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THREE TOPICS on FINANCIAL STATISTICS, STOCHASTICS and OPTIMIZATION

TOPIC I: Financial models and innovations in stochastic economics

(Brief survey)

The classical and neoclassical models of the dynamics of the prices driven by Brownian motion and Lévy processes. Stylized facts. Constructions based on the change of time, stochastic volatility.

The construction of the right probability-statistical models of the dynamics of prices of the basic financial instruments (bank account, bonds, stocks, etc.) is undoubtedly one of important steps for successful application of the results of mathematical finance and financial engineering.

Without adequate models for prices there is no successful risk management, portfolio optimization, allocation of funds, derivative pricing, *etc.*

The main accent in this lecture is made on the construction of the

HYPERBOLIC LÉVY PROCESSES,

which are widely used in econometric models of the dynamics of the financial indexes.

THE FIRST CLASSICAL MODELS FOR PRICE DYNAMICS

In the sequel,

$S = (S_t)_{t \geq 0}$ is the price of (for simplicity) *one* asset

L. Bachelier (1900). Théorie de la spéculation:

$$\boxed{S_t = S_0 + \mu t + \sigma B_t},$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, *i.e.*,
a Gaussian process with independent increments and continuous trajectories,
$B_0 = 0$, $\mathsf{E} B_t = 0$, $\mathsf{E}(B_t - B_s)^2 = t - s$.

M. Kendall (1953). The analysis of economic time series. Part 1. Prices (*J. Roy. Statist. Soc.*, **96**, 11–25):

The empirical analysis of prices $S = (S_n)_{n \geq 0}$ for

- wheat (monthly average prices on the Chicago market, 1883–1934),
- cotton (the New York Mercantile Exchange, 1816–1951)

did not reveal (contrary to common expectations) neither rhythms, nor cycles. The observed data look as if

"...the Demon of Chance drew a random number... and added it to the current price to determine the next... price":

$\boxed{S_n = S_0 e^{H_n}}$, where $H_n = h_1 + \cdots + h_n$ is the sum of independent random variables

("**random walk hypothesis**")

M. F. M. Osborne (1959). Brownian motion in the stock market. *Operation Research*, **7**, 145–153: $\boxed{S_t = S_0 + \mu t + \sigma B_t}$.

P. A. Samuelson (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Rev.*, **6**, 41–49:

$$\boxed{S_t = S_0 e^{H_t}, \quad H_t = \Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma B_t,}$$

$S = (S_t)_{t\geq 0}$ is an economic (geometric) Brownian motion;

$$dS_t = S_t(\mu\, dt + \sigma\, dB_t)$$

⟨This model underlies the Black–Scholes theory of option pricing.⟩

MARTINGALE APPROACH TO STUDYING THE MODELS

$$S = (S_n)_{n\geq 0}, \qquad S_n = S_0 e^{H_n}, \qquad H_n = h_1 + \cdots + h_n$$

($h_n = \log(S_n/S_{n-1})$ is a "return", "logarithmic return")

Doob's decomposition. Assume that "stochastics" of the market is described by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P})$ and $\mathsf{E}|H_n| < \infty$, $n \geq 0$.

$$\boxed{H_n = \sum_{k=1}^{n} \mathsf{E}(h_k \,|\, \mathcal{F}_{k-1}) + \sum_{k=1}^{n} \Big(h_k - \mathsf{E}(h_k \,|\, \mathcal{F}_{k-1})\Big)}\,, \quad \text{or}$$

$$h_n = \underbrace{\mathsf{E}(h_n \,|\, \mathcal{F}_{n-1})}_{\mu_n} + \underbrace{(h_n - \mathsf{E}(h_n \,|\, \mathcal{F}_{n-1}))}_{\delta_n}$$

μ_n are $\mathcal{F}_{n-1}$-measurable

δ_n are $\mathcal{F}_n$-measurable,
$\mathsf{E}(\delta_n \,|\, \mathcal{F}_{n-1}) = 0$,
(δ_n) is a martingale difference

1970s : *linear* models like AR, MA, ARMA with

⟨large time intervals – year, quarter, month⟩

$$\boxed{h_n = \mu_n + \sigma_n \varepsilon_n} \qquad (\textit{i.e.}, \ \delta_n = \sigma_n \varepsilon_n),$$

μ_n and σ_n are $\mathcal{F}_{n-1}$-measurable,

$\varepsilon_n \sim \mathcal{N}(0,1)$ are independent, $n \geq 1$.

AR(p) **model:**

$$\mu_n = a_0 + a_1 h_{n-1} + \cdots + a_p h_{n-p}, \qquad \sigma_n = \text{const}$$

MA(q) **model:**

$$\mu_n = b_0 + b_1 \varepsilon_{n-1} + \cdots + b_q \varepsilon_{n-q}, \qquad \sigma_n = \text{const}$$

ARMA(p,q) **model:**

$$\begin{aligned} \mu_n = & \left[a_0 + a_1 h_{n-1} + \cdots + a_p h_{n-p}\right] \\ & + \left[b_0 + b_1 \varepsilon_{n-1} + \cdots + b_q \varepsilon_{n-q}\right], \qquad \sigma_n = \text{const} \end{aligned}$$

1980s : *nonlinear* models ARCH, GARCH, CRR

⟨analysis of day data⟩

$$\boxed{S_n = S_0 \exp\{h_1 + \cdots + h_n\}}$$

ARCH(p) **model** – AutoRegressive Conditional Heteroskedastic model; P. Engle (1982):

$$h_n = \sigma_n \varepsilon_n, \qquad \sigma_n = \sqrt{\alpha_0 + \sum_{i=1}^{p} \alpha_i h_{n-i}^2} \quad \text{is random } \textbf{(!)}.$$

GARCH(p, q) **model** – Generalized ARCH model; T. Bollerslev (1986):

$$h_n = \sigma_n \varepsilon_n, \qquad \sigma_n = \sqrt{\left[\alpha_0 + \sum_{i=1}^{p} \alpha_i h_{n-i}^2\right] + \left[\sum_{j=1}^{q} \beta_j \sigma_{n-j}^2\right]}.$$

Binary CRR-**model** – Cox, Ross, Rubinstein (1979):

$$h_n = \log(1 + \rho_n), \qquad \rho_n \text{ takes two values, } \rho_n > -1.$$

1990s :

⟨intraday data analysis⟩

(a) Stochastic processes with discrete intervention of chance (piecewise constant trajectories with jumps at "close" times $\tau_1, \tau_2, \ldots$):

$$\boxed{H_t = \sum h_k I(\tau_k \leq t)}$$

(b) Data come almost continuously.

WEAKNESS of the MODEL $S_t = S_0 e^{H_t}$, $H_t = (\mu - \sigma^2/2)t + \sigma B_t$, based on a Brownian motion, *i.e.*, $dS_t = S_t(\mu\, dt + \sigma\, dB_t)$ with a constant volatility σ.

Really observable **smile effect** says that the volatility σ is NOT a constant.

Consider a call (buyer) option with pay-off function $(S_T - K)^+$:

$$C(t,x) = \mathsf{E}_{\tilde{\mathsf{P}}}\Big[(S_T - K)^+ \,|\, S_t = x\Big].$$

By the Black–Scholes formula we find $C(t,x) = C_{\mathsf{BS}}(t,x;T,K,\sigma)$. On option market there exist real prices

$$\widetilde{C}(t,x;T,K).$$

From $C_{\mathsf{BS}}(t,x;T,K,\sigma) \approx \widetilde{C}(t,x;T,K)$ we calculate the implied volatility $\tilde{\sigma} = \tilde{\sigma}(t,x;T,K)$. Fix t, x, T. It turns out that $\tilde{\sigma}(K)$ has a U-form (with $K_{\mathsf{min}} \approx x$) – smile effect.

1st CORRECTION

(R. Merton, 1973)

$$\sigma \quad \longrightarrow \quad \sigma(t)$$

$$\sigma(t) \quad \longrightarrow \quad \sigma(t, S_t)$$

2nd CORRECTION

(B. Dupire, 1994)

Pricing with a smile,

RISK, **7**, 18–20

One- and two-dimensional distributions of $H = (H_t)_{t \geq 0}$. The observable properties of $h_t^{(\Delta)} = \log(S_t/S_{t-\Delta})$

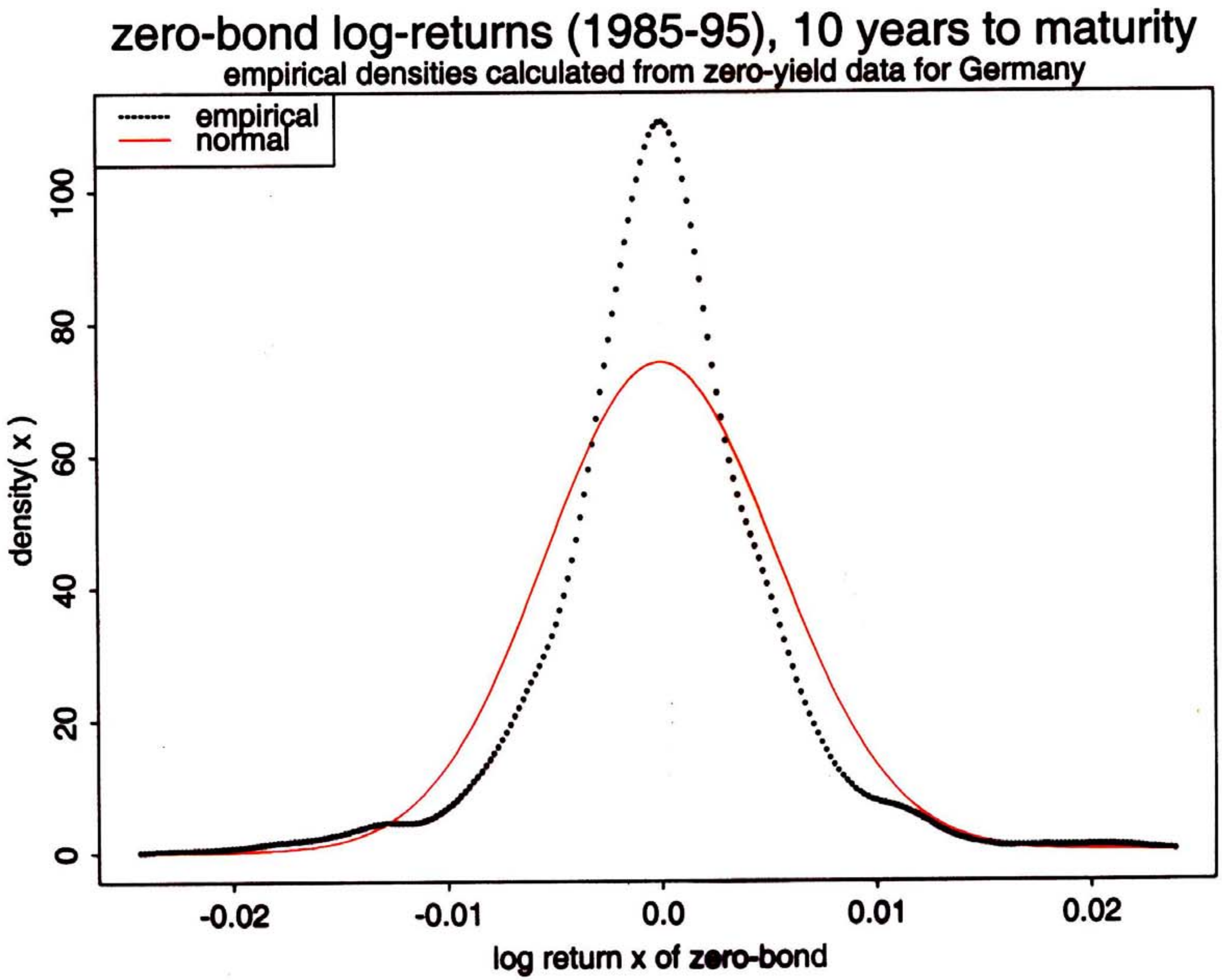
A. The behavior of empirical densities $\widehat{p}^{(\Delta)}(x)$, constructed upon $h_{\Delta}^{(\Delta)}, h_{2\Delta}^{(\Delta)}, \ldots$, is different from that of normal distribution. In a neighborhood of the central value, the densities $\widehat{p}^{(\Delta)}(x)$ are peak-like, and "heavy tails" are observed as $x \to \pm\infty$.

B. The empirical estimator of autocorrelation ($t = k\Delta$)

$$\rho(n\Delta) = \Big[\mathsf{E} h_t^{(\Delta)} h_{t+n\Delta}^{(\Delta)} - \mathsf{E} h_t^{(\Delta)} \mathsf{E} h_{t+n\Delta}^{(\Delta)}\Big] \Big/ \Big[\sqrt{\mathsf{D} h_t^{(\Delta)} \mathsf{D} h_{t+n\Delta}^{(\Delta)}}\,\Big]$$

shows that for small $n\Delta$ the value $\rho(n\Delta)$ is negative, while most of the values of $\rho(n\Delta)$ are close to zero (**noncorrelatedness**).

C. Analogous estimators for autocorrelation of absolute values $|h_t^{(\Delta)}|$ and $|h_{t+n\Delta}^{(\Delta)}|$ show that for small $n\Delta$ the autocorrelation is **positive** (**clustering effect**).



zero-bond log-returns (1985-95), 10 years to maturity
empirical densities calculated from zero-yield data for Germany
empirical
normal
density(x)
0
20
40
60
80
100
-0.02
-0.01
0.0
0.01
0.02
log return x of zero-bond

Searching for adequate statistical models which describe dynamics of the prices $S = (S_t)_{t \geq 0}$ led to

LÉVY PROCESSES.

Now these processes take the central place in modelling the prices of financial indexes, the latter displaying the jump character of changes.

MAIN MODELS

based on a Brownian motion

Exponential

Brownian model

$$S_t = S_0 \exp\{\mu t + \sigma B_t\}$$

$\downarrow$ $\downarrow$

Exponential

INTEGRAL

Brownian model:

$$S_t = S_0 \exp\Big\{\int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s\Big\}$$

Exponential

TIME-CHANGED

Brownian model:

$$S_t = S_0 \exp\big\{\mu T(t) + B_{T(t)}\big\}$$

Assuming that $\mu = 0$, one can rewrite these models in a **brief form**:

$$\boxed{S = S_0 e^{\sigma B}}$$

$$\downarrow \qquad \downarrow$$

$$\boxed{S = S_0 e^{\sigma \cdot B}} \qquad \boxed{S = S_0 e^{B \circ T}}$$

where
- $\sigma \cdot B$ is the stochastic integral $(\int_0^{\cdot} \sigma_s \, dB_s)$,
- $B \circ T$ is a time change in Brownian motion $(B_{T(t)})$.

A generalization of these “Brownian” models, which have being predominating in financial modelling for a long time, is based on the idea **to replace**

BROWNIAN MOTION by LÉVY PROCESSES :

$$B = (B_t)_{t \geq 0} \qquad\qquad L = (L_t)_{t \geq 0}$$

$$\begin{array}{c} \boxed{S = S_0 e^{\sigma L}} \\ \downarrow \qquad \downarrow \\ \boxed{S = S_0 e^{\sigma \cdot L}} \qquad \boxed{S = S_0 e^{L \circ T}} \end{array}$$

LÉVY PROCESS $L = (L_t)_{t \geq 0}$ is a process with stationary increments, $L_0 = 0$, which is continuous in probability.

Such processes have modifications whose trajectories

- are right-continuous (for $t \geq 0$) and
- have limits from the left (for $t > 0$).

Kolmogorov-Lévy-Khinchin's formula for characteristic functions:

$$\boxed{\mathsf{E} e^{i\lambda L_t} = \exp\left\{ t \left[i\lambda b - \frac{\lambda^2}{2} c + \int \left(e^{i\lambda x} - 1 - i\lambda h(x) \right) F(dx) \right] \right\}},$$

where: $h(x) = xI(|x| \leq 1)$ (classical "truncation" function),

$F(dx)$ is a σ-finite measure on $\mathbb{R} \setminus \{0\}$ such that $\int \min(1, x^2)\, F(dx) < \infty$,

$b \in \mathbb{R}$ and $c \geq 0$;

$(b, c, F) =: \mathbb{T}$ is a triplet of local characteristics of L.

The **Lévy–Itô representation** for trajectories of $L = (L_t)_{t \geq 0}$:

$$\boxed{L_t = B_t + L_t^c + \int_0^t \int h(x)\, d(\mu - \nu) + \int_0^t \int (x - h(x))\, d\mu},$$

- $B_t = bt$;
- L_t^c is a continuous component of L ($L_t^c = \sqrt{c}\, W_t$, where W_t is a Wiener process);
- μ is the measure of jumps: for $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$

$$\mu(\omega; (0,t] \times A) = \sum\nolimits_{0<s\leq t} I_A(\Delta L_s) \quad (\Delta L_s = L_s - L_{s-});$$

- ν is the compensator of the measure of jumps μ:

$$\nu((0,t] \times A) = tF(A), \qquad F(A) = \textstyle\int_A F(dx).$$

The measure μ is a <u>Poissonian</u> measure with

$$\mathsf{E} \exp\Big\{ i \sum\nolimits_{k \leq n} \lambda_k \mu(G_k) \Big\} = \exp\Big\{ \sum\nolimits_{k\leq n} (e^{i\lambda_k} - 1) \nu(G_k) \Big\}, \quad n \geq 1,$$

where G_k are sets from $\mathbb{R}_+ \times \mathbb{R}$ and $\nu(dt, dx) = dt\, F(dx)$.

EXAMPLES of LÉVY PROCESSES :

- Brownian motion,
- Poisson process,
- compound Poisson process $L_t = \sum_{k=1}^{N_t} \xi_k$, where

  $(N_t)_{t \geq 0}$ is a Poisson process,

  $(\xi_k)_{k \geq 1}$ is a sequence of independent and identically distributed random variables

In connection with financial econometrics, these are

HYPERBOLIC Lévy processes,

that are of a great interest, because they model well the really observable processes $H = (H_t)_{t \geq 0}$ for many underlying financial instruments (rate of exchange, stocks, *etc.*).

The credit of developing the theory of such processes and their applications is due to **E. Halphen**, **O. Barndorff-Nielsen**, **E. Eberlein**.

We will construct these processes, following mostly Chapters 9 and 12 of the monograph: O. Barndorff-Nielsen, A. N. Shiryaev, *Change of Time and Change of Measures*, World Scientific (in print).

For a Lévy process $(H_t)_{t \geq 0}$ we have

$$\mathsf{E} e^{i\lambda H_t} = (\mathsf{E} e^{i\lambda H_1})^t.$$

The properties of Lévy's processes imply that the random variable $h = H_1$ is infinitely divisible, *i.e.*, for any n one can find i.i.d. r.v.'s $\xi_1, \ldots, \xi_n$ such that

$$\mathrm{Law}(h) = \mathrm{Law}(\xi_1 + \cdots + \xi_n).$$

We will look for the infinitely divisible r.v.'s h having the form

$$h = \mu + \beta\sigma^2 + \sigma\varepsilon,$$

where ε is a standard Gaussian random variable, $\varepsilon \sim \mathcal{N}(0, 1)$,

$\sigma = \sigma(\omega)$ is the "volatility" (which does not depend on ε), for whose square, σ^2, we will construct the special distribution

$\mathbb{G}$IG – $\mathbb{G}$*eneralized Inverse Gaussian distribution*.

Strikingly, this distribution (on $\mathbb{R}_+$) is *infinitely divisible* and the distribution of $h = \mu + \beta\sigma^2 + \sigma\varepsilon$ (on $\mathbb{R}$) is *infinitely divisible* as well. Hence there exist Lévy processes $T = (T(t))_{t \geq 0}$ and $H^* = (H^*_t)_{t \geq 0}$ such that

$$\text{Law}(T(1)) = \text{Law}(\sigma^2) \quad \text{and} \quad \text{Law}(H^*_1) = \text{Law}(h).$$

As a realization of $H^* = (H^*_t)_{t \geq 0}$ one can take

$$H_t = \mu t + \beta T(t) + B_{T(t)},$$

where the "time change" $T = (T(t))_{t \geq 0}$ and the Brownian motion $B = (B_t)_{t \geq 0}$ are independent.

In the sequel, we do not distinguish between the processes H and H^*.

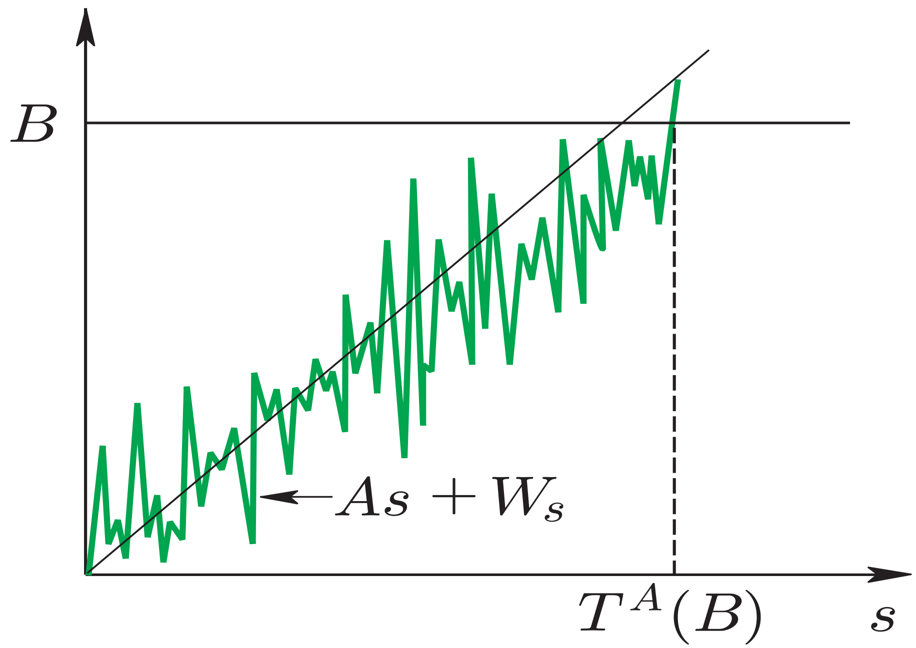
This process H, remarkable in many respect, bears the name

L($\mathbb{G}$H) – **Generalized Hyperbolic Lévy process**.

Let discuss the details of construction of $\mathbb{G}$IG-distributions for σ^2.

Let $W = (W_t)_{t \geq 0}$ be a Wiener process (standard Brownian motion). For $A \geq 0$, $B > 0$ introduce

$$T^A(B) = \inf\{s \geq 0 \colon As + W_s \geq B\}.$$



The formula for the density $p_{T^A(B)}(s) = d\mathsf{P}(T^A(B) \le s)/ds$ is well known:

$$p_{T^A(B)}(s) = \frac{B}{s}\varphi_s(B - As), \qquad \varphi_s(x) = \frac{1}{\sqrt{2\pi s}}\, e^{-x^2/(2s)}. \tag{1}$$

Herefrom we find the Laplace transform:

$$\mathsf{E}e^{-\lambda T^A(B)} = \exp\Big\{AB(1 - \sqrt{1 + 2\lambda/A^2})\Big\}.$$

Letting $b = B^2 > 0$ and $a = A^2 \ge 0$, we find from (1) the following formula for $p(s; a, b) = p_{T^{\sqrt{a}}(\sqrt{b})}(s)$:

$$\boxed{p(s; a, b) = c_1(a, b) s^{-3/2} e^{-(as+b/s)/2}}, \qquad c_1(a, b) = \sqrt{\frac{b}{2\pi}}\, e^{\sqrt{ab}}.$$

The distribution with density $p(s; a, b)$ is named

IG = IG(a, b) – **Inverse Gaussian distribution**.

Next important step: one define *ad hoc* the function

$$p(s;a,b,\nu) = c_2(a,b,\nu)\, s^{\nu-1} e^{-(as+b/s)/2}, \tag{2}$$

where parameters $a, b, \nu \in \mathbb{R}$ are chosen in such a way that $p(s;a,b,\nu)$ be probability density on $\mathbb{R}_+$:

$$\begin{array}{lll} a \geq 0, & b > 0, & \nu < 0 \\ a > 0, & b > 0, & \nu = 0 \\ a > 0, & b \geq 0, & \nu > 0 \end{array} \qquad \left(\Rightarrow \int_0^\infty s^{\nu-1} e^{-(as+b/s)/2}\, ds < \infty \right).$$

It is well known that $K_\nu(y) \equiv \frac{1}{2}\int_0^\infty s^{\nu-1} e^{-y(s+1/s)/2}\, ds$ is the modified third-kind Bessel function of order ν, which for $y > 0$ solves

$$y^2 f''(y) + y f'(y) - (y^2 + \nu^2) f(y) = 0.$$

The constant in (2) has the form $c_2(a,b,\nu) = \dfrac{(a/b)^{\nu/2}}{2K_\nu(\sqrt{ab})}$.

The distribution on $\mathbb{R}_+$ with density

$$\boxed{p(s; a, b, \nu) = \frac{(a/b)^{\nu/2}}{2K_\nu(\sqrt{ab})} s^{\nu-1} e^{-(as+b/s)/2}}$$

bears the name

$\mathbb{G}\mathrm{IG} = \mathbb{G}\mathrm{IG}(a, b)$ – **Generalized Inverse Gaussian distribution**.

IMPORTANT PROPERTIES of $\mathbb{G}$IG-DISTRIBUTION (for σ^2):

A. This distribution is infinitely divisible.

B. The density $p(s; a, b, \nu)$ is unimodal with the mode

$$m = \begin{cases} b \,/ \Big[2(1-\nu)\Big], & \text{if} \quad a = 0, \\ \Big[(\nu - 1) + \sqrt{ab + (\nu-1)^2}\Big]/a, & \text{if} \quad a > 0. \end{cases}$$

C. Laplace's transform $L(\lambda) = \int_0^\infty e^{-\lambda s} p(s; a, b, \nu)\, ds$ is given by

$$\boxed{L(\lambda) = \left(1 + \frac{2\lambda}{a}\right)^{-\nu/2} \frac{K_\nu(\sqrt{ab(1 + 2\lambda/a)})}{K_\nu(\sqrt{ab})}}.$$

As a by-product, one deduces the representation for the density $f(y)$ of Lévy measure $F(dy)$. (Note: $L(\lambda) = \exp\{\int_0^\infty (e^{-\lambda y} - 1) f(y)\, dy\}$.)

Particularly important **SPECIAL CASES** of $\mathbb{G}$IG-**distributions**:

I. $\underline{\underline{a \geqslant 0,\ b > 0,\ \nu = -1/2}}$: in this case $\boxed{\mathbb{G}\mathrm{IG}(a,b,-1/2) = \mathrm{IG}(a,b)}$

– **Inverse Gaussian distribution**.

Density: $p(s;a,b) = c_1(a,b)s^{-3/2}e^{-(as+b/s)/2}$, $c_1(a,b) = \sqrt{\frac{b}{2\pi}}\,e^{\sqrt{ab}}$.

Density of Lévy's measure: $f(y) = \sqrt{\frac{b}{2\pi}}\,\frac{e^{-ay/2}}{y^{3/2}}$.

II. $\underline{\underline{a > 0,\ b = 0,\ \nu > 0}}$: in this case $\boxed{\mathbb{G}\mathrm{IG}(a,0,\nu) = \mathrm{Gamma}(a/2,\nu)}$

– **Gamma distribution**. Density: $p(s;a,0,\nu) = \frac{(a/2)^\nu}{\Gamma(\nu)}\,s^{\nu-1}e^{-as/2}$.

Density of Lévy's measure: $f(y) = y^{-1}\nu e^{-ay/2}$.

III. $\underline{\underline{a > 0,\ b > 0,\ \nu = 1}}$: $\quad p(s;a,b,1) = \dfrac{\sqrt{a/b}}{2K_1(\sqrt{ab})}\,e^{-(as+b/s)/2}$

– PH – **Positive Hyperbolic distribution**, or H^+-**distribution**.

Since $\mathbb{G}$IG-distribution is infinitely divisible, if one take it as the distribution of squared volatility σ^2,

$$\mathrm{Law}(\sigma^2) = \mathbb{G}\mathrm{IG},$$

then one can construct a nonnegative nonincreasing Lévy process $T = (T(t))_{t\geq 0}$ (a subordinator) such that

$$\mathrm{Law}(T(1)) = \mathrm{Law}(\sigma^2) = \mathbb{G}\mathrm{IG}.$$

In the subsequent constructions, this process plays the role of

change of time, operational time,

business time.

As was explained above, the next step in construction of the return process $H = (H_t)_{t\geq 0}$, consists in the following.

Form the variable $h = \mu + \beta\sigma^2 + \sigma\varepsilon$, where $\mathrm{Law}(\sigma^2) = \mathbb{G}\mathrm{IG}$, $\mathrm{Law}(\varepsilon) = \mathcal{N}(0,1)$, σ^2 and ε are independent. It is clear that

$$\mathrm{Law}(h) = \mathsf{E}_{\sigma^2}\mathcal{N}(\mu + \beta\sigma^2, \sigma^2)$$

is a mixture of normal distributions, *i.e.*, the density $p_h(x)$ of h is of the form

$$p_h(x) = \int_0^\infty \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{(x-(\mu+\beta y))^2}{2y}\right\} p_{\mathbb{G}\mathrm{IG}}(y)\, dy.$$

Integrating and denoting $p_h(x)$ by $p^*(x; a, b, \mu, \beta, \nu)$, we find that

$$\boxed{p^*(x; a, b, \mu, \beta, \nu) = c_3(a, b, \beta, \nu)\, \frac{K_{\nu-1/2}\left(\alpha\sqrt{b + (x-\mu)^2}\right)}{\left(\sqrt{b + (x-\mu)^2}\right)^{1/2-\nu}}\, e^{\beta(x-\mu)}},$$

where $\alpha = \sqrt{a + \beta^2}$ and $c_3(a, b, \beta, \nu) = \dfrac{(a/b)^{\nu/2}\, \alpha^{\frac{1}{2}-\nu}}{\sqrt{2\pi}\, K_\nu(\sqrt{ab})}$.

The obtained distribution Law(h) with density $p^*(x; a, b, \mu, \beta, \nu)$ bears the name

Generalized Hyperbolic distribution, $\mathbb{G}$H $= \mathbb{G}$H$(a, b, \mu, \beta, \nu))$.

In the case $\nu = 1$ the graph of the function

$$\log p^*(x; a, b, \mu, \beta, 1) = \log c^*(a, b, \beta) - \alpha\sqrt{b + (x-\mu)^2} + \beta(x-\mu)$$

is a **hyperbola** with asymptotes $\alpha|x-\mu| + \beta(x-\mu)$.

This is why the distribution for h in the case $\nu = 1$ is called hyperbolic, which explains the name "generalized hyperbolic distribution" in the case of arbitrary ν.

SOME PROPERTIES of $\mathbb{G}$H-**distribution** (for h):

A*. This distribution is infinitely divisible.

B*. If $\beta = 0$, then the distribution is unimodal with mode $m = \mu$.
(In the general case m is determined as a root of a certain transcendental equation.)

C*. Laplace's transform $L^*(\lambda) = \int_0^\infty e^{\lambda x} p(x; a, b, \mu, \beta, \nu)\, dx$
(for complex λ such that $|\beta + \lambda| < \alpha$, $\alpha = \sqrt{a + \beta^2}$)
is given by

$$\boxed{L^*(\lambda) = e^{\lambda\mu} \left[\frac{a}{\alpha^2 - (\beta+\lambda)^2}\right]^{\nu/2} \frac{K_\nu\left(\sqrt{b[\alpha^2 - (\beta+\lambda)^2]}\right)}{K_\nu(\sqrt{ab})}}\,.$$

THREE important **SPECIAL CASES** of $\mathbb{G}$H-**distributions**:

I*. $a \geqslant 0$, $b > 0$, $\nu = -1/2$: in this case

$\mathbb{G}$IG$(a, b, -1/2) =$ IG(a, b) is **Inverse Gaussian** distribution.

The corresponding $\mathbb{G}$H-distribution is commonly named

$\mathbb{N}$ormal Inverse Gaussian

and denoted by $\mathbb{N} \circ$IG. The density has the form

$$\boxed{p^*(x; a, b, \mu, \beta, -\tfrac{1}{2}) = \frac{\alpha b}{\pi}\, e^{\sqrt{ab}}\, \frac{K_1(\alpha\sqrt{b + (x-\mu)^2})}{\sqrt{b + (x-\mu)^2}}\, e^{\beta(x-\mu)}}, \quad x \in \mathbb{R}.$$

Laplace's transform:

$$L^*(\lambda) = \exp\Big\{\lambda\mu + \sqrt{b}\big(\sqrt{a} - \sqrt{a - 2\beta\lambda - \lambda^2}\big)\Big\}.$$

II*. $\underline{\underline{a > 0,\ b = 0,\ \nu > 0}}$: in this case

$$\mathbb{G}\mathrm{IG}(a, 0, \nu) = \mathrm{Gamma}(a/2, \nu)$$ – **Gamma distribution**.

The corresponding $\mathbb{G}$H-distribution is named

$\mathbb{N}$ormal Gamma distribution

(notation: $\mathbb{N} \circ \mathrm{Gamma}$) or

$\mathbb{V}$G-distribution

(notation: $\mathbb{V}$G [Variance Gamma]). Density:

$$\boxed{\begin{aligned} p^*(x; a, 0, \mu, \beta, \nu) = \frac{a^\nu}{\sqrt{\pi}\Gamma(\nu)(2\alpha)^{\nu-1/2}} & |x-\mu|^{\nu-1/2} \\ & \times K_{\nu-1/2}(\alpha|x-\mu|)\, e^{\beta(x-\mu)} \end{aligned}}\ .$$

Laplace's transform: $L^*(\lambda) = e^{\mu\lambda}(a/[a - 2\beta\lambda - \lambda^2])^\nu$.

III*. $\underline{\underline{a > 0,\ b > 0,\ \nu = 1}}$: in this case

$$\mathbb{G}\mathrm{IG}(a, b, 1) = \mathrm{H}^+(a, b) \text{ – } \textbf{Positive hyperbolic} \text{ distribution.}$$

The corresponding $\mathbb{G}$H-distribution for h is the hyperbolic distribution H called also

"**NORMAL positive hyperbolic** distribution"

(notation: $\mathbb{H}$ or $\mathbb{N} \circ \mathrm{H}^+$). Density:

$$\boxed{p^*(x; a, b, \mu, \beta, 1) = \frac{a}{2b\alpha K_1(\sqrt{ab})} \exp\left\{-\alpha\sqrt{b + (x-\mu)^2} + \beta(x-\mu)\right\},}$$

where $\alpha = \sqrt{a + \beta^2}$.

Having $\mathbb{G}$IG-distributions for σ^2 and $\mathbb{G}$H-distributions for h, we turn to construction of the Lévy process $H = (H_t)_{t \geq 0}$ used in representation of the prices $S_t = S_0 e^{H_t}$, $t \geq 0$.

TWO POSSIBILITIES

↓	↓
the fact that h has infinitely divisible distribution allows one to construct, using the general theory, the Lévy process $H^* = (H_t^*)_{t \geq 0}$ such that $\mathrm{Law}(H_1^*) = \mathrm{Law}(h)$	using the already constructed process $T = (T_t)_{t \geq 0}$, one forms the process $H = (H_t)_{t \geq 0}$: $H_t = \mu t + \beta T(t) + B_{T(t)}$, where Brownian motion B and process T are taken to be independent.

The process $H = (H_t)_{t \geq 0}$ bears the name

L($\mathbb{G}$H) – "**GENERALIZED hyperbolic Lévy distribution**".

In the cases $\mathbf{I^*}$, $\mathbf{II^*}$, and $\mathbf{III^*}$ mentioned above the corresponding Lévy processes have the special names:

$$\mathsf{L}(\mathbb{N} \circ \mathsf{IG})\text{-process,}$$
$$\mathsf{L}(\mathbb{N} \circ \mathsf{H}^+)\text{- or } \mathsf{L}(\mathsf{H})\text{-process,}$$
$$\mathsf{L}(\mathbb{N} \circ \mathsf{Gamma})\text{- or } \mathsf{L}(\mathbb{V}\mathsf{G})\text{-process.}$$

It is interesting to note that $\mathsf{L}(\mathbb{N} \circ \mathsf{IG})$- and $\mathsf{L}(\mathbb{N} \circ \mathsf{Gamma})$-processes have an important property:

$$\boxed{\text{Law}(H_t) \text{ belongs to the same type of distributions as Law}(H_1)}$$

(this follows immediately from the formulae for Laplace's transforms).

CONCLUDING NOTES.

Densities of distributions of h $(= H_1)$ are determined by **FIVE** parameters (a, b, μ, β, ν), that gives a great freedom in determining parameters which would fit well the empirical data.

In this connection it is appropriate to recall that in statistics, there exists a well-known method of "Pearson's curves", which is widely used to construct (one-dimensional) densities of distributions upon independent observations over a random variable ξ. K. Pearson itself (1894) constructed such densities as solutions $f(x)$ of the system of nonlinear equations

$$f'(x) = \frac{(x-a)f(x)}{b_0 + b_1 x + b_2 x^2}.$$

These densities are determined by **FOUR** parameters (a, b_0, b_1, b_2).

The density $p^*(x; a, b, \mu, \beta, \nu)$ of $\mathbb{G}$H-distributions of (constructively built) variables $h = \mu + \beta\sigma^2 + \sigma\varepsilon$ is determined by **FIVE** parameters. (It is known that these densities lie between Pearson's curves of type III and type V.)

The essential advantage of $\mathbb{G}$H-distributions consists in their

infinite divisibility

(this is not the case for distributions from the Pearson system), which enables us to construct processes $H = (H_t)_{t \geq 0}$ which describe adequately the time dynamics of logarithmic return of the prices $S = (S_t)_{t \geq 0}$.

Albert N. SHIRYAEV

THREE TOPICS on FINANCIAL STATISTICS, STOCHASTICS and OPTIMIZATION

TOPIC II: On the duality principle in option pricing: semimartingale setting

(Brief survey)

Some new ideas on the option pricing based on change of measures (duality approaches)

The purpose of this lecture is to describe the appropriate mathematical tools for the study of the

CALL-PUT DUALITY

in option pricing for models where prices are described by general exponential semimartingales. Particular cases of these models are the ones which are driven

by **Brownian motions** and by **Lévy processes**,

which have been considered in many papers.

Generally speaking the **DUALITY PRINCIPLE** states that

> the calculation of the price of a CALL option for a model with price process $\boxed{S = e^{H}}$ (w.r.t. the measure P)

is equivalent to

> the calculation of the price of a PUT option for a suitable DUAL model $\boxed{S' = e^{H'}}$ (w.r.t. some measure P′).

From our lecture it will be clear that appealing to the general

exponential semimartingale models

leads to a deeper insight into the essence of the duality principle.

There is a long list of articles dealing with the **put–call duality**, in various different models and for several different pay-off functions:

Carr (1994)
Chesney & Gibson (1995)
Bates (1997)
Schröder (1999)
Peskir & Shiryaev (2002)
Eberlein & Papapantoleon (2005)
Fajardo & Mordecki (2006)

to cite just a few.

§ 1. INTRODUCTION

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, \mathsf{P})$ is a filtered probability space

$S = e^H$, i.e., $S_t = e^{H_t}$, where $H = (H_t)_{t\le T} \in \mathrm{Sem}(\mathsf{P})$

$$\boxed{\mathbb{T}(H \,|\, \mathsf{P}) = (B, C, \nu)}$$

For **Lévy** processes: $B_t = bt$, $C_t = ct$, $\nu(dt, dx) = dt\, F(dx)$.

For **diffusion** processes:
$$dH_t = b(t, H_t)\, dt + \sigma(t, H_t)\, dW_t,$$
$$B_t = \int_0^t b(s, H_s)\, ds,$$
$$C_t = \int_0^t \sigma^2(s, H_s)\, ds.$$

(B, S)-market with $B \equiv 1$.

The **payoff function**:

for a standard **call** option $f_T = (S_T - K)^+$,

for a standard **put** option $f_T = (K - S_T)^+$,

K is the strike price.

CALL–PUT PARITY

$$(S_T - K)^+ = (K - S_T)^+ + S_T - K$$

$$\Downarrow$$

$$\boxed{\mathbb{C}_T(S,K) = \mathbb{P}_T(K,S) + 1 - K} \qquad (1)$$

where $\quad \mathbb{C}_T(S,K) = \mathsf{E}(S_T - K)^+, \qquad \mathbb{P}_T(K,S) = \mathsf{E}(K - S_T)^+,$

E$(\cdot)$ is the expectation w.r.t. a **martingale** measure P

(S is P-martingale)

Generally speaking the **DUALITY PRINCIPLE** states that

$$\boxed{\mathbb{C}_T(S,K) = K\mathbb{P}'_T(K',S'), \quad \mathbb{P}_T(K,S) = K\mathbb{C}'_T(S',K')} \qquad (2)$$

From

$$\mathbb{C}_T(S,K) = \mathbb{P}_T(K,S) + 1 - K \quad \text{and} \tag{1}$$

$$\mathbb{C}_T(S,K) = K\mathbb{P}'_T(K',S'), \quad \mathbb{P}_T(K,S) = K\mathbb{C}'_T(S',K') \tag{2}$$

we get the

CALL–CALL and **PUT–PUT** **parity**:

$$\boxed{\begin{aligned} \mathbb{C}_T(S,K) &= K\mathbb{C}'_T(S',K') + 1 - K, \\ \mathbb{P}_T(K,S) &= K\mathbb{P}'_T(K',S') + K - 1 \end{aligned}}$$

$$\boxed{S = e^H}$$

$$H = (H_t)_{t \le T} \in \mathrm{Sem}(\mathsf{P})$$

$\uparrow$

compound

interest

representation

$$H_t = A_t + M_t, \qquad \Delta H = H - H_-$$

$$\begin{array}{cc} \rightsquigarrow & \rightsquigarrow \\ \mathcal{V} & M_{\mathrm{loc}} \end{array}$$

$$\boxed{S = e^H = \mathcal{E}(\widetilde{H})}$$

$$\boxed{d\mathcal{E}(X) = \mathcal{E}(X)_- \, dX} \tag{3}$$

$S = \mathcal{E}(\widetilde{H})$

simple

interest

representation

$$\mathcal{E}(X)_t = e^{X_t - \frac{1}{2}\langle X^c \rangle_t} \times \prod_{0 < s \le t} (1 + \Delta X_s) e^{-\Delta H_s}$$

$$S = e^{H} = \mathcal{E}(\widetilde{H}) \qquad d\mathcal{E}(X) = \mathcal{E}(X)_{-}\, dX \tag{3}$$

From (3) $\boxed{H = \log \mathcal{E}(\widetilde{H}), \quad \widetilde{H} = \mathcal{L}\mathrm{og}(e^{H})}$

Stochastic logarithm of X (> 0):

$$(\mathcal{L}\mathrm{og}\, X)_t \stackrel{\text{def}}{=} \int_0^t \frac{dX_s}{X_{s-}}$$

So: $\boxed{\begin{aligned} \mathcal{L}\mathrm{og}\, \mathcal{E}(X) &= X, \\ \log e^{X} &= X \end{aligned}}$

$$\mathcal{L}\mathrm{og}\, X = \log X + \frac{1}{2X_{-}^{2}} \langle X^{c} \rangle - \sum_{0 \le s \le \cdot} \left[\log\left(\left| 1 + \frac{\Delta X_s}{X_{s-}} \right| \right) - \frac{\Delta X_s}{X_{s-}} \right]$$

$$(X > 0, \quad X_0 = 1)$$

USEFUL FORMULAE:

$$\widetilde{H} = H + \frac{1}{2}\langle H^c \rangle + \sum_{0<s\le\cdot} \Big(e^{\Delta H_s} - 1 - \Delta H_s \Big),$$

$$H = \widetilde{H} - \frac{1}{2}\langle \widetilde{H}^c \rangle + \sum_{0<s\le\cdot} \Big(\log(1 + \Delta \widetilde{H}_s) - \Delta \widetilde{H}_s \Big)$$

$$(e^H = \mathcal{E}(\widetilde{H})).$$

If $\mu^H = \mu^H(\omega; ds, dx)$ and $\mu^{\widetilde{H}} = \mu^{\widetilde{H}}(\omega; ds, dx)$ are the random **measures of jumps** of H and $\widetilde{H}$, then

$$\widetilde{H} = H + \frac{1}{2}\langle H^c \rangle + (e^x - 1 - x) * \mu^H,$$

$$H = \widetilde{H} - \frac{1}{2}\langle \widetilde{H}^c \rangle + \Big(\log(1 + x) - x \Big) * \mu^{\widetilde{H}},$$

where $W * \mu = \int_0^{\cdot} \int W(\omega; s, x)\, \mu(\omega; ds, dx)$.

In the **DISCRETE-TIME** setting

$$S_n = e^{H_n}, \quad H_n = h_1 + \cdots + h_n, \quad H_0 = 0 \qquad (\textbf{compound} \text{ interest})$$

$$S_n = \mathcal{E}(\widetilde{H})_n = \prod_{k \le n} (1 + \tilde{h}_k), \quad \tilde{h}_k = e^{h_k} - 1 \qquad (\textbf{simple} \text{ interest})$$

CANONICAL REPRESENTATION:

$$\boxed{H = H_0 + B + H + h(x) * (\mu - \nu) + (x - h(x)) * \mu}$$

or, in detail,

$$H_t = H_0 + B_t + H_t^c + \int_0^t \int_{\mathbb{R}\setminus\{0\}} h(x)\, d(\mu - \nu) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (x - h(x))\, d\mu,$$

where h, B, H^c, and ν are defined as follows.

- $\boxed{h = h(x)}$ is a truncation function: $h(x) = x$ in a neighborhood of zero and usually with compact support; the canonical choice is $h(x) = xI(|x| \le a)$;

- $\boxed{B = (B_t)}$ is a predictable process of bounded variation;

- $\boxed{H^c = (H_t^c)}$ is the continuous martingale part of H;

- $\boxed{\nu = \nu(\omega; ds, dx)}$ is the predictable compensator of the random measure of jumps $\mu = \mu(\omega; ds, dx)$ of H.

$$\boxed{\mathbb{T}(H\,|\,\mathsf{P}) = (B, C, \nu)} \qquad C = \langle H^c \rangle$$

the **TRIPLET of predictable characteristics** of H

If $\mathbb{T}(H\,|\,\mathsf{P}) = (B, C, \nu)$; $\mathbb{T}(\widetilde{H}\,|\,\mathsf{P}) = (\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ $[\widetilde{H} = \mathcal{L}\mathrm{og}(e^H)]$, then

$$\boxed{\begin{aligned} &\widetilde{B} = B + C/2 + \Big(h(e^x - 1) - h(e^x)\Big) * \nu, \\ &\widetilde{C} = C, \\ &I_A(x) * \widetilde{\nu} = I_A(e^x - 1) * \nu, \qquad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}), \end{aligned}}$$

$$\boxed{\begin{aligned} &B = \widetilde{B} - C/2 + \Big(h(\log(1+x)) - h(x)\Big) * \widetilde{\nu}, \\ &C = \widetilde{C}, \\ &I_A(x) * \nu = I_A\Big(\log(1+x)\Big) * \widetilde{\nu}. \end{aligned}}$$

If H is a **Lévy process** with the **triplet of local characteristics** (b, c, F), then $\widetilde{H} = \log(e^H)$ is also a Lévy process with triplet $(\tilde{b}, \tilde{c}, \widetilde{F})$:

$$\begin{cases} \tilde{b} = b + \dfrac{c}{2} + \displaystyle\int_{\mathbb{R}\setminus\{0\}} \Big(h(e^x - 1) - h(e^x)\Big)\, F(dx), \\ \tilde{c} = c, \\ \widetilde{F}(A) = \displaystyle\int_{\mathbb{R}\setminus\{0\}} I_A(e^x - 1)\, F(dx), \qquad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}); \end{cases}$$

$$\begin{cases} b = \tilde{b} - \dfrac{c}{2} + \displaystyle\int_{\mathbb{R}\setminus\{0\}} \Big(h(\log(1+x)) - h(x)\Big)\, \widetilde{dx}, \\ c = \tilde{c}, \\ F(A) = \displaystyle\int_{\mathbb{R}\setminus\{0\}} I_A\Big(\log(1+x)\Big)\, \widetilde{dx}, \qquad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}). \end{cases}$$

Assumption: $h(-x) = -h(x)$. We have $B(h) - B(h') = (h - h') * \nu$.

§ 2. Martingale measures and dual martingale measures

$\mathcal{M}_{\mathrm{loc}}(\mathsf{P})$ stands for the class of all local martingales

$$\boxed{H \in \mathcal{M}_{\mathrm{loc}}(\mathsf{P}) \quad \Leftrightarrow \quad B + (x - h(x)) * \nu = 0}$$

These implications follow from the canonical representation:

$$H = H_0 + \underbrace{B} + \underbrace{H^c + h * (\mu - \nu)}_{\mathcal{M}_{\mathrm{loc}}} + \underbrace{(x - h(x)) * \mu}$$

$$\downarrow$$

$$\underbrace{(x-h) * \nu} + \underbrace{(x-h)(\mu - \nu)}_{\mathcal{M}_{\mathrm{loc}}}$$

$$\Downarrow$$

$$H - \Big[H_0 + H^c + h * (\mu - \nu) + (x-h)(\mu-\nu)\Big] = \underbrace{B + (x - h(x)) * \nu}_{=0}$$

Thus

$$\widetilde{H} \in \mathcal{M}_{\mathsf{loc}}(\mathsf{P}) \quad \Longleftrightarrow \quad \tilde{B} + (x - h(x)) * \tilde{\nu} = 0.$$

In the sequel we will assume that the following condition is satisfied:

$$\boxed{\mathbb{ES}: \quad \text{The process } I(x>1)e^x * \nu \text{ has bounded variation}}$$

Under $\mathbb{ES}$-condition

$$\widetilde{H} \in \mathcal{M}_{\mathsf{loc}}(\mathsf{P}) \quad \Longleftrightarrow \quad B + \frac{C}{2} + (e^x - 1 - h(x)) * \nu = 0$$

$$\downarrow$$

$$S = e^H = \mathcal{E}(\widetilde{H}) \in \mathcal{M}_{\mathsf{loc}}(\mathsf{P}) \quad \Longleftrightarrow \quad B + \frac{C}{2} + (e^x - 1 - h(x)) * \nu = 0$$

From now on we assume that S is not only a local martingale but also a **MARTINGALE** ($S \in \mathcal{M}(\mathsf{P})$) on $[0, T]$. Thus $\mathsf{E} S_T = 1$.

Define on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \le T})$ a **NEW** probability measure P' with

$$d\mathsf{P}' = S_T \, d\mathsf{P}.$$

Since $S > 0$ (P-a.s.) we have

$$\mathsf{P} \ll \mathsf{P}' \quad \text{and} \quad d\mathsf{P} = \frac{1}{S_T} \, d\mathsf{P}'.$$

Introduce the process

$$S' = \frac{1}{S}.$$

Then with $H' = -H$

$$S' = e^{H'}.$$

LEMMA. *Suppose* $S = e^H \in \mathcal{M}(\mathsf{P})$, *i.e.,* S *is a* P-*martingale. Then* $S' \in \mathcal{M}(\mathsf{P}')$, *i.e., the process* S' *is a* P'-*martingale.*

[If we denote $Z = \frac{d\mathsf{P}'}{d\mathsf{P}}$ then $S' \in \mathcal{M}(\mathsf{P}')$ if and only if $S'Z \in \mathcal{M}(\mathsf{P})$. In our case $Z = S$ and $S'S = 1$.]

THEOREM. *The triplet* $\mathbb{T}(H' \,|\, \mathsf{P}') = (B', C', \nu')$ *can be expresses via the triplet* $\mathbb{T}(H \,|\, \mathsf{P}) = (B, C, \nu)$ *by the following formulae:*

$$\boxed{\begin{aligned} &B' = -B - C - h(x)(e^x - 1) * \nu \\ &C' = C \\ &I_A(x) * \nu' = I_A(-x)e^x * \nu' \end{aligned}} \qquad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$

SCHEME OF THE PROOF:

$$
\begin{tikzcd}
 & \mathbb{T}(H \,|\, \mathsf{P}') \arrow[dr, "\text{(c)}"] & \\
\mathbb{T}(H \,|\, \mathsf{P}) \arrow[ur, "\text{(a)}", "\text{(G)}"'] \arrow[dr, "\text{(b)}"'] & & \mathbb{T}(H' \,|\, \mathsf{P}') \\
 & \mathbb{T}(H' \,|\, \mathsf{P}) \arrow[ur, "\text{(G)}", "\text{(d)}"'] &
\end{tikzcd}
$$

(a) $\mathbb{T}(H\,|\,\mathsf{P}) \to \mathbb{T}(H\,|\,\mathsf{P}') \stackrel{\text{def}}{=} (B^+, C^+, \nu^+)$, where

$$B^+ = B + \beta^+ \cdot C + h(x)(Y^+ - 1) * \nu, \qquad C^+ = C, \qquad \nu^+ = Y^+ \cdot \nu.$$

For β^+ and Y^+ we have:

- $\langle S^c, H^c \rangle = (S_- \beta^+) \cdot C \;\Rightarrow\; \boxed{\beta^+ \equiv 1}$;
- $Y^+ = M^{\mathsf{P}}_{\mu^H}\left(\dfrac{S}{S_-}\,\Big|\,\widetilde{\mathcal{P}}\right)$, where $M^{\mathsf{P}}_{\mu^H}(d\omega, ds, dx) = \mu^H(\omega, ds, dx)\,\mathsf{P}(d\omega)$, $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ is the σ-field of predictable sets in $\widetilde{\Omega} = \Omega \times [0,1] \times \mathbb{R}$,

  whence we find that $\boxed{Y^+ = e^x}$.

Thus $$\boxed{B^+ = B + C + h(x)(e^x - 1) * \nu, \qquad C^+ = C, \qquad \nu^+ = e^x \cdot \nu}$$

where $(B^+, C^+, \nu^+) = \mathbb{T}(H\,|\,\mathsf{P}')$.

(b) $\mathbb{T}(H\,|\,\mathsf{P}) \to \mathbb{T}(H'\,|\,\mathsf{P}) = \mathbb{T}(-H\,|\,\mathsf{P}) = (B^-, C^-, \nu^-)$, where

$$\boxed{B^- = -B, \quad C^- = C, \quad I_A(x) * \nu^- = I_A(-x) * \nu}$$

Remark. Let

$$J = \int_0^{\cdot} f\,dH,$$

where f is a predictable and, *for simplicity*, bounded process. Then $\mathbb{T}(J\,|\,\mathsf{P}) = (B_J, C_J, \nu_J)$ with

$$B_J = f \cdot B + \Big[h(fx) - fh(x)\Big] * \nu,$$

$$C_J = f^2 \cdot C,$$

$$I_A(x) * \nu_J = I_A(fx) * \nu.$$

(In our case $f = -1$, $h(-x) + h(x) = 0$.)

EXAMPLES

1. BROWNIAN CASE: $\nu = 0$, $S = e^H \Leftrightarrow B + \frac{C}{2} = 0$,

$$\mathbb{T}(H' \,|\, \mathsf{P}') = (B', C', 0) \quad \text{with} \quad B' = -(B + C) \quad \text{and} \quad C' = C.$$

(So, $B' + \frac{C'}{2} = -\left(B + \frac{C}{2}\right) = 0 \Rightarrow S' \in \mathcal{M}_{\text{loc}}(\mathsf{P}')$.)

If

$$S_t = e^{\sigma W_t - \sigma^2 t/2}, \qquad \text{i.e.,} \quad H_t = \sigma W_t - \sigma^2 t/2 \quad \text{and} \quad \boxed{dS_t = \sigma S_t \, dW_t},$$

then for $S' = e^{H'} = e^{-H}$

$$dS'_t = -\sigma S'_t \,(dW_t - \sigma \, dt).$$

We have that $W'_t = W_t - \sigma t \in \mathcal{M}_{\text{loc}}(\mathsf{P}')$ and W' is a P'-Brownian motion indeed. So,

$$\boxed{dS'_t = -\sigma S'_t \, dW'_t}.$$

2. POISSONIAN CASE:

$$S = e^H \qquad \text{with} \quad H_t = \alpha \pi_t - \lambda(e^\alpha - 1)t, \quad \alpha \neq 0,$$

where $\pi = (\pi_t)$ is a Poisson process with parameter λ. Here

$$\begin{aligned} B_t &= -\lambda(e^\alpha - 1)t, \\ C_t &= 0, \\ \nu(dt\,dx) &= \lambda I_{\{\alpha\}}(dx)\,dt, \end{aligned}$$

and

$$B'_t = \lambda(e^\alpha - 1)t \quad \text{and} \quad (e^x - 1) * \nu'_t = \lambda(1 - e^\alpha)t$$

$$\Rightarrow B' + (e^x - 1) * \nu' = 0$$

$$\Rightarrow S' = e^{-H} \in \mathcal{M}(\mathsf{P}').$$

§ 3. The CALL-PUT duality in option pricing

1. We assume the initial measure P to be a **martingale** measure, and all our calculations of $\mathsf{E}_{\mathsf{P}} f_T$ will be done w.r.t. this measure P.

In the case of an **incomplete** market the option price

$$\boxed{\mathsf{E}_{\mathsf{P}} f_T}$$

can be called a **quasi-rational** option price.

2. For a standard call option $f_T = (S_T - K)^+$, $K > 0$,
whereas for a put option it is $f_T = (K - S_T)^+$, $K > 0$.

OPTION PRICES:

$$\mathbb{C}_T(S,K) = \mathsf{E}(S_T - K)^+, \qquad \mathbb{P}_T(K,S) = \mathsf{E}(K - S_T)^+.$$

For $S = e^H$ we get

$$\begin{aligned} \mathbb{C}_T(S,K) &= \mathsf{E} S_T \frac{f_T}{S_T} = \mathsf{E}' \frac{f_T}{S_T} = \mathsf{E}'(1 - K S'_T)^+ \\ &= K\mathsf{E}'(\tfrac{1}{K} - S'_T)^+ = K\mathsf{E}'(K' - S'_T)^+, \quad K' = \tfrac{1}{K}. \end{aligned}$$

Thus

$$\tfrac{1}{K}\mathbb{C}_T(S,K) = \mathbb{P}'_T(K',S'), \quad \tfrac{1}{K}\mathbb{P}_T(K,S) = \mathbb{C}'_T(S',K'),$$

where $K' = \frac{1}{K}$.

3. Floating strike lookback call and put options

Suppose $S \in \mathcal{M}(\mathsf{P})$, $\alpha \geq 1$. For a floating case we get

$$\mathbb{C}_T(S; \alpha \inf S) = \mathsf{E}_\mathsf{P}\Big(S_T - \overbrace{\alpha \inf_{t \leq T} S_t}^{\text{floating strike}}\Big)^+$$

$$= \mathsf{E}\Big[S_T\Big(1 - \frac{\alpha \inf_{t\leq T} S_t}{S_T}\Big)^+\Big] = \mathsf{E}'\Big(1 - \alpha e^{\inf_{t\leq T} H_t - H_T}\Big)^+$$

$$= \mathsf{E}'\Big(1 - \alpha e^{H'_T - \sup_{t\leq T} H'_t}\Big)^+ = \alpha\, \mathsf{E}'\Big(\tfrac{1}{\alpha} - e^{H'_T - \sup_{t\leq T} H'_t}\Big)^+.$$

Assume the following **reflection principle**:

$$\boxed{\mathsf{Law}\Big(\sup_{t\le T} H_t - H'_T \,\Big|\, \mathsf{P}'\Big) = \mathsf{Law}\Big(-\inf_{t\le T} H'_t \,\Big|\, \mathsf{P}'\Big)}$$

(It holds, for example, if H' is a Lévy process w.r.t. P'.)

Then

$$\begin{aligned}
\tfrac{1}{\alpha}\mathbb{C}_T(S;\alpha \inf S) &= \mathsf{E}'\Big(\tfrac{1}{\alpha} - e^{\inf_{t\le T} H'_t}\Big)^{+} \\
&= \mathsf{E}\Big(\tfrac{1}{\alpha} - \inf_{t\le T} S'_t\Big)^{+} = \mathbb{P}'_T\Big(\tfrac{1}{\alpha};\inf S'\Big).
\end{aligned}$$

So,

$$\frac{1}{\alpha}\,\mathbb{C}_T(S;\alpha \inf S) = \mathbb{P}'_T\Big(\overbrace{\;\frac{1}{\alpha}\;}^{\textbf{fixed strike}};\inf S'\Big).$$

Similarly, assuming the **reflection principle**

$$\boxed{\mathsf{Law}\Big(H'_T - \inf_{t\le T} H_t \,\Big|\, \mathsf{P}'\Big) = \mathsf{Law}\Big(\sup_{t\le T} H'_t \,\Big|\, \mathsf{P}'\Big)}$$

we get the following duality

$$\tfrac{1}{\beta}\,\mathbb{P}_T(\beta \sup S; S) = \mathbb{C}'_T(\sup S; \tfrac{1}{\beta}).$$

4. Floating strike Asian options

Assume again that $S \in \mathcal{M}(\mathsf{P})$ and consider the price

$$
\begin{aligned}
\mathbb{C}_T\Big(S; \frac{1}{T}\int S\Big) &= \mathsf{E}\Big(S_T - \frac{1}{T}\int_0^T S_t\,dt\Big)^+ = \mathsf{E}\Big[S_T\Big(1 - \frac{1}{T}\int_0^T \frac{S_t}{S_T}\,dt\Big)^+\Big] \\
&= \mathsf{E}'\Big[1 - \frac{1}{T}\int_0^T \frac{S'_T}{S'_t}\,dt\Big]^+ = \mathsf{E}'\Big[1 - \frac{1}{T}\int_0^T e^{H'_T - H'_t}\,dt\Big]^+ \\
&= \mathsf{E}'\Big[1 - \frac{1}{T}\int_0^T e^{H'_T - H'_{T-u}}\,du\Big]^+.
\end{aligned}
$$

If H' is a Lévy process then

$$
\mathrm{Law}(H'_T - H'_{(T-t)-};\ 0 \le t < T \,|\, \mathsf{P}') = \mathrm{Law}(H'_t;\ 0 \le t < T \,|\, \mathsf{P}').
$$

Hence we conclude

$$\mathbb{C}_T\Big(S; \frac{1}{T}\int S\Big) = \mathsf{E}'\Big(1 - \frac{1}{T}\int_0^T S'_u\,du\Big)^+ = \mathbb{P}'_T\Big(1; \frac{1}{T}\int S\Big).$$

Similarly,

$$\mathbb{P}_T\Big(\frac{1}{T}\int S; S\Big) = \mathbb{C}'_T\Big(\frac{1}{T}\int S; 1\Big).$$

Therefore, if H is a Lévy process, then the duality holds:

the calculation of prices of
Asian call and put option with **floating** strikes $\frac{1}{T}\int S$

CAN BE REDUCED TO

the calculation of prices $\mathbb{P}'_T(1; \frac{1}{T}\int S')$ and $\mathbb{C}'_T(\frac{1}{T}\int S'; 1)$ of
Asian put and call options with **fixed** strike.

5. Standard American call and put options

Denote
$$\widehat{\mathbb{C}}_T(S;K) = \sup_{\tau\in\mathfrak{M}_T} \mathsf{E} e^{-\lambda\tau}(S_\tau - K)^+,$$
$$\widehat{\mathbb{P}}_T(K;S) = \sup_{\tau\in\mathfrak{M}_T} \mathsf{E} e^{-\lambda\tau}(K - S_\tau)^+.$$

Similarly to the case of European options, for $f_\tau = (S_\tau - K)^+$

$$\begin{aligned}
\widehat{\mathbb{C}}_T(S;K) &= \sup_{\tau\in\mathfrak{M}_T} \mathsf{E}\Big(e^{-\lambda\tau} f_\tau \frac{S_\tau}{S_T}\Big) = \sup \mathsf{E}'\Big(e^{-\lambda\tau}\frac{f_\tau}{S_T}\Big) = \sup \mathsf{E}'\big(e^{-\lambda\tau} f_\tau S'_T\big) \\
&= \sup \mathsf{E}'\Big(e^{-\lambda\tau} f_\tau\, \mathsf{E}'(S'_T \,|\, \mathcal{F}_\tau)\Big) = \sup \mathsf{E}'\Big(e^{-\lambda\tau} f_\tau S'_\tau\Big) \\
&= \sup \mathsf{E}'\Big(e^{-\lambda\tau}(S_\tau - K)^+ S'_\tau\Big) = \sup \mathsf{E}'\Big(e^{-\lambda\tau}(1 - K S'_\tau)^+\Big) \\
&= K \sup \mathsf{E}'\Big(e^{-\lambda\tau}(K' - S'_\tau)^+\Big) = K\widehat{\mathbb{P}}_T(K';S').
\end{aligned}$$

So,

$$\boxed{\frac{1}{K}\,\widehat{\mathbb{C}}_T(S;K) = \widehat{\mathbb{P}}'_T(K';S')}, \text{ and similarly } \boxed{\frac{1}{K}\,\widehat{\mathbb{P}}_T(K;S) = \widehat{\mathbb{C}}'_T(S';K')}.$$

6. Forward-start options

$$\mathbb{C}_T^t(S;S) = \mathsf{E}(S_T - S_t)^+ = \mathsf{E}\Big[S_T \frac{(S_T - S_t)^+}{S_T}\Big] = \mathsf{E}'\Big(1 - \frac{S_t}{S_T}\Big)^+$$
$$= \mathsf{E}'\Big(1 - \frac{S'_T}{S'_t}\Big)^+ = \mathsf{E}'\Big(1 - e^{H'_T - H'_t}\Big)^+$$
$$= \mathsf{E}'\Big(1 - e^{H'_T - H'_{T-u}}\Big)^+, \qquad u = T - t. \qquad (*)$$

For a Lévy process H' (w.r.t. P'):

$$\mathrm{Law}(H'_T - H'_{(T-u)-};\ 0 \le u < T \,|\, \mathsf{P}') = \mathrm{Law}(H'_u;\ 0 \le u < T \,|\, \mathsf{P}').$$

Then from $(*)$

$$\mathbb{C}_T^t(S;S) = \mathsf{E}'(1 - e^{H'_u})^+ = \mathbb{P}'_u(1;S') \quad \text{with} \quad u = T - t.$$

Similarly,

$$\mathbb{P}_T^t(S;S) = \mathbb{C}'_u(S';1) \quad \text{with} \quad u = T - t.$$

Therefore, if H is a Lévy process, then again the duality holds:

the calculation of prices $\mathbb{C}_T^t(S;S)$ and $\mathbb{P}_T^t(S;S)$ of **forward-start** call and put options

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via formulae $\boxed{\begin{array}{l} \mathbb{C}_T^t(S;S) = \mathbb{P}'_{T-t}(1;S') \\ \mathbb{P}_T^t(S;S) = \mathbb{C}'_{T-t}(S';1) \end{array}}$ **TO**

the calculation of prices $\mathbb{P}'_{T-t}(1;S')$ and $\mathbb{C}'_{T-t}(S';1)$ of **vanilla European** put and call options.

Albert N. SHIRYAEV

THREE TOPICS on FINANCIAL STATISTICS, STOCHASTICS and OPTIMIZATION

Topic III. A:

Toward the Technical Analysis: Prediction, Estimation of the Changes in Stock Prices, and Price Range Charts

(Brief survey)

The machinery of the technical analysis of financial mathematics and engineering.

The main motivation of this lecture is based on idea to obtain a mathematical explanation of some practical methods ("when to buy, when to sell", *etc.*) of the Technical Analysis which have as usual only a descriptive character.

As is well known, the "fundamentalists" are trying to explain

WHY the stock price moves;

they make their decisions by looking at the state of the "economy at large"; they define a stock value and calculate proper stock prices in view of its estimated future values; they build their analysis upon the assumption that the actions of market operators are "rational".

As to the "technicians" they concentrate on the local peculiarities of the markets, they emphasize "mass behavior", "market moods"; they start their analysis from an idea that stock price movement is "the product of supply and demand"; their basic concept is the following: the analysis of past stock prices helps us to see future prices because past prices take future prices into account; they try to explain

HOW the stock prices move.

CONTENTS

§ 1. "Mathematization" of the "Kagi charts" and "Renko charts" from the Technical Analysis

§ 2. Prediction of time of maximum value of prices observable on time interval $[0, T]$

§ 3. Quickest detection of time of appearing of arbitrage

§ 4. Drawdowns as the characteristics of risk

§ 1. "Mathematization" of the "Kagi charts" and "Renko charts" from the Technical Analysis

Let $X = (X_t)_{t \geq 0}$ be a stock price.

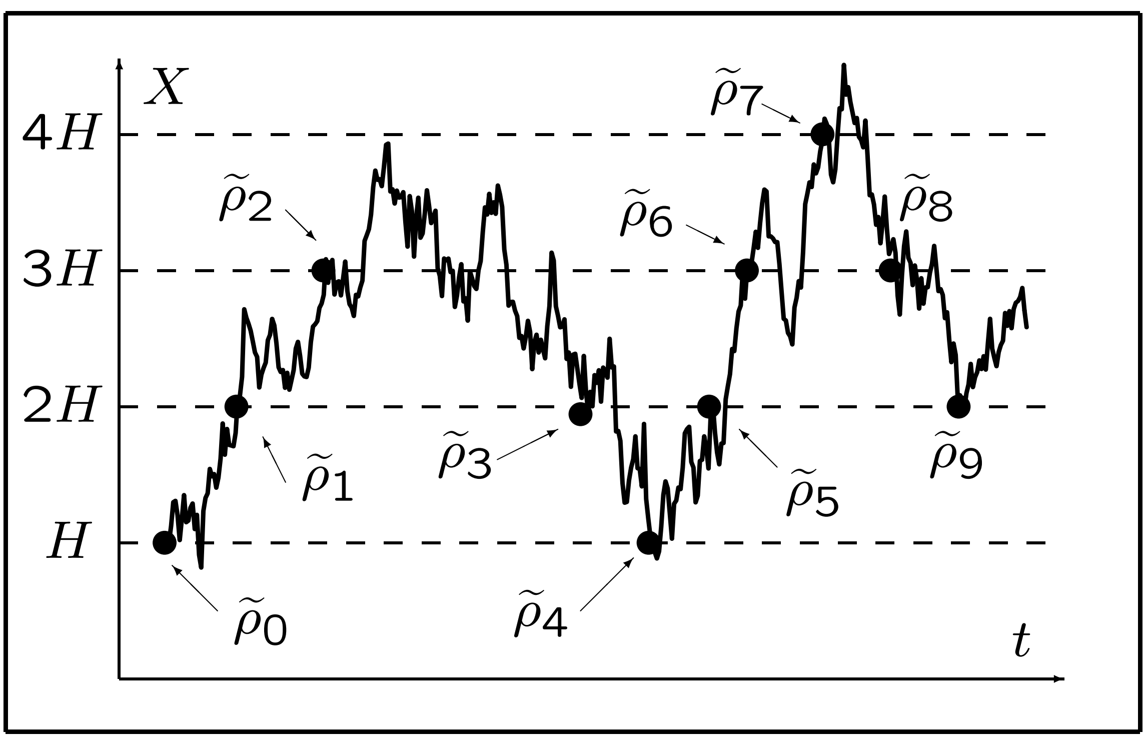
The Japanese "Kagi chart" and "Renko chart" (also called the **price ranges**) give methods to forecast price trends from price changes which exceed either a certain range H or a certain rate H. The price range or rate H is determined in advance. (In Japan, popular price ranges are ¥5, 10, 20, 50, 100, 200.) Greater price ranges are used for stocks with higher prices because their upward and downward movements are larger.

$$R \to |X|$$
$$K \to \max X - X$$

Renko-construction:

Step I: We construct $(\widetilde{\rho}_i)$:

$$\widetilde{\rho}_0 = 0,$$
$$\widetilde{\rho}_{n+1} = \inf\Big\{t > \widetilde{\rho}_n : |X_t - X_{\widetilde{\rho}_n}| = H\Big\}, \quad n \geq 1.$$

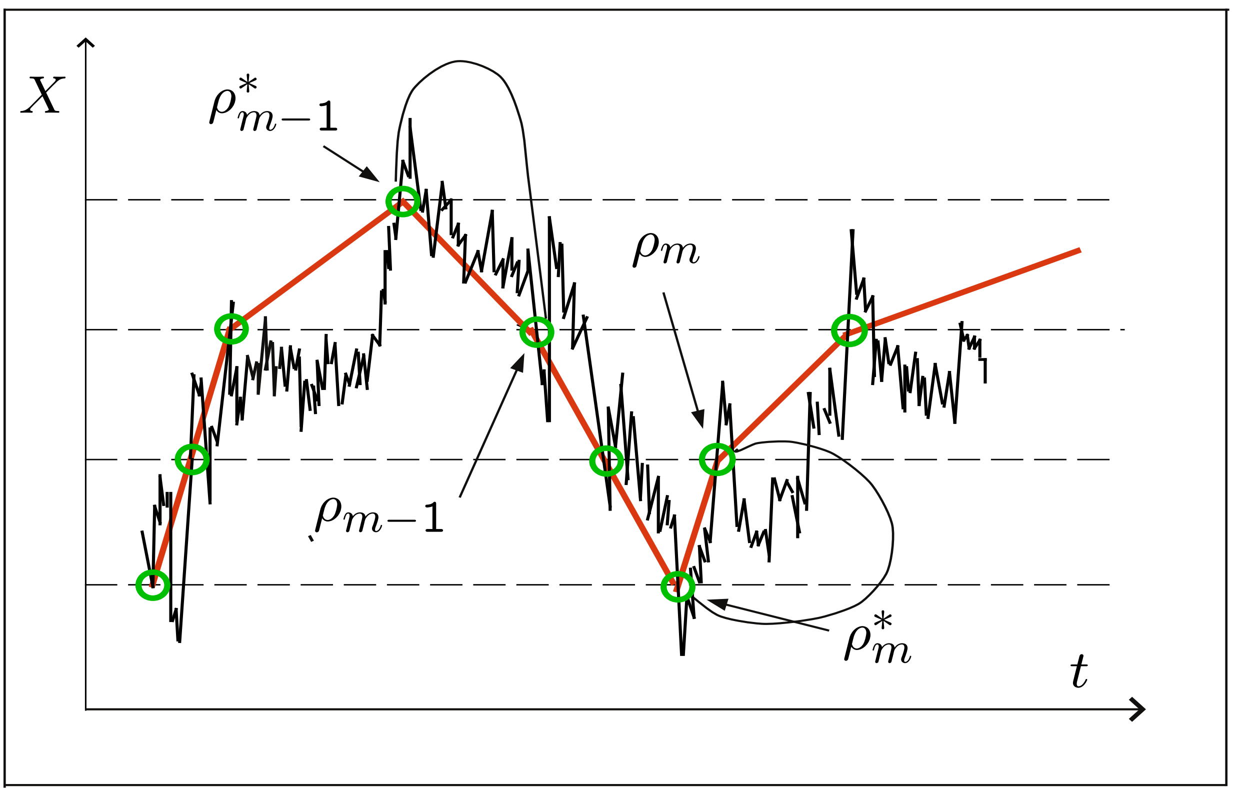


Step II: Construction

$$(\widetilde{\rho}_n) \longrightarrow (\rho_m, \rho_m^*).$$

We look at all $\widetilde{\rho}_n$ such that

$$\left(X_{\widetilde{\rho}_n} - X_{\widetilde{\rho}_{n-1}}\right)\left(X_{\widetilde{\rho}_{n-1}} - X_{\widetilde{\rho}_{n-2}}\right) < 0.$$



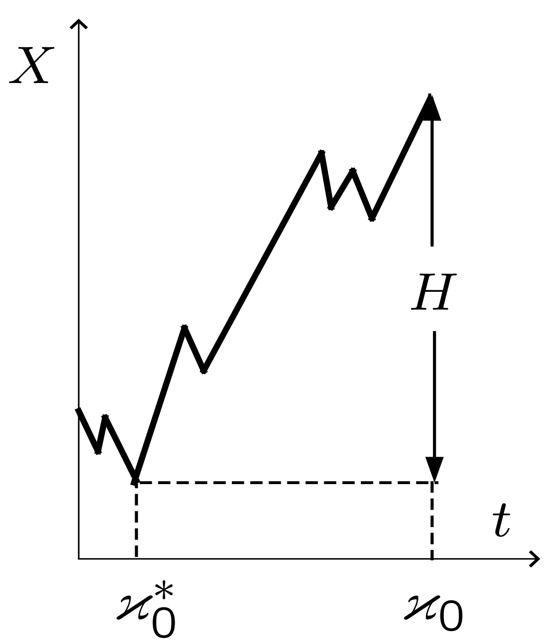
ρ_m is a Markov time

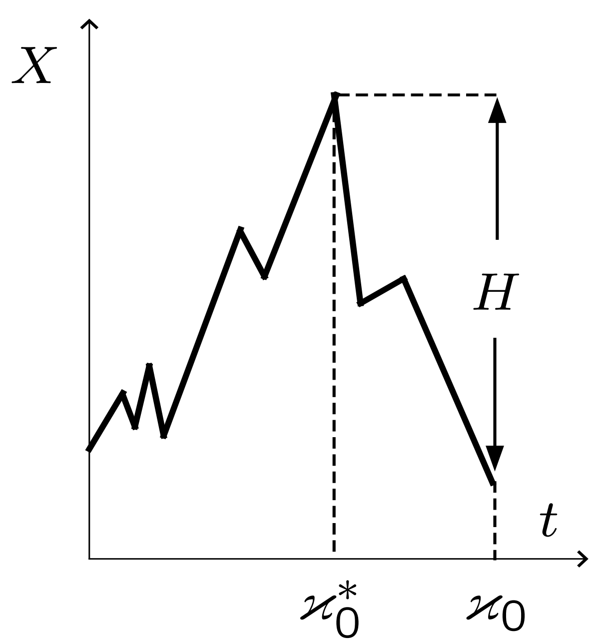
ρ_m^* is a non-Markov time

Kagi construction:

$$\varkappa_0 = \inf\left\{u > 0 : \max_{[0,u]} X - \min_{[0,u]} X = H\right\}$$

$$\varkappa_0^* = \begin{cases} \inf\left\{u \in [0,\varkappa_0] : X_u = \min\limits_{[0,\varkappa_0]} X\right\} \\ \qquad \text{if } X_{\varkappa_0} = \max\limits_{[0,\varkappa_0]} X \\ \inf\left\{u \in [0,\varkappa_0] : X_u = \max\limits_{[0,\varkappa_0]} X\right\} \\ \qquad \text{if } X_{\varkappa_0} = \min\limits_{[0,\varkappa_0]} X \end{cases}$$





Next step: we define by induction

$$\varkappa_{n+1} = \begin{cases} \inf\left\{u > \varkappa_n : \max\limits_{[x_n,u]} X - X_u = H\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = H \\ \\ \inf\left\{u > \varkappa_n : \max\limits_{[x_n,u]} X - X_u = H\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = -H \end{cases}$$

$$\varkappa_{n+1}^* = \begin{cases} \inf\left\{u \in [\varkappa_n, \varkappa_{n+1}] : X_u = \max\limits_{[\varkappa_n,\varkappa_{n+1}]} X\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = H \\ \\ \inf\left\{u \in [\varkappa_n, \varkappa_{n+1}] : X_u = \min\limits_{[\varkappa_n,\varkappa_{n+1}]} X\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = -H \end{cases}$$

Kagi and Renko variation (on $[0,T]$):

$$K_T(X;H) = \sum_{n=1}^{N} |X_{\varkappa_n^*} - X_{\varkappa_{n-1}^*}|, \qquad N = N_T(X;H),$$

$$R_T(X;H) = \sum_{n=1}^{M} |X_{\rho_n^*} - X_{\rho_{n-1}^*}|, \qquad M = M_T(X;H).$$

Kagi and Renko volatilities (on $[0,T]$):

$$k_T(X;H) = \frac{K_T(X;H)}{M_T(X;H)},$$

$$r_T(X;H) = \frac{R_T(X;H)}{M_T(X;H)}.$$

Theorem. *If $X = \sigma B$ then*

1) $k_T(\sigma B;H) \sim 2H$, $\quad N_T \sim \dfrac{T\sigma^2}{H^2}$ (P-*a.s.*), *and*

$$K_T = k_T N_T \overset{\mathsf{P}}{\sim} \frac{2T\sigma^2}{H^2};$$

2) $r_T(\sigma B;H) \overset{\mathsf{P}}{\sim} 2H$, $\quad M_T \sim \dfrac{T\sigma^2}{2H^2}$ (P-*a.s.*), *and*

$$R_T = r_T M_T \overset{\mathsf{P}}{\sim} \frac{T\sigma^2}{H}.$$

Results of the statistical analysis of some stock prices

$X = (X_t)_{t \geq 0}$ — Future on Index SP500
(Emini-SP500 Futures)
1 point = $ 50

2002-2003 (471 trading days)

$\Delta = 1$ sec

X_t is the value of the last transaction at time t.

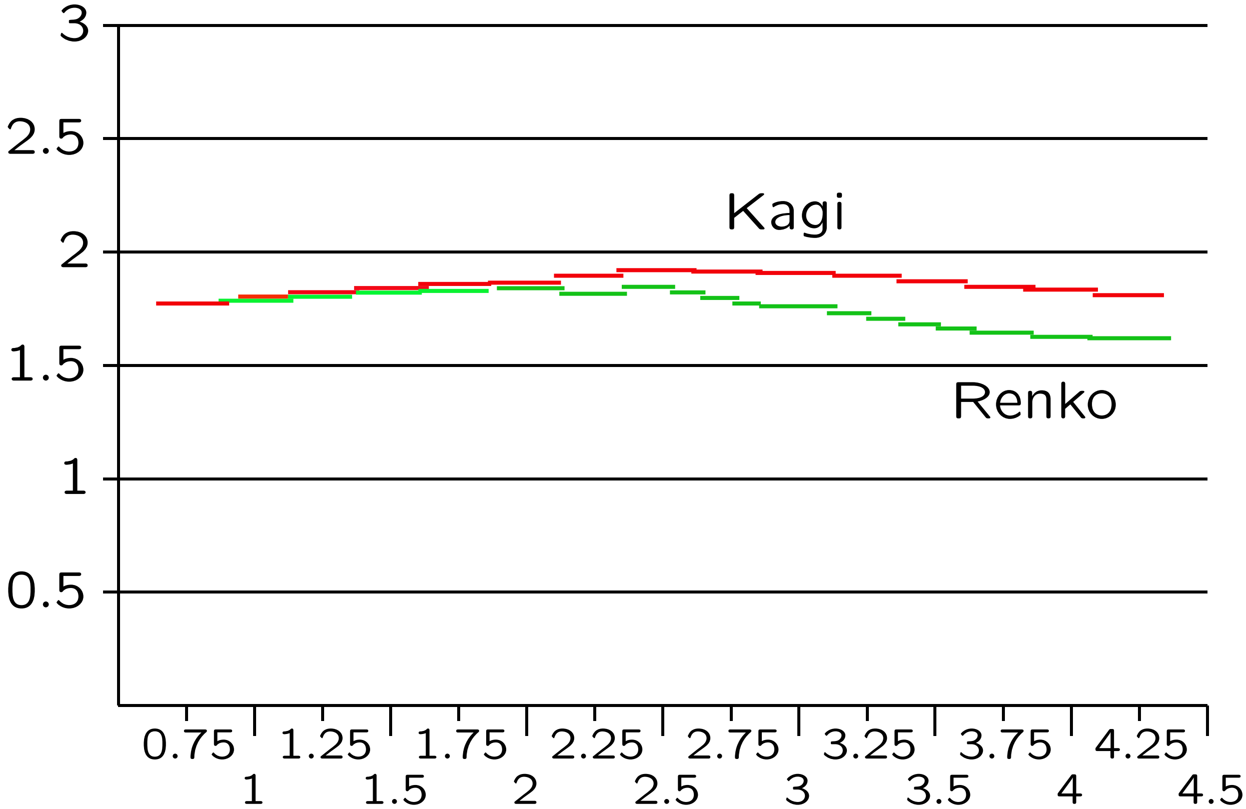
RENKO

H	1	1.25	1.5	2	2.25	2.5	3	4
$\frac{r_T(X;H)}{H}$	1.83	1.84	1.86	1.88	1.86	1.88	1.80	1.69

KAGI

H	1	1.25	1.5	2	2.25	2.5	3	4
$\frac{k_T(X;H)}{H}$	1.83	1.85	1.85	1.89	1.91	1.93	1.92	1.87

Almost the same results are valid for Futures on Index Nasdaq 100 (Emini-Nasdag100 Futures), 1 point = $ 20



For EESR (United Energy System of Russia)

$$\frac{r_T(X;H)}{H} \sim 1.99 \sim \frac{k_T(X;H)}{H}.$$

Let us say that X-market has

$$r(H)\text{-property if } \mathsf{E} r_T(X;H) \sim r(H)\cdot H$$
$$k(H)\text{-property if } \mathsf{E} k_T(X;H) \sim k(H)\cdot H$$

(For a Brownian motion $r(H) = k(H) = 2$.)

Define Renko strategy $\gamma^R = (\gamma_t^R)_{t\geq 0}$ with

$$\gamma_t^R = \sum_{n\geq 1} \operatorname{sgn}\Big(X_{\rho_{n-1}} - X_{\rho^*_{n-1}}\Big) I_{[\rho_{n-1},\rho^*_{n-1})}(t)$$
$$\times \Big(I(k(H)\geq 2) - I(k(H)<2)\Big), \quad t \geq 0,$$

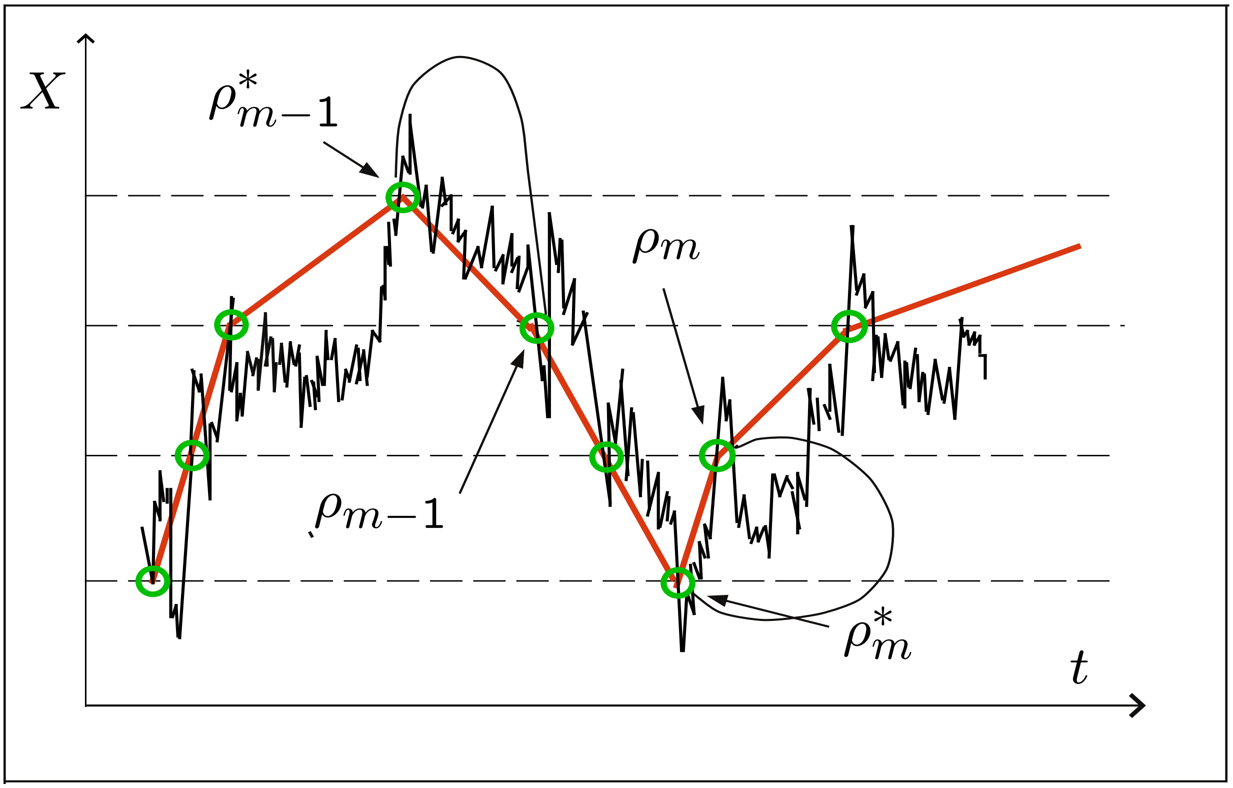
and the corresponding capital

$$C_t^{\gamma^R} = \int_0^t \gamma_u^R \, dX_u - \lambda \int_0^t |d\gamma_u^R|.$$

Then

$$\lim_{t\to\infty} \mathsf{E}\frac{C_t^{\gamma^R}}{M_t} = |r(H) - 2| \cdot H - 2\lambda.$$

The similar result is valid for the Kagi strategy $\gamma^K = (\gamma_t^K)_{t\geq 0}$.



If $R(H) > 2$	we buy in times $\rho_{m-2}, \rho_m, \ldots$
	we sell in times $\rho_{m-1}, \rho_{m+1}, \ldots$
	$(\uparrow\uparrow) \quad (\downarrow\downarrow)$

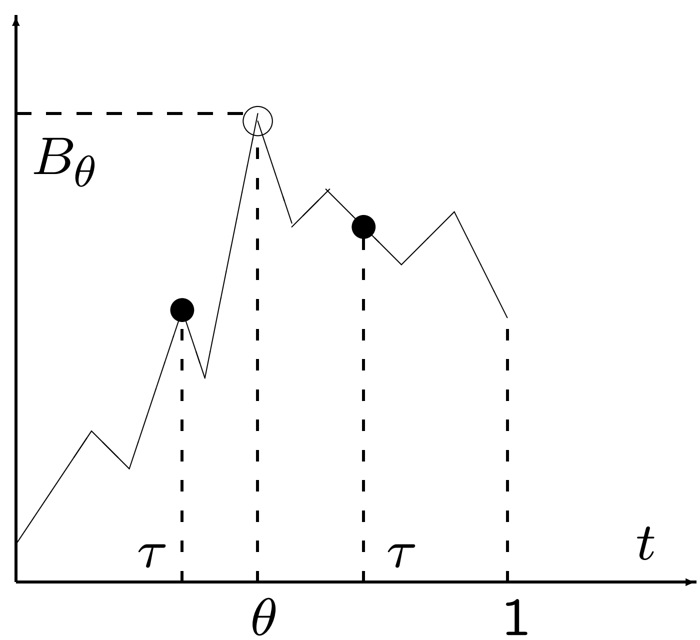
If $R(H) < 2$	we buy in times $\rho_{m-1}, \rho_{m+1}, \ldots$
	we sell in times $\rho_{m-2}, \rho_m, \ldots$
	$(\uparrow\downarrow) \quad (\downarrow\uparrow)$

§ 2. Prediction of time of maximum value of prices observable on time interval $[0, T]$

We would like to present now several our probability and statistical approaches to solving some other problems of the technical analysis.

Problem. *When to sell stock optimally?*

We shall describe prices by a Brownian motion $B = (B_t)_{0 \le t \le 1}$; θ is a point of maximum of B: $B_\theta = \max\limits_{0 \le t \le 1} B_t$.



Suppose that we begin to observe this process at time $t = 0$ ("morning time"), and, using only past observations, we stop at time τ declaring "alarm" about selling. It is very natural to try to solve the following problem: to find "optimal" times τ^* and τ^{**} such that either

$$\inf_{0\le\tau\le1} \mathsf{E}|B_\theta - B_\tau|^2 = \mathsf{E}|B_\theta - B_{\tau^*}|^2$$

or

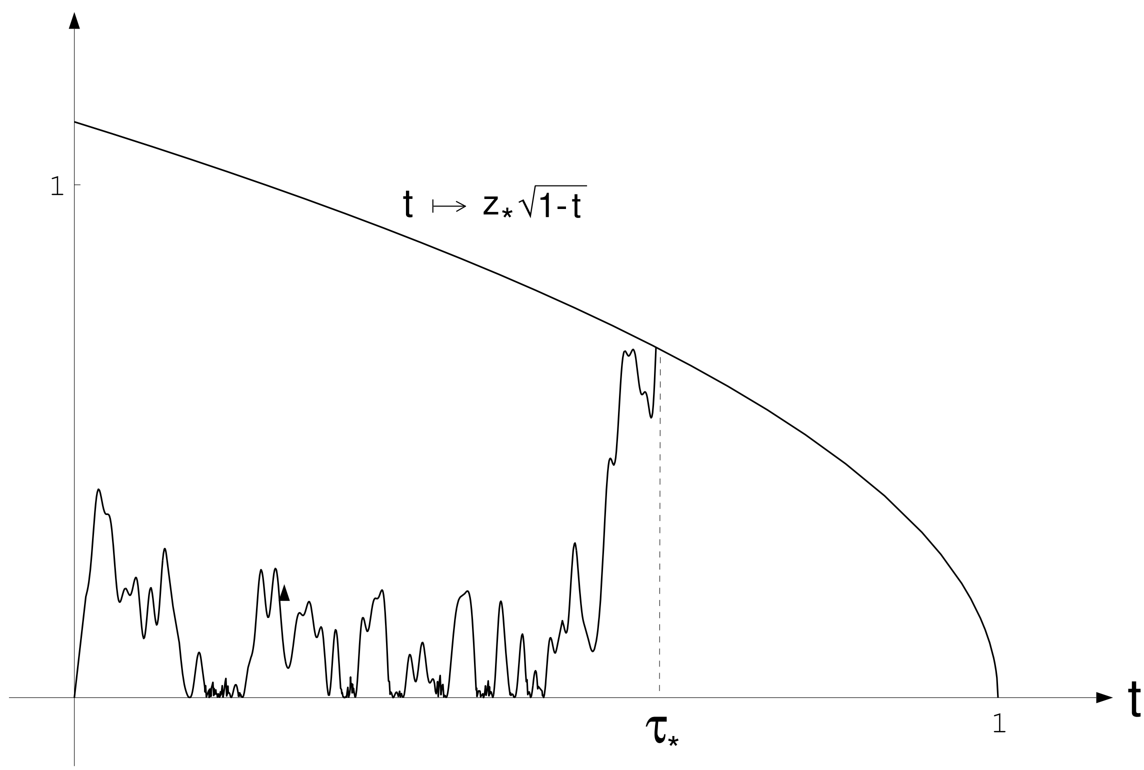
$$\inf_{0\le\tau\le1} \mathsf{E}|\theta - \tau| = \mathsf{E}|\theta - \tau^{**}|.$$

For us it was a little bit surprising that here the optimal stopping times coincide: $\tau^{**} = \tau^*$. The solution shows that

$$\tau^* = \inf\left\{t \le 1 : \max_{s\le t} B_s - B_t \ge z_* \sqrt{1-t}\right\},$$

where z_* is a certain (known) constant ($z_* = 1.12\ldots$).

This problem belongs to the theory of optimal stopping and method of its solution is based on reducing to the special **free-boundary** problem.



It is interesting to note that

$$\mathsf{E}\tau^* = 0.55\ldots, \qquad \mathsf{D}\tau^* = 0.05\ldots.$$

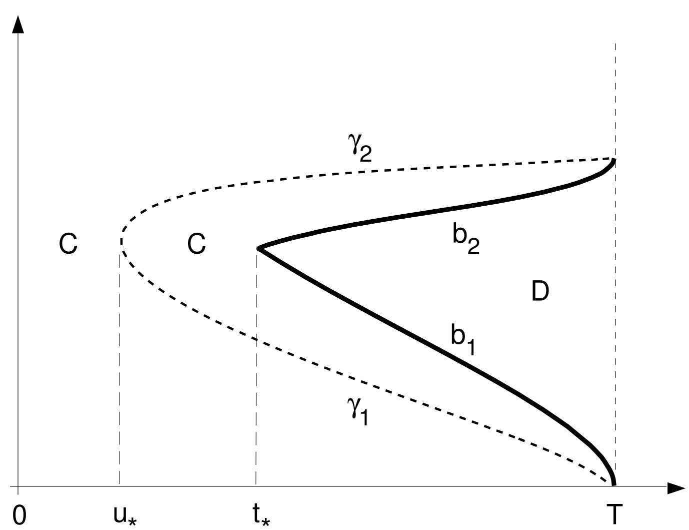
The cases $B_t^\mu = \mu t + B_t$ instead of B_t are more complicated. If $\mu > 0$ and μ is away from 0, then

$$\tau^* = \inf\{t \leq 1 : b_1(t) \leq S_t^\mu - B_t^\mu \leq b_2(t)\}$$

where

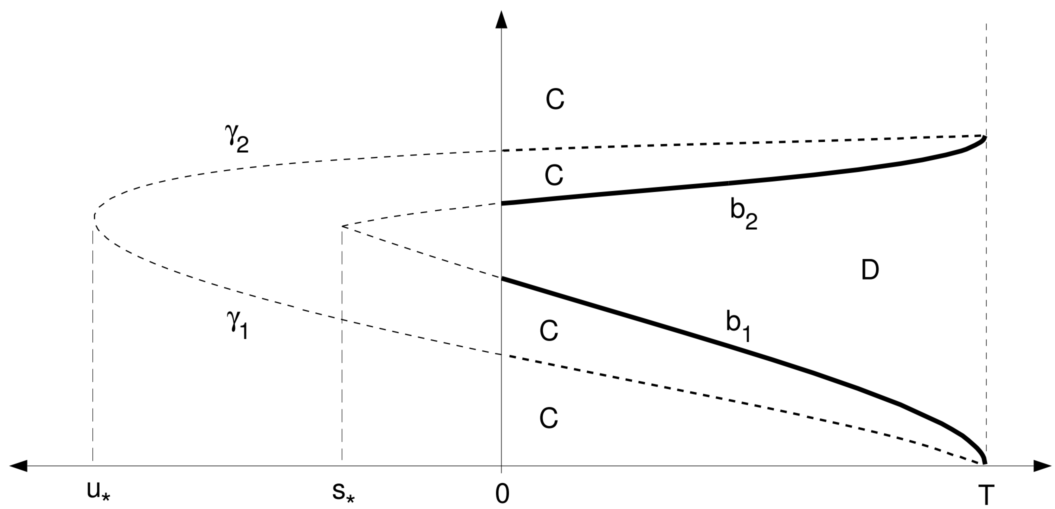
$$B_t^\mu = \mu t + B_t, \qquad S_t^\mu = \max_{u \leq t} B_u^\mu,$$

and $b_1(t)$ and $b_2(t)$ have the following form:

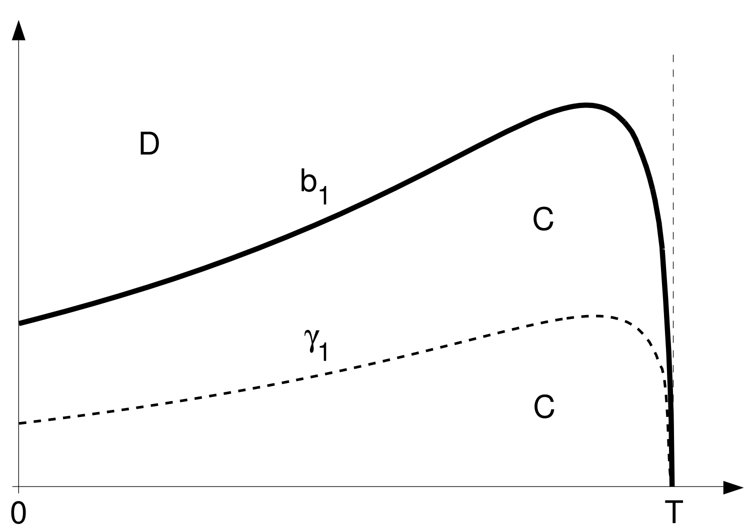


⟨Here C is the area of continuation of observations, D is the stopping area.⟩

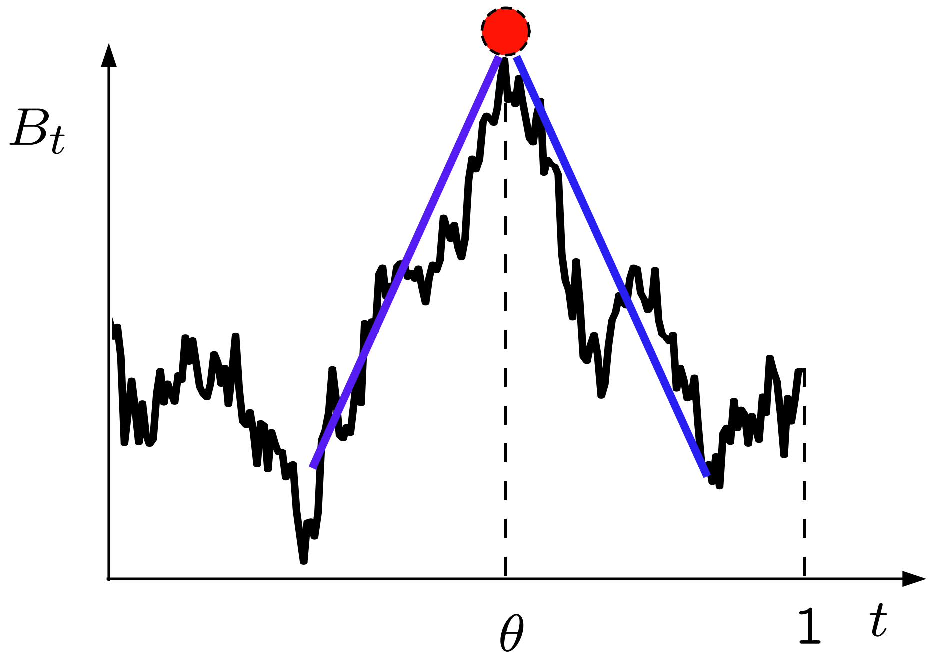
If $\mu > 0$ and μ is close to 0, then the corresponding picture has the following form:



For $\mu < 0$ and if μ is far from 0, the picture is as follows:



In the considered problem, the time θ is a "change point" of the changing of the directions of trend



Solution of the problem

$$\text{“}\inf_{\tau} \mathsf{E}|B_\tau - B_\theta|^2\text{”}$$

or the problem "$\inf_\tau \mathsf{E}|\tau - \theta|$" depends, of course, on the construction at any time t a "good" prediction of the change point θ. The natural estimate of θ should be based on the *a posteriori* probability

$$\pi_t = \mathsf{P}(\theta \le t \,|\, \mathcal{F}_t^B),$$

where $\mathcal{F}_t^B = \sigma(B_s, s \le t)$.

Stochastic analysis shows that

$$\pi_t = 2\varphi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) - 1, \qquad S_t = \max_{u \le t} B_u,$$

that explains appearing of the expression

$$\frac{S_t - B_t}{\sqrt{1-t}}$$

which is involved above in the definition of optimal stopping time

$$\tau^* = \inf\left\{t \le 1 : \frac{S_t - B_t}{\sqrt{1-t}} \ge z_*\right\}.$$

Statistics $S_t - B_t$ is appearing in many problems of the financial mathematics and financial engineering (and, generally, in the mathematical statistics under name *CUSUM statistics*).

Now we are going to tackle the following problem, which is interesting, e.g., from the point of view of the quickest detection of arbitrage.

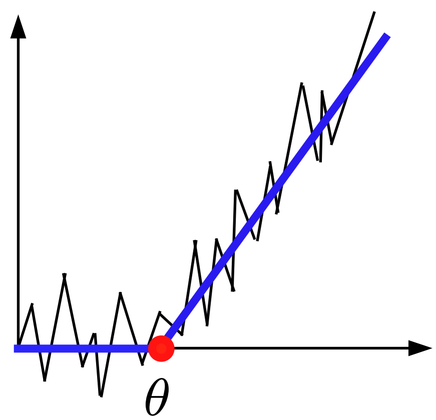
§ 3. Quickest detection of time of appearing of arbitrage

Problem. Suppose we observe the prices

$$X_t = r(t-\theta)^+ + \sigma B_t$$

or

$$dX_t = \begin{cases} \sigma\, dB_t, & t \le \theta, \\ r\, dt + \sigma\, dB_t, & t > \theta. \end{cases}$$

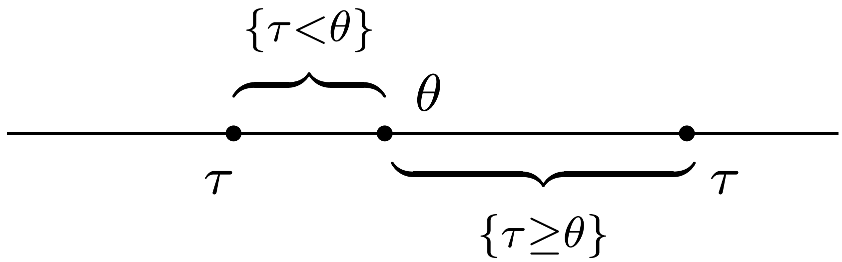


Here a "change point" θ is considered as a time of appearing of arbitrage. (Brownian motion's prices correspond to the *non-arbitrage* situation. Brownian motion with drift corresponds to a case of *arbitrage*.)

One very difficult question here is "what is θ?". There are two approaches. In the first one we assume that θ is a *random variable*. Suppose that τ is time of "alarm" θ. Consider two events

$$\{\tau < \theta\} \quad \text{and} \quad \{\tau \ge \theta\}.$$

The set $\{\tau < \theta\}$ is the event of a false alarm with a (false alarm) probability $\mathsf{P}(\tau < \theta)$.



From a financial point of view an interesting characteristic of the event $\{\tau \geq \theta\}$ is a delay time $\mathsf{E}(\tau - \theta \,|\, \tau \geq \theta)$ or $\mathsf{E}(\tau - \theta)^+$. These considerations lead to the following problem: in the class $\mathfrak{M}_\alpha = \{\tau : \mathsf{P}(\tau < \theta) \leq \alpha\}$, i.e., in the class of stopping times with the probability of false alarm $\mathsf{P}(\tau < \theta)$ which less or equal the fixed level α, one need to find optimal stopping $\tau_\alpha^* \in \mathfrak{M}_\alpha$ such that

$$\inf_{\tau \in \mathfrak{M}_\alpha} \mathsf{E}(\tau - \theta)^+ = \mathsf{E}(\tau_\alpha^* - \theta)^+.$$

It turned out that it is not a simple problem if we consider an arbitrary distribution for θ. However, there exists one case when we may solve this problem in implicit form. This case is the following.

Assume that θ has the *exponential* distribution:

$$\mathsf{P}(\theta = 0) = \pi \quad \text{and} \quad \mathsf{P}(\theta > t \,|\, \theta > 0) = e^{-\lambda t},$$

where λ is a given positive constant and $\pi \in [0,1)$. This assumption is very reasonable. Indeed, for $A < a < b < B$

$$\lim_{\lambda \to 0} \mathsf{P}\Big(\theta \in (a,b) \mid \theta \in (A,B)\Big) = \frac{|b-a|}{|B-A|}.$$

It means that in limit ($\lambda \to 0$) the conditional distribution of θ is *uniform*, that is, in some sense the *worst* possible case from point of view of uncertainty of time of appearing of a change point θ.

We describe now the results about structure of the optimal stopping time τ_α^*.

Denote

$$\pi_t = \mathsf{P}(\theta \le t \,|\, \mathcal{F}_t^X),$$

where $\mathcal{F}_t^X = \sigma(X_s, s \le t)$.

This process satisfies the following nonlinear stochastic differential equation:

$$d\pi_t = \left(\lambda - \frac{r^2}{\sigma^2}\,\pi_t^2\right)(1-\pi_t)\,dt + \frac{r}{\sigma^2}\,\pi_t(1-\pi_t)\,dX_\tau$$

with $\pi_0 = \pi$.

Then it turns out that an optimal stopping time τ_α^* has the following form:

$$\tau_\alpha^* = \inf\{t : \pi_t \ge B_\alpha^*\},$$

where (for case $\pi = 0$, for simplicity)

$$B_\alpha^* = 1 - \alpha.$$

Second formulation of the quickest detection of arbitrage assumes that θ is simply a *parameter* from $[0,\infty)$. In this case we denote by P_θ the distribution of the process X under the assumption that a change point is occurred at time θ.

By P_∞ we denote the distribution of X under assumption that there is no change point at all.

Denote for given $T > 0$

$$\mathfrak{M}_T = \{\tau : \mathsf{E}_\infty \tau \leq T\}$$

the class of stopping time for which the mean time $\mathsf{E}_\infty \tau$ before (false) alarm is less or equal to T.

Put also

$$\mathbb{C}(T) = \inf_{\tau \in \mathfrak{M}_T} \sup_\theta \operatorname{ess\,sup}_\omega \mathsf{E}_\theta \Big[(\tau - \theta)^+ \,|\, \mathcal{F}_\theta \Big](\omega).$$

We proved that for each $T > 0$ in the class $\mathfrak{M}_T$ there exists an optimal strategy with the following structure: declare alarm at time

$$\tau_T^* = \inf \Big\{ t : \max_{u \leq t} X_u - X_t \geq a^*(T) \Big\},$$

where $a^*(T)$ is a certain constant. It is interesting to note that (if $r^2/(2\sigma^2) = 1$)

$$\mathbb{C}(T) \sim \log T, \qquad T \to \infty.$$

The given method, based on the

"CUSUM statistics $\max X - X$",

also is *asymptotically* optimal for more tractable criteria

$$\mathbb{D}(T) = \inf_{\tau \in \mathfrak{M}_T} \sup_{\theta} \mathsf{E}_\theta(\tau - \theta \,|\, \tau \geq \theta)$$

(We don't know what is an optimal method for $\mathbb{D}(T)$-criterion.) Asymptotically, again

$$\mathbb{D}(T) \sim \log T, \qquad T \to \infty.$$

§ 4. Drawdowns, downfalls

From given above exposition we observe importance of the "max $X - X$"-characteristics for taking optimal decisions. Now we would like to discuss that statistics and related ones in the problems of measure of *risk*. There is a special terminology for such an object which related to words "drawdown", or "downfall".

In practice a "drawdown" on time interval $[0, t]$ is defined as the percent change in a manager's net asset value

- from any newly established peak to a subsequent through,
- from a high "water mark" to the next low "water mark".

From the theoretical point of view,

Drawdown is a statistical measure of risk for investments; a competitor to the standard measure of risk such as return probability, VaR, Sharpe ration, etc.

There are many different definitions of drawdown's characteristics, which measure the decline in net asset value from the historic high point.

In one financial paper we read that

> ... Measuring risk through extreme losses is a very appealing idea. This is indeed how financial companies perceive risks. This explains the popularity of loss statistics as the maximum drawdown and maximum loss...

and

> ... it does not seem possible to derive exact results for the expected maximum drawdown.

Looking forward:

- What kinds of drawdowns should we expect over any given investment horizon?
- How many drawdowns should be experienced?
- How big?

Under the

Commodity Futures Trading Commission's
(CFTC)

mandatory disclosure regime managed futures advisors are obliged to disclose, as part of their capsule performance records, their

"*worst peak-to-valley drawdown*".

We shall demonstrate here some our theoretical calculations related to drawdowns.

Let $B_t^\mu = \mu t + \sigma W_t$ be a Brownian motion with drift, $W_0 = 0$.

There are several interesting characteristics related to

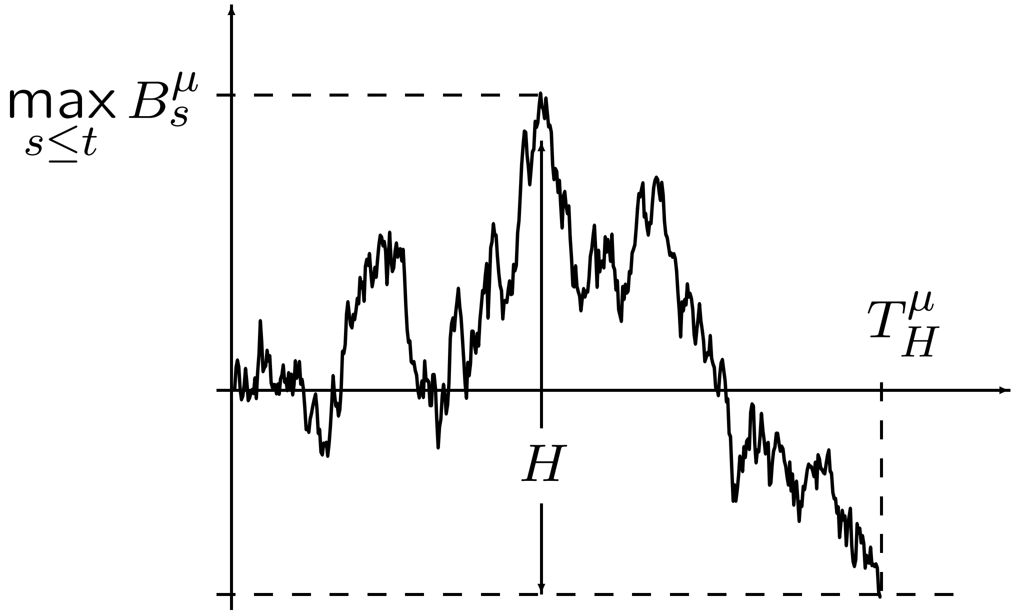
Range,
Drawdowns,
Downfalls,. . .

Range:

$$\boxed{R_t^\mu = \max_{s\le t} B_s^\mu - \min_{s\le t} B_s^\mu}$$

Statistics T_H^μ for B^μ:

$$\boxed{T_H^\mu = \inf\left\{t \ge 0 : \max_{s\le t} B_s^\mu - B_t^\mu \ge H\right\}}$$



If $\mu = 0$:

$$\boxed{\begin{aligned} \mathsf{E}T_H^0 &= \left(\frac{H}{\sigma}\right)^2, \qquad & \mathsf{E}\max_{t\le T_H^0} B_t^0 &= H, \\ \mathsf{D}T_H^0 &= \frac{2}{3}\left(\frac{H}{\sigma}\right)^4, \qquad & \mathsf{E}e^{-\lambda T_H^0} &= \frac{1}{\cosh\left(\frac{H}{\sigma}\sqrt{2\lambda}\right)}. \end{aligned}}$$

If $\mu \neq 0$:

$$\boxed{\begin{aligned} \mathsf{E}T_H^\mu &= \frac{\sigma^2}{2\mu^2}\left[\exp\left\{\frac{2\mu}{\sigma^2}H\right\} - 1 - \frac{2\mu}{\sigma^2}H\right], \\ \mathsf{E}\max_{t\le T_H^\mu} B_t^\mu &= \frac{\sigma^2}{2\mu}\left[\exp\left\{\frac{2\mu}{\sigma^2}H\right\} - 1\right]. \end{aligned}}$$

Towards a problem from Kolmogorov's diary (1944):

...For free (or not) random walk: How X_t drops when X_t falls for the first time (on $(t-\Delta, t)$) from above to some level ξ? To all appearance, certainly very steeply!..

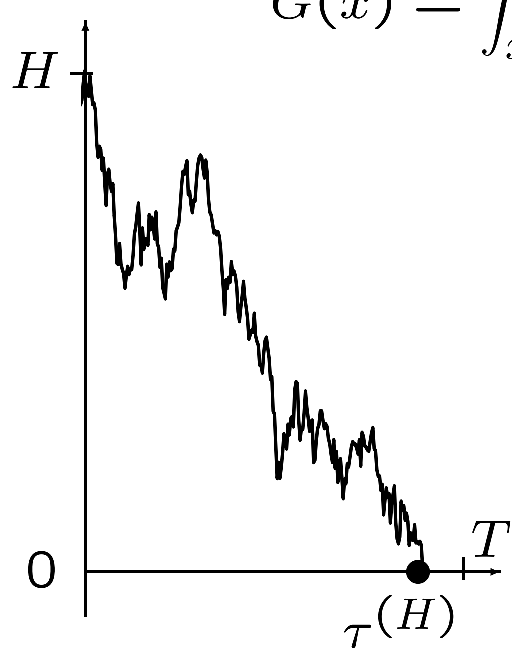
$$B_t^{(H)} = H + B_t, \quad B_0^{(H)} = H, \quad B_t = B_t^0$$

$$\tau^{(H)} = \inf\{u : B_u^{(H)} = 0\}$$

$$F(t) = \mathsf{P}\Big(\tau^{(H)} \le t \mid \min_{s\le T} B_s \le 0\Big)$$

$$\begin{aligned} f(t) &= \frac{dF(t)}{dt} \\ &= \frac{H\sqrt{T}}{2G(H/\sqrt{T})} t^{-3/2} e^{-H^2/2t}, \quad t \le T \end{aligned}$$

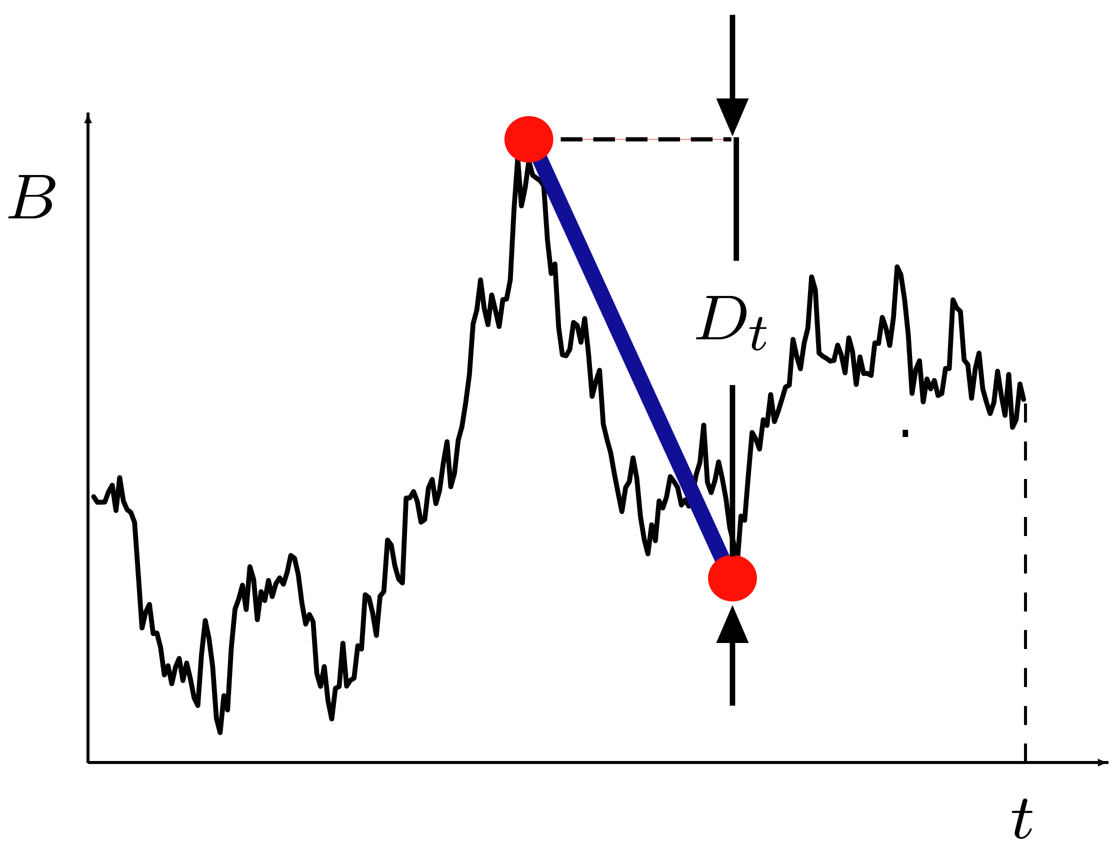
$$G(x) = \int_x^\infty e^{-u^2/2}\, du$$



The following three characteristics of draw-downs are the most important:

Maximum drawdown $\longrightarrow$ $\boxed{D_t = \max_{0 \le s \le s' \le t} (B_s - B_{s'})}$

(cf. $R_t = \max_{0 \le s, s' \le t} (B_s - B_{s'})$; so $D_t \le R_t$).



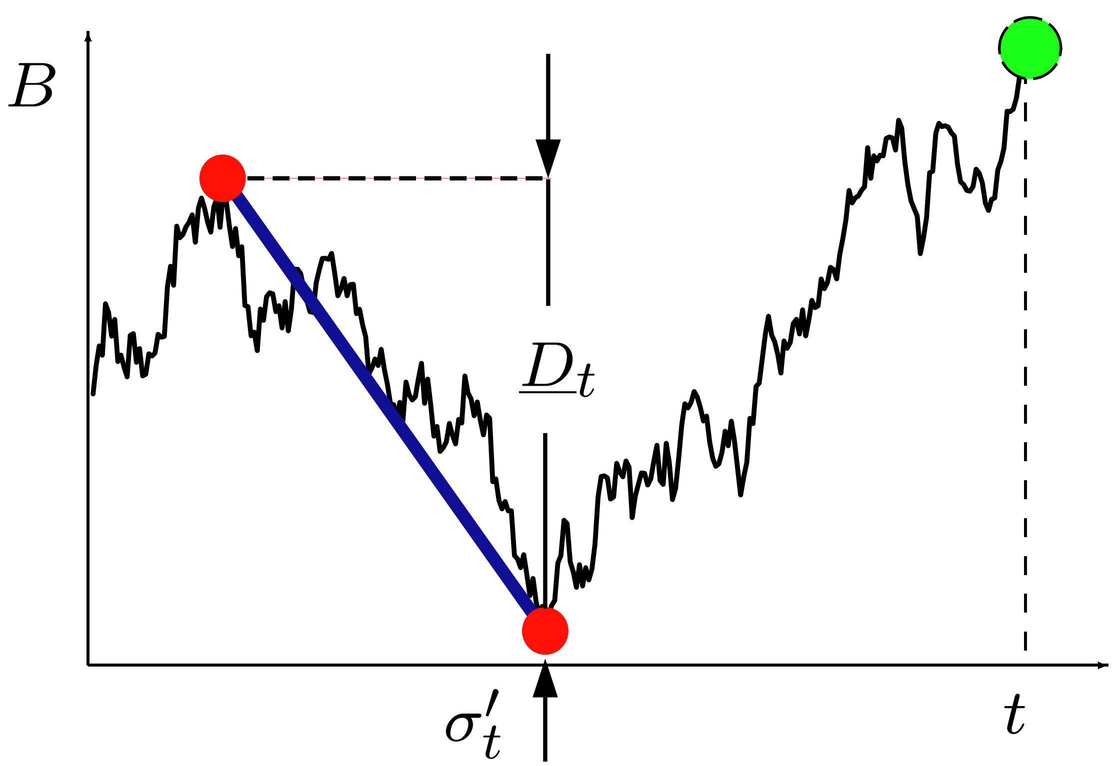
Drawdown from high "water mark" to the next low "water mark" $\longrightarrow$

$$\boxed{\overline{D}_t = B_{\sigma_t} - \min_{\sigma_t \le s' \le t} B_{s'}}$$
$$= \max_{0 \le s \le t} B_s - \min_{\sigma_t \le s' \le t} B_{s'}$$

(where $\sigma_t = \inf\{s \le t : B_s = \max_{u \le t} B_u\}$.

Drawdown from previous high "water mark" to the lowest "water mark" $\longrightarrow$ $\boxed{\underline{D}_t = \max_{0 \le s \le \sigma'_t} B_s - B_{\sigma'_t}}$

(where $\sigma'_t = \inf\{s \le t : B_s = \min_{u \le t} B_u\}$.



General results on D, $\overline{D}$, $\underline{D}$ for B:

$$(1)\quad \overline{D}_t = \underline{D}_t$$

$$(2)\quad D_t = \max(\overline{D}_t, \underline{D}_t)$$

$$(3)\quad D_t \overset{\text{law}}{=} \max_{s \le t} |B_s|$$

$$(4)\quad \overline{D}_t \overset{\text{law}}{=} \max_{g_t \le s \le t} |B_s|$$

where $g_t = \sup\{s \le t : B_s = 0\}$.

Distributional results on D_1, $\overline{D}_1$ for a standard Brownian motion $B = B^\circ$:

$$(5)\quad \mathsf{P}(D_1 \le x) = \mathsf{P}\Big(\max_{s \le 1} |B_s| \le x\Big)$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left\{ -\frac{\pi^2(2n+1)^2}{8x^2} \right\}$$

$$\boxed{\mathsf{E}D_1 = \mathsf{E}\max_{s \le 1} |B_s| = \sqrt{\tfrac{\pi}{2}} = 1.2533\ldots}$$

$$\mathsf{E}D_t = \sigma\sqrt{t}\sqrt{\tfrac{\pi}{2}} \quad (\text{for } \sigma B^\circ \text{ on } [0,t])$$

$$(6) \quad \mathsf{P}(\overline{D}_1 \le x) = \mathsf{P}\left(\max_{g_1 \le s \le 1} |B_s| \le x\right) = F_{\overline{D}_1}(x)$$

$$\begin{aligned} f_{\overline{D}_1}(x) &= \frac{dF_{\overline{D}_1}(x)}{dx} \\ &= \sqrt{\tfrac{8}{\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k e^{-\frac{1}{2}k^2 x^2} \end{aligned}$$

$$\boxed{\mathsf{E}\overline{D}_1 = \sqrt{\tfrac{8}{\pi}} \log 2 = 1.1061}$$

Note that

$$f_{R_1}(x) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-k^2x^2/2}, \quad x > 0,$$

$$F_{K_1}(x) = \mathsf{P}\Big(\max_{s\le 1} |b_s| \le x \Big),$$

$$f_{K_1}(x) = \frac{dF_{K_1}(x)}{dx} = 8x \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-k^2x^2},$$

where $b = (b_s)_{s\le 1}$ is a Brownian bridge ($b_s = B_s - sB_1$).

We note that

$$f_{R_1}(x) = \sqrt{\tfrac{2}{\pi}}\, \tfrac{1}{x}\, f_{K_1}(x).$$

So

$$\mathsf{E}R_1 = \sqrt{\tfrac{8}{\pi}} = 1.5957\ldots$$

and

$$\begin{array}{ccccc} \mathsf{E}\overline{D}_1 & \le & \mathsf{E}D_1 & \le & \mathsf{E}R_1 \\ \sqrt{\dfrac{8}{\pi}}\log 2 & \le & \sqrt{\dfrac{\pi}{2}} & \le & \sqrt{\dfrac{8}{\pi}} \\ 1.1061\ldots & \le & 1.2533\ldots & \le & 1.5957\ldots \end{array}$$

Lemma.

$$(1) \quad \overline{\mathbb{D}}_t \overset{\text{law}}{=} \underline{\mathbb{D}}_t$$

$$(2) \quad \mathbb{D}_t = \begin{cases} \overline{\mathbb{D}}_t = \underline{\mathbb{D}}_t & \textit{on } \{\sigma_t \le \sigma'_t\} \\ \max(\overline{\mathbb{D}}_t, \underline{\mathbb{D}}_t) & \textit{on } \{\sigma_t > \sigma'_t\} \end{cases}$$

$$(3) \quad \max(\overline{\mathbb{D}}_t, \underline{\mathbb{D}}_t) = \mathbb{D}_t \le R_t$$

<u>Known results about</u> R_t <u>and</u> T_d

$R = R_1$: $t = 1$, $\mu = 0$, $\sigma = 1$

W. Feller (1951) got for $f_R(x) = \dfrac{d\mathsf{P}(R \le x)}{dx}$, $x > 0$, the following formula:

$$f_R(x) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-\frac{k^2 x^2}{2}}.$$

Remark. If $b(t) = B_t - tB_1$, $t \le 1$, is a Brownian bridge, then for Kolmogorov's distribution $F_{\mathsf{K}}(x) = \mathsf{P}\Big(\sup_{t\le 1} |b(t)| \le x\Big)$ we have

$$F_{\mathsf{K}}(x) = 1 - 2\sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2x^2}$$
$$= \frac{\sqrt{2\pi}}{x} \sum_{k=1}^{\infty} e^{-(2k-1)^2\pi^2/x^2}$$
$$\Downarrow \quad (\theta\text{-function})$$
$$f_{\mathsf{K}}(x) = 8x \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-2k^2x^2}$$

Together with

$$f_R(x) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-\frac{k^2x^2}{2}}$$

we get

$$f_R(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{x}\, f_{\mathsf{K}}(x),$$

so

$$\mathsf{E}R = \sqrt{\frac{8}{\pi}} \quad (= 1.5957691216\ldots)$$

Theorem 1. $(t = 1,\ \underline{\underline{\mu = 0}},\ \sigma = 1.)$

(a) $\boxed{\mathbb{D}_1 \overset{\text{law}}{=} \max_{0 \le t \le 1} |B_t|}$

(b) *If* $F_{\mathbb{D}_1}(x) = \mathsf{P}(\mathbb{D}_1 \le x)$ *then* (*it is well known*)

$$F_{\mathbb{D}_1}(x) = 1 - \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \int_{-x}^{x} \left[e^{-\frac{(y+4kx)^2}{2}} - e^{-\frac{(y+2x+4kx)^2}{2}} \right] dy$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, e^{-\frac{\pi^2(2n+1)^2}{8x^2}}$$

(c) $\mathsf{E}\mathbb{D}_1 = \mathsf{E} \max_{0 \le t \le 1} |B_t| = \sqrt{\frac{\pi}{2}}$

$(\mathsf{E}\mathbb{D}_T = \sigma\sqrt{T}\sqrt{\frac{\pi}{2}})$

Proof. (a): Denote

$$M_t = \max_{s \le t} B_s, \qquad L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(|B_s| \le \varepsilon)\, ds.$$

By Lévy's theorem

$$(M_t - B_t, M_t; t \le 1) \overset{\text{law}}{=} (|B_t|, L_t; t \le 1).$$

Hence

$$\begin{aligned} \mathbb{D}_1 &= \max_{0 \le s \le s' \le 1} (B_s - B_{s'}) \\ &= \max_{0 \le s' \le 1} \Big(\max_{0 \le s \le s'} B_s - B_{s'} \Big) \\ &= \max_{0 \le s' \le 1} (M_{s'} - B_{s'}) \overset{\text{law}}{=} \max_{0 \le t \le 1} |B_t|. \end{aligned}$$

<u>Proof</u>. (c): We give two proofs. Let $\beta = (\beta_t)_{t\geq 0}$ be a Brownian motion. From self-similarity

$$(\beta_{at}; t \geq 0) \overset{\text{law}}{=} (a^{1/2}\beta_t; t \geq 0).$$

So if $s_1 = \inf\{t \geq 0 : |\beta_t| = 1\}$, then

$$\begin{aligned}
\mathsf{P}\Big(\sup_{0\leq t\leq 1} |\beta_t| \leq x\Big) &= \mathsf{P}\Big(\sup_{t\leq 1} |\beta_{t/x^2}| \leq 1\Big) \\
&= \mathsf{P}\Big(\sup_{t\leq 1/x^2} |\beta_t| \leq 1\Big) \\
&= \mathsf{P}\Big(s_1 \geq \frac{1}{x^2}\Big) = \mathsf{P}\Big(\frac{1}{\sqrt{s_1}} \leq x\Big),
\end{aligned}$$

i.e.,

$$\sup_{t\leq 1} |\beta_t| \overset{\text{law}}{=} \frac{1}{\sqrt{s_1}}.$$

The normal distribution property

$$\sqrt{\frac{2}{\pi}}\int\limits_0^\infty e^{-\frac{x^2}{2\sigma^2}}\,dx = \sigma$$

$$\Downarrow$$

$$\mathsf{E}\,\mathbb{D} = \mathsf{E}\sup_{0\le t\le 1}|\beta_t| = \mathsf{E}\frac{1}{\sqrt{s_1}} = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty \mathsf{E}e^{-\frac{x^2 s_1}{2}}\,dx$$

We have $\mathsf{E}e^{-\lambda s_1} = \dfrac{1}{\cosh\sqrt{2\lambda}}$. Hence

$$\mathsf{E}\,\mathbb{D} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{dx}{\cosh x} = 2\sqrt{\frac{2}{\pi}}\int\limits_0^\infty \frac{e^x\,dx}{e^{2x}+1}$$

$$= 2\sqrt{\frac{2}{\pi}}\int\limits_1^\infty \frac{dy}{1+y^2} = 2\sqrt{\frac{2}{\pi}}\,\arctan(x)\Big|_1^\infty$$

$$= 2\sqrt{\frac{2}{\pi}}\frac{\pi}{4} = \sqrt{\frac{\pi}{2}} \quad\Rightarrow\quad \mathsf{E}\,\mathbb{D} = \sqrt{\frac{\pi}{2}}$$

Second proof of the equality $\mathsf{E}\mathbb{D} = \sqrt{\frac{\pi}{2}}$ is based on the fact that

$$\sup_{t \le 1} |\beta_t| \overset{\text{law}}{=} \frac{1}{2} \int_0^1 \frac{du}{R_u^{(2)}},$$

where $R_s^{(2)}$ is a Bessel-2:

$$R_s^{(2)} = \widehat{\beta}_s + \frac{1}{2} \int_0^s \frac{du}{R_u^{(2)}}.$$

Thus,

$$\mathsf{E}\,\mathbb{D} = \mathsf{E} \sup |\beta_t| = \mathsf{E} R_1^{(2)} = \mathsf{E}\sqrt{\xi_1^2 + \xi_2^2} = \sqrt{\frac{\pi}{2}},$$

$$\xi_1 \perp\!\!\!\perp \xi_2, \qquad \xi_i \sim \mathcal{N}(0,1).$$

Theorem 2. ($t = 1$, $\mu = 0$, $\sigma = 1$, $\overline{\mathbb{D}}_1 = B_{\sigma_1} - \min_{\sigma_1 \le s' \le 1} B_{s'}$.)

(a) $\overline{\mathbb{D}}_1 \overset{\text{law}}{=} \sup_{g_1 \le s \le 1} |B_s|$, *where*

$$g_1 = \sup\{t \le 1 : B_t = 0\}.$$

(b) $f_{\overline{\mathbb{D}}_1}(x) = \sqrt{\frac{8}{\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k e^{-\frac{k^2 x^2}{2}}, \quad x > 0.$

(c) $\mathsf{E}\overline{\mathbb{D}}_1 = \sqrt{\frac{8}{\pi}} \log 2 \; (= 1.1061\ldots)$,

$$\begin{array}{ccccc} \mathsf{E}\overline{\mathbb{D}}_1 & \le & \mathsf{E}\mathbb{D}_1 & \le & \mathsf{E}R \\ \sqrt{\frac{8}{\pi}}\log 2 & \le & \sqrt{\frac{\pi}{2}} & \le & \sqrt{\frac{8}{\pi}} \\ 1.1061... & \le & 1.2533... & \le & 1.5957... \end{array}$$

Proof (a) By Lévy's theorem

$$\begin{pmatrix} M_t - B_t, M_t, B_t; \\ t \leq 1 \end{pmatrix} \overset{\text{law}}{=} \begin{pmatrix} |B_t|, L_t, L_t - |B_t|; \\ t \leq 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} M_t - B_t, M_t, B_t; \\ \sigma_1 \leq t \leq 1 \end{pmatrix} \overset{\text{law}}{=} \begin{pmatrix} |B_t|, L_t, L_t - |B_t|; \\ g_1 \leq t \leq 1 \end{pmatrix}$$

where $\sigma_1 = \min\left\{ s \leq 1 \,:\, B_s = \max_{u \leq 1} B_u \right\}$.

Therefore

$$\Big(B_{\sigma_1}, \max_{\sigma_1 \le t \le 1}(M_t - B_t - M_t)\Big)$$

$$\stackrel{\text{law}}{=} \Big(L_{g_1} - |B_{g_1}|, \max_{g_1 \le t \le 1}(|B_t| - L_t)\Big)$$

$$= \Big(L_{g_1}, \max_{g_1 \le t \le 1} |B_t| - L_{g_1}\Big)$$

(since $B_{g_1} = 0$ and $L_t = L_{g_1}$ for $g_1 \le t \le 1$).

Finally,

$$\overline{\mathbb{D}}_1 = B_{\sigma_1} - \min_{\sigma_1 \le t \le 1} B_t = B_{\sigma_1} + \max_{\sigma_1 \le t \le 1}(-B_t)$$

$$= B_{\sigma_1} + \max_{\sigma_1 \le t \le 1}(M_t - B_t - M_t)$$

$$\stackrel{\text{law}}{=} L_{g_1} + \max_{g_1 \le t \le 1} |B_t| - L_{g_1} = \max_{g_1 \le t \le 1} |B_t|$$

Albert N. SHIRYAEV

THREE TOPICS on

FINANCIAL STATISTICS, STOCHASTICS and OPTIMIZATION

Topic III. B: A stochastic version of the rule "BUY & HOLD" (Brief survey)

Optimization in the portfolio and rebalancing problems (results based on using optimal stopping theory; stochastic "Buy and Hold rule").

In this lecture we describe the structure of the

OPTIMAL ONE-TIME REBALANCING STRATEGY

on the (B,S)-market in the case of the possible spontaneous change of its parameter.

Let us begin with the (B,S)-model (**Black–Scholes**)

$$\boxed{\begin{aligned} dB_t &= rB_t\,dt, & B_0 &= 1 \\ dS_t &= S_t\,(\mu\,dt + \sigma\,dW_t), & S_0 &= 1 \end{aligned}} \tag{1}$$

$[W = (W_t)_{t\le T}$ is a standard Wiener process (Brownian motion)].

Discounted prices $P_t = S_t/B_t$ solve the equation

$$dP_t = P_t\Big((\mu - r)\,dt + \sigma\,dW_t\Big), \qquad P_0 = 1,$$

and

$$P_t = \exp\{\nu t + \sigma W_t\}, \quad \text{where} \ \ \nu = \mu - r - \tfrac{1}{2}\sigma^2.$$

Let $U = U(x)$ be a utility function (*e.g.*, $U(x) = \log x$ or $U(x) = x$).

In the paper:

A. Shiryaev, Z. Xu, X. Y. Zhou, **Thou Shalt Buy and Hold**,
[*Quantitative Finance*, **8**:8 (2008), 765–776],

the following problem was considered:

To find an optimal stopping time τ^* such that

$$\boxed{\mathsf{E}\, U\left(\frac{P_{\tau^*}}{M_T}\right) = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E}\, U\left(\frac{P_\tau}{M_T}\right),}$$

where $\mathfrak{M}_T$ is the class of all stopping times taking values in $[0, T]$ and $M_T = \sup_{t \le T} P_t$.

The main result of our paper mentioned above implies that

for the **LINEAR** function $U(x) = x$ the (degenerated (!)) stopping time

$$\boxed{\tau^* = \begin{cases} T & \text{if } \nu > 0, \\ 0 & \text{if } \nu \le -\sigma^2/2, \end{cases}}$$

is **optimal**.

(In the case $-\sigma^2/2 < \nu \le 0$ the optimal stopping time is $\tau^* = 0$; this was shown by **J. du Toit, G. Peskir***.)

* "Selling a stock at the ultimate maximum", *Working paper*, Manchester University, 2008.

In the case of the logarithmic function $U(x) = \log x$

$$\sup_{\tau\in\mathfrak{M}_T} \mathsf{E}\log\frac{P_\tau}{M_T} = \sup_{\tau\in\mathfrak{M}_T} \mathsf{E}\Big[\nu\tau + \sigma W_\tau - M_T\Big] = \sup_{\tau\in\mathfrak{M}_T} \mathsf{E}[\nu\tau + \sigma W_\tau] - \mathsf{E}M_T$$

$$= \sup_{\tau\in\mathfrak{M}_T} \mathsf{E}\nu\tau - \mathsf{E}M_T = \begin{cases} \nu T - \mathsf{E}M_T & \text{if} \quad \nu > 0, \\ -\mathsf{E}M_T & \text{if} \quad \nu \le 0. \end{cases}$$

So, in this logarithmic case an optimal stopping time is

$$\tau^* = \begin{cases} T & \text{if} \quad \nu > 0, \\ 0 & \text{if} \quad \nu \le 0, \end{cases}$$

which is the same as for the linear utility function.

In the present lecture we consider the following generalization of the above problem using the model which has been proposed a long time ago by the author in these papers on the "quickest detection problem of the spontaneously appearing effects".

More exactly, we shall consider the following two models (I and II):

$$\textbf{I:} \quad \boxed{dS_t^{(\mathrm{I})} = S_t^{(\mathrm{I})}\left(\mu^{(\mathrm{I})}(t,\theta)\,dt + \sigma\,dW_t\right), \qquad S_0 = 1,}$$

and

$$\textbf{II:} \quad \boxed{dS_t^{(\mathrm{II})} = S_t^{(\mathrm{II})}\left(\mu^{(\mathrm{II})}(t,\theta)\,dt + \sigma\,dW_t\right), \qquad S_0 = 1,}$$

where $\theta = \theta(\omega)$ is a random variable which is independent of W and assumes values in $\mathbb{R}_+ = [0,\infty)$ and

$$\mu^{(\mathrm{I})}(t,\theta) = \begin{cases} \mu_1, & t < \theta, \\ \mu_2, & t \ge \theta, \end{cases} \qquad \mu^{(\mathrm{II})}(t,\theta) = \begin{cases} \mu_2, & t < \theta, \\ \mu_1, & t \ge \theta; \end{cases}$$

the parameters μ_1 and μ_2 are assumed such that $\mu_1 > \mu_2$ and

$$\begin{aligned} \nu_1 &\equiv \mu_1 - r - \tfrac{1}{2}\sigma^2 > 0, \\ \nu_2 &\equiv \mu_2 - r - \tfrac{1}{2}\sigma^2 < 0, \end{aligned} \quad \text{so that} \quad \mu_2 - \tfrac{1}{2}\sigma^2 < r < \mu_1 - \tfrac{1}{2}\sigma^2.$$

Consider the **MODEL (I):** $(B, S^{(\mathrm{I})})$. In this case, starting from the initial time $t = 0$ the driving parameter is μ_1. If this value remains unchanged on the whole interval $[0, T]$ and $\nu_1 \equiv \mu_1 - r - \frac{1}{2}\sigma^2 > 0$, then by the previous result we should

hold the stock until time $t = T$ and sell it at this time.

But in fact the model (I) admits that at a certain random time θ the regime switches from μ_1 to μ_2, and if $\nu_2 \equiv \mu_2 - r - \frac{1}{2}\sigma^2 < 0$, then, again by the previous results, we should

sell the stock at this time θ.

However, this time is unobservable and so the time of selling must depend on the "correct" estimation of the time θ.

All these considerations lead to the following optimal problem:

To find "**one-time rebalancing**" stopping time τ_T^* such that

$$V_T^{(\mathrm{I})} \equiv \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \log \frac{P_\tau^{(\mathrm{I})}}{M_T^{(\mathrm{I})}} = \mathsf{E} \log \frac{P_{\tau_T^*}^{(\mathrm{I})}}{M_T^{(\mathrm{I})}}.$$

Since

$$P_t^{(\mathrm{I})} = \exp\Bigg\{ \overbrace{\int_0^t \underbrace{\nu^{(\mathrm{I})}(s,\theta)}_{:=\,\mu^{(\mathrm{I})}(s,\theta)-r-\frac{1}{2}\sigma^2} \, ds}^{(=:\,H_t^{(\mathrm{I})})} + \sigma W_t \Bigg\}$$

and $\mathsf{E} W_\tau = 0$ for $\tau \in \mathfrak{M}_T$, we deduce that

$$V_T^{(\mathrm{I})} = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \int_0^\tau \nu^{(\mathrm{I})}(s,\theta)\, ds - \mathsf{E} \log M_T^{(\mathrm{I})}.$$

It is interesting that this "$V_T^{(\mathrm{I})}$-**criterion**" is equivalent to the following "$W_T^{(\mathrm{I})}$-**criterion**":

$$\boxed{W_T^{(\mathrm{I})} = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \log \left[S_\tau^{(\mathrm{I})} \frac{B_T}{B_\tau} \right]}.$$

The value $S_\tau^{(\mathrm{I})} B_T / B_\tau$ determines the capital at time T, if at time τ we sell the stock and put the gained value $S_\tau^{(\mathrm{I})}$ on the bank account. Since

$$S_\tau^{(\mathrm{I})} \frac{B_T}{B_\tau} = \frac{S_\tau^{(\mathrm{I})}}{B_\tau} B_T = P_\tau^{(\mathrm{I})} B_T,$$

we see that

$$W_T^{(\mathrm{I})} = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \log P_\tau^{(\mathrm{I})} + \mathsf{E} \log B_T.$$

Hence

$$\boxed{W_T^{(\mathrm{I})} = V_T^{(\mathrm{I})} + \mathsf{E} \log M_T^{(\mathrm{I})} + \mathsf{E} \log B_T}.$$

Consider now the **MODEL (II)**, where $\mu_2 \to \mu_1$ at time θ. In this case, our "one-time rebalancing" problem can be reformulated as the problem of buying (after the time θ) the stock at minimal price:

$$\boxed{V_T^{(\mathrm{II})} = \inf_{\tau \in \mathfrak{M}_T} \mathsf{E} \log \frac{P_\tau^{(\mathrm{II})}}{\min_{t \in [0,T]} P_t^{(\mathrm{II})}}}\,.$$

The corresponding "$W^{(\mathrm{II})}$-**problem**" is defined by the criterion:

$$W_T^{(\mathrm{II})} = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \log \left[B_\tau \frac{S_T^{(\mathrm{II})}}{S_\tau^{(\mathrm{II})}} \right] = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \log \left[\frac{B_\tau}{S_\tau^{(\mathrm{II})}} S_T^{(\mathrm{II})} \right]$$
$$= \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \log \frac{1}{P_\tau^{(\mathrm{II})}} + \mathsf{E} \log S_T^{(\mathrm{II})} = - \inf_{\tau \in \mathfrak{M}_T} \mathsf{E} \log P_\tau^{(\mathrm{II})} + \mathsf{E} \log S_T^{(\mathrm{II})}.$$

Hence

$$\boxed{W_T^{(\mathrm{II})} = -V_T^{(\mathrm{II})} + \mathsf{E} \log S_T^{(\mathrm{II})} - \mathsf{E} \log \min_{t \in [0,T]} P_t^{(\mathrm{II})}}\,.$$

Consider in details the “$V_T^{(\mathrm{I})}$-**criterion**”, where, as we have seen,

$$V_T^{(\mathrm{I})} = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \int_0^\tau \nu^{(\mathrm{I})}(s,\theta)\,ds - \mathsf{E} \log M_T^{(\mathrm{I})}.$$

To find an optimal stopping time in the problem

$$\sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \int_0^\tau \nu^{(\mathrm{I})}(s,\theta)\,ds,$$

we need an assumption about the distribution of the (hidden) random variable θ. Let us make the following assumption:

$$\boxed{\mathsf{P}(\theta = 0) = \pi, \qquad \mathsf{P}(\theta > t \,|\, \theta > 0) = e^{-\lambda t}, \quad \text{where } \lambda > 0 \text{ is known}}\,.$$

Having this distribution it is easy to find the *aposteriori* probability $\pi_\tau = \mathsf{P}(\theta \le t \,|\, \mathcal{F}_\tau)$, where $\mathcal{F}_t = \sigma(H_s^{(\mathrm{I})},\, s \le t)$.

Indeed, introduce the processes

$$\varphi_t = \frac{\pi_t}{1-\pi_t} \quad \text{and} \quad L_t(H) = \frac{d(\mathsf{P}^{\nu_1}\,|\,\mathcal{F}_t^H)}{d(\mathsf{P}^{\nu_2}\,|\,\mathcal{F}_t^H)};$$

$L_t(H)$ is the likelihood process, *i.e.*, the Radon–Nikodým derivative of the measure $\mathsf{P}^{\nu_1}|\mathcal{F}_t^H$ w.r.t. the measure $\mathsf{P}^{\nu_2}|\mathcal{F}_t^H$, where P^{ν_i} is the distribution of the process $(H_t^i)_{t\ge 0}$ with

$$dH_t^i = \nu_i\,dt + \sigma\,dW_t.$$

It is well known that for each $i=1,2$

$$L_t(H^i) = \exp\left\{\frac{\nu_2-\nu_1}{\sigma^2}\,H_t^i - \frac{1}{2}\,\frac{\nu_2^2-\nu_1^2}{\sigma^2}\,t\right\} \quad (\mathsf{P}^{\nu_i}\text{-a.s.}).$$

By Itô's formula, this implies that

$$dL_t(H^i) = L_t(H^i)\frac{\nu_2-\nu_1}{\sigma^2}\,\{dH_t^i - \nu_1\,dt\}.$$

By Bayes' formula, taking for simplicity $\pi = 0$, we get

$$\varphi_t = \lambda \int_0^t \frac{e^{\lambda t} L_t(H^{(\mathrm{I})})}{e^{\lambda u} L_u(H^{(\mathrm{I})})}\, du.$$

From this representation, again by Itô's formula, we find

$$d\varphi_t = \left[\lambda(1+\varphi_t) - \varphi_t \nu_1 \frac{\nu_2 - \nu_1}{\sigma^2}\right] dt + \varphi_t \frac{\nu_2 - \nu_1}{\sigma^2}\, dH_t^{(\mathrm{I})}$$

and

$$\begin{aligned} d\pi_t = (1-\pi_t)\left[\lambda - \nu_1 \frac{\nu_2 - \nu_1}{\sigma^2}\, \pi_t - \frac{(\nu_2 - \nu_1)^2}{\sigma^2}\, \pi_t^2\right] dt \\ + \frac{\nu_2 - \nu_1}{\sigma^2}\, \pi_t (1-\pi_t)\, dH_t^{(\mathrm{I})}. \end{aligned}$$

Let $\overline{W}_t^{(\mathrm{I})} := \dfrac{1}{\sigma}\Big[H_t^{(\mathrm{I})} - \nu_1 t - (\nu_2 - \nu_1)\int_0^t \pi_s\, ds\Big]$. This "innovation" process is remarkable: it is known to be a *Wiener process* (with respect to $(\mathcal{F}_t^{H^{(\mathrm{I})}})_{t\ge 0}$). Taking into account the definition of $\overline{W}_t^{(\mathrm{I})}$, we find that

$$d\pi_t = \lambda(1-\pi_t)\,dt + \frac{\nu_2-\nu_1}{\sigma}\,\pi_t(1-\pi_t)\,d\overline{W}_t^{(\mathrm{I})}.$$

Return to the expectation $\mathsf{E}\int_0^\tau \nu^{(\mathrm{I})}(s,\theta)\,ds$. We have

$$H_t^{(\mathrm{I})} =: \int_0^\tau \nu^{(\mathrm{I})}(s,\theta)\,ds + \sigma W_t.$$

So, for $\tau \in \mathfrak{M}_T$, from the definition of $\overline{W}_t^{(\mathrm{I})}$, $t \le T$, we obtain that

$$\begin{aligned}\mathsf{E}\int_0^\tau \nu^{(\mathrm{I})}(s,\theta)\,ds = \mathsf{E}H_\tau^{(\mathrm{I})} &= \mathsf{E}\Big\{\sigma\overline{W}_\tau^{(\mathrm{I})} + \nu_1\tau + (\nu_2-\nu_1)\int_0^\tau \pi_s\,ds\Big\}\\ &= \mathsf{E}\int_0^\tau \Big[\nu_1 - (\nu_2-\nu_1)\pi_s\Big]\,ds.\end{aligned}$$

This implies that we need find a stopping time τ_T^* such that

$$\sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \int_0^\tau \nu_1(s,\theta)\,ds = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \int_0^\tau \Big[\nu_1 + (\nu_2 - \nu_1)\pi_t\Big]\,dt$$

$$= \mathsf{E} \int_0^{\boldsymbol{\tau_T^*}} \Big[\nu_1 + (\nu_2 - \nu_1)\pi_t\Big]\,dt, . \qquad (*)$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

since the process π_t is continuous, then if $\pi_0 < \frac{\nu_1}{\nu_2 - \nu_1}$, one should continue observations at least till the time

$$\boxed{\widetilde{\boldsymbol{\tau}}_{\boldsymbol{T}} = \min\left\{0 \le t \le T \colon \pi_t \ge \widetilde{\boldsymbol{A}}_{\boldsymbol{\infty}} := \frac{\nu_1}{\nu_1 - \nu_2}\right\}}, \quad \text{i.e.,} \quad \boldsymbol{\tau_T^*} \ge \widetilde{\boldsymbol{\tau}}_{\boldsymbol{T}}.$$

Note also that

- if $\pi_t \equiv 0$ (i.e., from the beginning we have parameter $\nu_1 > 0$), then it follows from $(*)$ that $\tau_T^* = T$;
- If $\pi_t \equiv 1$ (i.e., from the beginning we have parameter $\nu_2 > 0$), then evidently $(*)$ implies that $\tau_T^* = 0$.

These deterministic Buy & Hold rules were already described above.

To find an optimal stopping time τ_T^*, let us consider more carefully the expression

$$I(t) = \int_0^t \Big[\nu_1 + (\nu_2 - \nu_1)\pi_t\Big]\,dt.$$

Since

$$d\pi_t = \lambda(1-\pi_t)\,dt + \frac{\nu_2-\nu_1}{\sigma}\,\pi_t(1-\pi_t)\,d\overline{W}_t^{(\mathrm{I})},$$

we see (taking $\pi_0 = 0$) that

$$\lambda t = \pi_t + \lambda\int_0^t \pi_s\,ds - \int_0^t \frac{\nu_2-\nu_1}{\sigma}\,\pi_s(1-\pi_s)\,d\overline{W}_s^{(\mathrm{I})}.$$

So,

$$t = \frac{\pi_t}{\lambda} + \int_0^t \pi_s\,ds - \int_0^t \frac{\nu_2-\nu_1}{\lambda\sigma}\,\pi_s(1-\pi_s)\,d\overline{W}_s^{(\mathrm{I})}$$

and

$$\nu_1 t = \frac{\nu_1}{\lambda}\pi_t + \nu_1\int_0^t \pi_s\,ds - \int_0^t \frac{(\nu_2-\nu_1)\nu_1}{\lambda\sigma}\,\pi_s(1-\pi_s)\,d\overline{W}_s^{(\mathrm{I})}. \qquad (4)$$

Using the representation (4), we find that

$$\mathsf{E}I(\tau) = \mathsf{E}\int_0^\tau \Big[\nu_1 + (\nu_2 - \nu_1)\pi_t\Big]\,dt = \mathsf{E}\Big\{\frac{\nu_1}{\lambda}\pi_\tau + \nu_2\int_0^\tau \pi_t\,dt\Big\}$$

$$= \frac{\nu_1}{\lambda}\Big\{\pi_\tau + \frac{\nu_2}{\nu_1}\lambda\int_0^\tau \pi_t\,dt\Big\} = -\frac{\nu_1}{\lambda}\Big\{-\pi_\tau + \frac{|\nu_2|}{\nu_1}\lambda\int_0^\tau \pi_t\,dt\Big\}.$$

Hence, letting $c = (|\nu_2|/\nu_1)\lambda$, we get

$$\sup_{\tau\in\mathfrak{M}_T} \mathsf{E}I(\tau) = -\frac{\nu_1}{\lambda}\inf_{\tau\in\mathfrak{M}_T}\mathsf{E}\Big\{-\pi_\tau + c\int_0^\tau \pi_t\,dt\Big\}$$

$$= -\frac{\nu_1}{\lambda}\bigg\{-1 + \inf_{\tau\in\mathfrak{M}_T}\mathsf{E}\Big\{(1-\pi_\tau) + c\int_0^\tau \pi_t\,dt\Big\}\bigg\}$$

$$= \frac{\nu_1}{\lambda} - \frac{\nu_1}{\lambda}\underbrace{\inf_{\tau\in\mathfrak{M}_T}\mathsf{E}\Big\{(1-\pi_\tau) + c\int_0^\tau \pi_t\,dt\Big\}}_{=:\,\boldsymbol{V_T}}.$$

$$\boxed{V_T = \inf_{\tau \in \mathfrak{M}_T} \mathsf{E}\Big\{(1-\pi_\tau) + c\int_0^\tau \pi_t\,dt\Big\}}$$

The **case $T=\infty$** was investigated in an author's monograph*, from which it follows that an optimal stopping time is

$$\tau^*_\infty = \inf\{t\colon \pi_t \ge A^*_\infty\},$$

where A^*_∞ is a unique root of the equation

$$\boxed{C\int_0^{A^*_\infty} \exp\Big\{-\Lambda[H(\pi)-H(y)]\Big\}\frac{dy}{y(1-y)^2} = 1,} \qquad \textbf{(**)}$$

where

$$C = \frac{c}{\rho}, \quad \Lambda = \frac{\lambda}{\rho}, \quad H(y) = \log\frac{y}{1-y} - \frac{1}{y}, \quad \rho = \frac{(\nu_1-\nu_2)^2}{2\sigma^2}.$$

$\Big[A^*_\infty$ can be shown to satisfy $A^*_\infty > \widetilde{A}_\infty \ \Big(= \dfrac{\nu_1}{\nu_1-\nu_2} = \dfrac{\nu_1}{\nu_1+|\nu_2|}\Big).\Big]$

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*A. N. Shiryaev. Optimal Stopping Rules, Springer, 1978, 2008; Chap. 4, § 4.4 "The problem of disruption for a Wiener process".

In the **case $T < \infty$** the optimal stopping time τ_T^* has the following form:

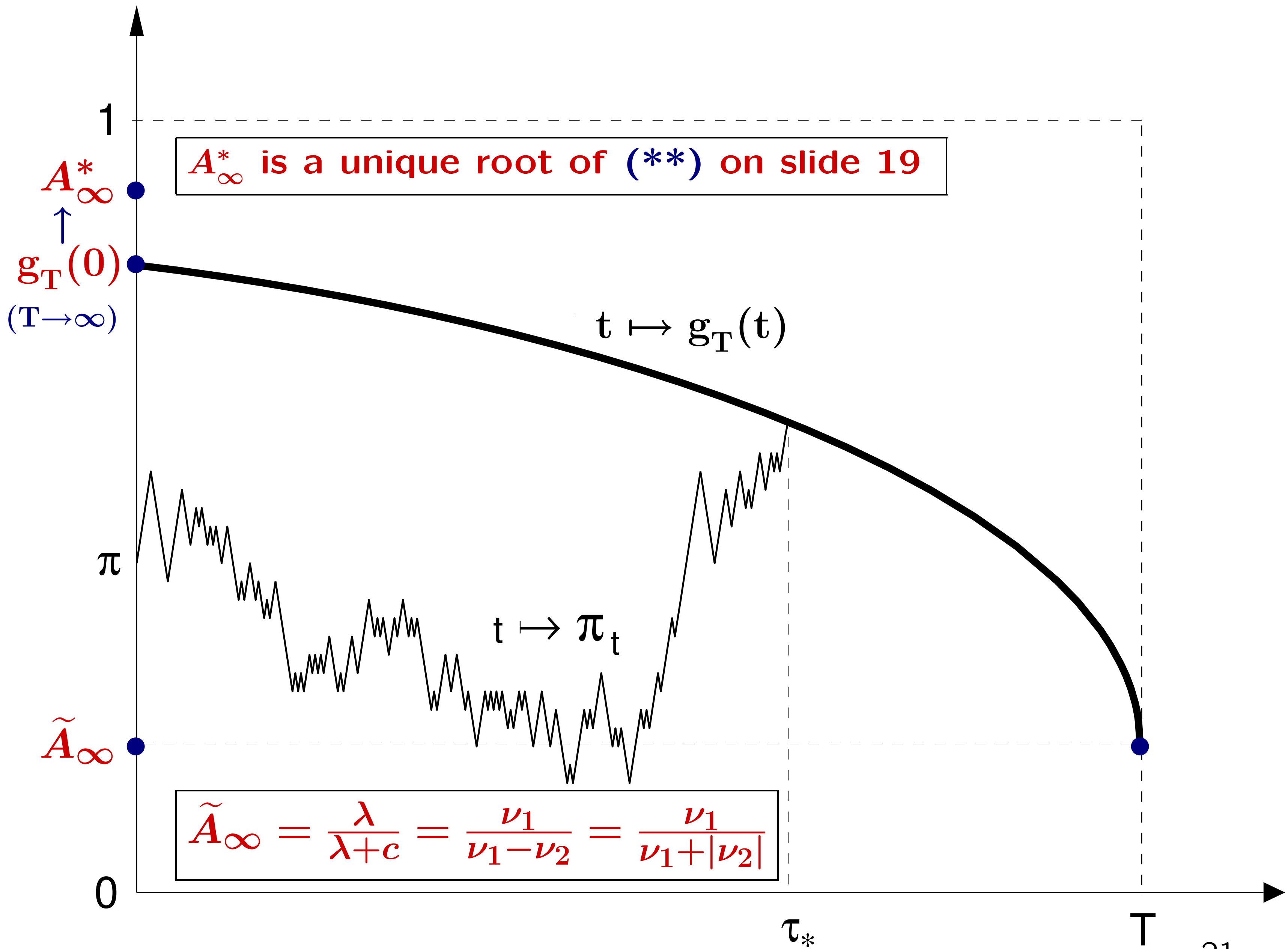
$$\tau_T^* = \inf\{0 \le t \le T \colon \pi_t \ge g_T(t)\},$$

where the optimal stopping boundary $g_T(t) =: g(t)$, $0 \le t \le T$, is a unique solution of the nonlinear integral equation*:

$$\begin{aligned} \mathsf{E}_{t,g(t)}\pi_T = g(t) &+ c \int_0^{T-t} \mathsf{E}_{t,g(t)}\Big[\pi_{t+u}\, I(\pi_{t+u} < g(t+u))\Big]\, du \\ &+ \lambda \int_0^{T-t} \mathsf{E}_{t,g(t)}\Big[(1-\pi_{t+u})\, I(\pi_{t+u} < g(t+u))\Big]\, du. \end{aligned}$$

Note that $g_T(t) \uparrow A_\infty^*$ as $T \to \infty$ for all $t \ge 0$.

*See details in: G. Peskir, A. Shiryaev. Optimal Stopping and Free-Boundary problems, Birkhäuser, 2006; Chap. VI, Sec. 22, Fig. VI.2.



1
A^*_∞ is a unique root of (**) on slide 19
A^*_∞
$\uparrow$
$g_T(0)$
$(T\to\infty)$
$t \mapsto g_T(t)$
π
$t \mapsto \pi_t$
$\widetilde{A}_\infty$
$\widetilde{A}_\infty = \frac{\lambda}{\lambda+c} = \frac{\nu_1}{\nu_1-\nu_2} = \frac{\nu_1}{\nu_1+|\nu_2|}$
0
τ_*
T

$\boxed{W_T^{(\mathrm{I})} = \sup_{\tau \in \mathfrak{M}_T} \mathsf{E} \log \left[S_\tau^{(\mathrm{I})} \dfrac{B_t}{B_\tau} \right]}$: as we have seen, for this criterion the optimal stopping time is the same as for $V_T^{(\mathrm{I})}$-criterion. Hence

the above stopping time $\tau_T^* = \inf\{0 \le t \le T \colon \pi_t \ge g(t)\}$ is optimal.

- Let us mention that the $W_T^{(\mathrm{I})}$-criterion was considered by **Ch. Blanchet-Scalliet, A. Diop, R. Gibson, D. Talay, E. Taure**.* In particular, they proposed ad hoc to use for each $T > 0$ the stopping time $\tau_\infty^* = \inf\{t \le T \colon \pi_t \ge A^*\}$. From the above results it follows that their method is only **almost optimal**. (The **optimal** stopping time, as we have demonstrated, is the time τ_T^*.)

Criteria $V_T^{(\mathrm{II})}$ and $W_T^{(\mathrm{II})}$ can be investigated in a similar way.

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* "Technical analysis compared to mathematical models based methods under parameters mis-specification", J. Banking Fin., 31 (2007), 1351–1373.

It is interesting to note that the process (π_t) proves to be optimal in other formulations of the problem "when sell the stock". For example, **M. Beibel and H. R. Lerche**[*] showed that the stopping time

$$\sigma_\infty^* = \inf\{t \geq 0 \colon \pi_t \geq B_\infty^*\},$$

where B_∞^* is a certain constant, is optimal in the following problem:

$$\sup_{\sigma \in \mathfrak{M}_\infty} \mathsf{E}\, \frac{S_\sigma}{B_\sigma}$$

provided that $\mu_2 < r < \mu_1$ and $\mu_1 < \lambda + r$.

[*] "A new look at optimal stopping problems related to mathematical finance", Statistica Sinica, 7:1 (1997), 93–108.