BETA-GAMMA ALGEBRA, DISCOUNTED CASH-FLOWS, AND BARNES' LEMMAS

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1. Let $G^{(p)}$ denote a variable with a $\mathbf{Gamma}(p,1)$ distribution. If $G_1^{(a)}, G_2^{(b)}$ are independent, then

$$G_1^{(a)} + G_2^{(b)} \stackrel{\mathrm{d}}{=} G_3^{(a+b)}. \qquad (1a)$$

This is the simplest result from the so-called "beta-gamma algebra". Other ones are:

$$B_1^{(a,b)} G_1^{(a+b)} \stackrel{\mathrm{d}}{=} G_2^{(a)}, \qquad (G_1^{(2a)})^2 \stackrel{\mathrm{d}}{=} 4G_2^{(a)} G_3^{(a+\frac{1}{2})}, \qquad (1b)$$

where $B_1^{(a,b)}$ is a variable with a $\mathbf{Beta}(a,b)$ distribution, and all variables are independent.

2. Sums of discounted cash flows such as

$$\sum A_1 \cdots A_n C_n$$

occur naturally in various contexts, including actuarial science. Gerber (*An Introduction to Mathematical Risk Theory*, 1979) mentions that if $C_j \sim \mathbf{Exp}(1)$, $j = 1, 2, \ldots$, are independent and also independent of $\{T_k\}$, a Poisson point process with parameter λ, then, for a discount rate $r > 0$,

$$\sum e^{-rT_n} C_n \stackrel{\mathrm{d}}{=} G^{(\frac{\lambda}{r})}. \qquad (2)$$

It has been known for a while that (2) and the first identity in (1b) are indeed the same.

3. Barnes (1908, 1910) proved two results that have come to be known as his first and second "lemmas", the first one of which being

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\, \Gamma(A+z)\Gamma(B+z)\Gamma(C-z)\Gamma(D-z) = \frac{\Gamma(A+B)\Gamma(A+C)\Gamma(B+C)\Gamma(B+D)}{\Gamma(A+B+C+D)}. \qquad (3)$$

It will be shown that (3) is equivalent to (1a), that is, one implies the other. Moreover, Barnes' second lemma is equivalent to another identity from the beta-gamma algebra. All this leads to new beta-gamma identities, which in turn imply new explicit distributions for sums such as (2). There are connections with the Bessel function K_ν as well.

The main tool here is a formula for the Mellin transform $\mathsf{E}(X+Y)^s$, in terms of $\mathsf{E}X^s$ and $\mathsf{E}Y^s$.