several GARCH (1,1) series from the model (1) with the following parameterizations:

1. $(c, a, b) \in \{(1, 0.3, 0.6), (1, 0.1, 0.8), (1, 0.05, 0.9)\}$;
2. $f(x) \in \{N(0,1), MN(2,-2), L, t(5), t(7), t(9), \chi_6^2, \chi_{12}^2\}$, which are, respectively, referred to the densities of standardized (mean 0 and variance 1) distributions from Normal, Mixed Normal with means $(2, -2)$, Laplace, student distributions with degree of freedom 5, 7 and 9 and chi-squared distribution with 6 and 12 degrees of freedom.

Table 1. Integrated Mean Squared Error ($\times 10^{-3}$) of Distribution Function Estimators

	$n = 500$			$n = 1000$		
	$\widehat{F}(x)$	$\widehat{F}_N(x)$	$\widehat{F}_w(x)$	$\widehat{F}(x)$	$\widehat{F}_N(x)$	$\widehat{F}_w(x)$
N	0.3365	0.1199	0.1212	0.1616	0.0580	0.0583
MN	0.3286	0.1412	0.0916	0.1622	0.0687	0.0462
L	0.3313	0.2188	0.1603	0.1692	0.1092	0.0810
$t(5)$	0.3419	0.2157	0.1635	0.1657	0.1055	0.0797
$t(7)$	0.3255	0.1594	0.1458	0.1695	0.0791	0.0708
$t(9)$	0.3336	0.1439	0.1361	0.1664	0.0730	0.0687
χ_6^2	0.3308	0.1479	0.1217	0.1692	0.0721	0.0605
χ_{12}^2	0.3297	0.1335	0.1213	0.1692	0.0654	0.0595

Table 2. Comparison of quantile estimators for $q_{0.05}$ (true errors are available)

	$n = 1000$					
	$Bias(\times 10^{-3})$			$MSE(\times 10^{-3})$		
	$\widehat{q}_\alpha$	$\widehat{q}_{N\alpha}$	$\widehat{q}_{w\alpha}$	$\widehat{q}_\alpha$	$\widehat{q}_{N\alpha}$	$\widehat{q}_{w\alpha}$
N	-1.2	-0.6	0.1	4.5	2.0	2.0
MN	-0.8	-1.1	0.0	1.3	1.0	0.7
L	2.6	2.7	4.6	9.4	4.9	4.6
$t(5)$	-1.4	2.7	4.9	6.9	4.8	3.9
$t(7)$	0.6	2.9	3.5	6.3	3.3	3.2
$t(9)$	3.5	2.1	3.1	5.8	2.9	2.8
χ_6^2	-2.3	0.2	-1.0	0.7	1.1	0.6
χ_{12}^2	-2.0	-1.2	-0.8	1.4	1.3	1.0

Empirical Study: Backtest VaR Models comparison 1. EWMA, MA, HS, GARCH(1,1) and GARCH-ELW

Model	Violation Ratio	Volatility
EWMA	2.2084	0.0185
MA	3.4541	0.0080
HS	2.2650	0.0130
GARCH(1,1)	2.3216	0.0174
GARCH-ELW(our model)	1.1891	0.0185