On the Market Viability under Proportional Transaction Costs

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Outline

- FTAP and Market Viability for Continuous Time Models
  - Frictionless Market: FTAP (NA, NFLVR)
  - Frictionless Market: Market Viability (NUPBR, NA1)
  - Market with Frictions: FTAP (Robust NFLVR)
- Motivation
- Preparation (some new definitions)
- Main Results (A new equivalence result)
- Sketch of the Proof
- Some Future Work

Frictionless Market: FTAP and NFLVR

- Market: One riskless asset (numeraire) and d risky assets S which is a d-dimensional semimartingale.

- Definition: The semimartingale S satisfies the condition of
  (i) **No Arbitrage** (NA) if $C \cap \mathbb{L}^{\infty}_{+} = \{0\}$.
  (ii) **No Free Lunch with Vanishing Risk** (NFLVR) if $\bar{C} \cap \mathbb{L}^{\infty}_{+} = \{0\}$.

  Here, we define

$$K_0 = \{(H \cdot S)_T : \quad H \text{ is admissible}\},$$

  and $C_0 = K_0 - \mathbb{L}^0_+$, i.e., the cone of random variables dominated by elements of K_0. $C = C_0 \cap \mathbb{L}^{\infty}$ and $\bar{C}$ is the closure of C w.r.t. the norm topology of $\mathbb{L}^{\infty}$.

Frictionless Market: FTAP and NFLVR

- Definition: An **Equivalent Local Martingale Measure** (ELMM) is a probability measure $\mathbb{Q} \sim \mathbb{P}$, such that S is a $\mathbb{Q}$-local martingale.

- FTAP: *Delbaen and Schachermayer* (1994), (1998)

$$\text{(NFLVR)} \;\Leftrightarrow\; \text{Existence of (ELMM).}$$

Frictionless Market: Market Viability and NUPBR

- Market Viability: The utility maximization problem on the terminal wealth admits the optimal solution in the given market.

- Observations: Some financial problems are well-defined and solvable even if ($NFLVR$) or (NA) are not satisfied.

- Definition: The semimartingale S satisfies **No Unbounded Profit with Bounded Risk** if the set

$$K = \{X_T : X_t = 1 + (H \cdot S)_t \geq 0, \quad H \text{ is 1-admissible}\},$$

is bounded in probability.

Frictionless Market: Market Viability and NUPBR

- Definition : The nonnegative process Y is called a local martingale deflator (LMD) of S, if SY is a $\mathbb{P}$-local martingale.

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Recall that

$$\text{(NA)} + \text{(NUPBR)} \;\equiv\; \text{(NFLVR)}.$$

Markets with Transaction costs

- Market: one risk-free bond B, normalized to be 1, and one risky asset S.
- Probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ that satisfies the usual conditions of right continuity and completeness.
- Bid-ask spread: $[(1-\lambda)S, (1+\lambda)S]$.

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  $(S_t)_{t \in [0,T]}$ is adapted to $(\mathcal{F}_t)_{t \in [0,T]}$ with strictly positive and locally bounded càdlàg paths.

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- S may not be a semimartingale. For e.g. fractional Brownian motion is kosher.

Consistent Price System

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- Definition: Given the stock price $(S_t)_{t\in[0,T]}$ with transaction cost $(\lambda_t)_{t\in[0,T]}$ such that $0 < \lambda_t < 1$ a.s. for all $t \in [0,T]$. The pair $(\tilde{S}, \mathbb{Q})$ is called a CPS if

$$(1-\lambda_t)S_t \leq \tilde{S}_t \leq (1+\lambda_t)S_t, \quad \text{a.s.} \quad \forall t \in [0,T],$$

where $(\tilde{S}_t)_{t\in[0,T]}$ is a local martingale under $\mathbb{Q}$ and $\mathbb{Q} \sim \mathbb{P}$.

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$$\inf_{t\in[0,T]} \left(\lambda_t S_t - |S_t - \tilde{S}_t|\right) > 0, \quad \text{a.s.},$$

the pair $(\tilde{S}, \mathbb{Q})$ is said to be a strictly consistent price system (SCPS).

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- With transaction costs, CPS plays the same role as ELMM as the dual element to option pricing and optimal investment problems.

FTAP in Continuous Time Models

FTAP: *Guasoni, Lepinette and Rasonyi* (2012)

$$\text{(RNFLVR)} \;\Leftrightarrow\; (SCPS).$$

Our Motivation

- Frictionless Market: (NFLVR) $\Leftrightarrow$ (ELMM) $\longrightarrow$

  Market with Frictions: (NA) for any $\lambda > 0 \Leftrightarrow$ (CPS) for any $\lambda > 0$.
  or (RNFLVR) $\Leftrightarrow$ (SCPS) for simple strategies.

- Frictionless Market: (NFLVR) $\Leftrightarrow$ (ELMM) $\longrightarrow$

  Frictionless Market: (NUPBR) $\Leftrightarrow$ (LMD)

- Question: What can we say about the Market Viability with proportional transaction costs ?

A new definition

Extend the definition of CPS to local martingales:

Definition: Given the stock price $(S_t)_{t\in[0,T]}$ with transaction cost λ_t such that $0 < \lambda_t < 1$ a.s. for all $t \in [0, T]$. The pair $(\tilde{S}, Z)$ is called a consistent local martingale system (CLMS) if $\tilde{S}$ is a semimartingale satisfying

$$(1-\lambda_t)S_t \leq \tilde{S}_t \leq (1+\lambda_t)S_t, \quad \text{a.s.}, \quad \forall t \in [0, T],$$

and there exists a strictly positive local martingale Z_t with $Z_0 = 1$ such that $\tilde{S}_t Z_t$ is a local martingale. We shall denote $\mathcal{Z}_{\text{loc}}(\lambda)$ the set of all CLMS with the transaction cost $(\lambda_t)_{t\in[0,T]}$.

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$$\inf_{t\in[0,T]} \left(\lambda_t S_t - |S_t - \tilde{S}_t|\right) > 0, \quad \text{a.s.},$$

we shall call the pair $(\tilde{S}, Z)$ a SCLMS. And we denote $\mathscr{Z}^s_{\mathrm{loc}}(\lambda)$ the set of all SCLMS.

Admissible Strategies

Definition: A self-financing trading strategy is a pair of predictable, finite variation processes $(\phi^0_t, \phi^1_t)_{0\le t\le T}$ with

(1) $\phi^0_0 = \phi^1_0 = 0$,

(2) denoting by $\phi^0_t = \phi^{0,\uparrow}_t - \phi^{0,\downarrow}_t$ and $\phi^1_t = \phi^{1,\uparrow}_t - \phi^{1,\downarrow}_t$ such that $\phi^{0,\uparrow}_0 = \phi^{0,\downarrow}_0 = \phi^{1,\uparrow}_0 = \phi^{1,\downarrow}_0 = 0$, these canonical decompositions satisfy

$$\phi^{0,\uparrow}_t \le \int_0^t (1-\lambda_u) S_u d\phi^{1,\downarrow}_u, \qquad \phi^{0,\downarrow}_t \ge \int_0^t (1+\lambda_u) S_u d\phi^{1,\uparrow}_u, \quad 0 \le t \le T,$$

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where the two integrals above are defined as predictable Stieltjes integrals.

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If S is cadlag and ϕ is predictable and of finite variation, the predictable Stieltjes integral (Guasoni, Lepinette and Rasonyi (2012)) can be defined by

$$\int_0^t S_u d\phi_u \triangleq \int_0^t S_u d\phi_{u-} - \sum_{s \le t} (\phi_s - \phi_{s-})(S_s - S_{s-}).$$

Predictable Stieltjes Integrals

Some important results: (Guasoni, Lepinette and Rasonyi (2012))

- If S is a càdlàg semimartingale and ϕ is predictable and of finite variation. We have integration by parts:

$$\int_0^T S_u d\phi_u = \phi_T S_T - \phi_0 S_0 - \int_0^T \phi_u dS_u.$$

- If $\phi^n \to \phi$ pointwise and $\sup_n \|\phi^n\|_T < \infty$, then $\int_0^T S_u d\phi^n_u \to \int_0^T S_u d\phi_u$ pointwise.

- If $\phi^n \to \phi$ pointwise and $S \geq 0$, then $\liminf_n \int_0^T S_u d\|\phi^n\|_u \geq \int_0^T S_u d\|\phi\|_u$ pointwise.

Liquidation Value Process

- We assume that the investor starts with the initial position $(x, 0)$ in bond and stock assets for the given constant $x \geq 0$. The trading strategy $\phi = (\phi^0, \phi^1)$ is called *x-admissible* if the liquidation value $V_t^{\text{liq},x}$ satisfies

$$V_t^{\text{liq},x}(\phi^0, \phi^1) \triangleq x + \phi_t^0 + (\phi_t^1)^+(1 - \lambda_t)S_t - (\phi_t^1)^-(1 + \lambda_t)S_t \geq 0,$$

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  $\mathbb{P}$-a.s. for $t \in [0, T]$.

- We shall denote $\mathcal{A}_x(\lambda)$ as the set of all x-admissible portfolios with transaction cost λ. Moreover, we will also denote $\mathcal{V}_x(\lambda)$ as the set of the terminal liquidation value V_T^{liq} under the admissible portfolio $(\phi^0, \phi^1) \in \mathcal{A}_x(\lambda)$.

Example of CLMS which is not a CPS

Let Y be a **compensated** $\mathbb{P}^0$**-Poisson process with intensity** $\beta = \frac{1}{T} \leq 1$ **started from one**, stopped when it hits zero or when it first jumps. Denote by τ the first hitting time of zero and by ρ the first jump time and $S = Y$.

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Let the initial wealth be $x = 1 - e^{-1}$ and define the portfolio $\phi^* = (\phi^{0,*}, \phi^{1,*})$ by

$$\phi_t^{1,*} = e^{-1+\beta t}\mathbf{1}_{\{t \leq \tau \wedge \rho\}}, \qquad \phi_t^{0,*} = -\int_0^t (1+\lambda) S_t d\phi_t^{1,*}.$$

Example cont.

We obtain that if $\lambda \leq \frac{1}{4e-2}$, then

$$V_T^{\mathrm{liq},x}(\phi^{0,*}, \phi^{1,*}) \geq \mathbf{1}_{\{\rho \leq \tau\}} = \mathbf{1}_{\{S_T > 0\}}.$$

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$$V_T^{\mathrm{liq},x}(\phi^{0,*}, \phi^{1,*}) \geq \mathbf{1}_{\{\rho \leq \tau\}} = \mathbf{1}_{\{S_T > 0\}}.$$

Moreover, for any $t < \tau \wedge \rho$,

$$V_t^{\mathrm{liq},x}(\phi^{0,*}, \phi^{1,*}) > 0.$$

Example cont.

- We define the probability $\mathbb{P}$ by

$$\frac{d\mathbb{P}}{d\mathbb{P}^0} = Y_T.$$

  $\frac{1}{Y}$ is a positive $\mathbb{P}$-strict local martingale with $\mathbb{P}(\frac{1}{Y_T} > 0) = 1$. Since the process $\frac{1}{Y}$ is a $\mathbb{P}$-local martingale and $\frac{S}{Y} = 1$ is a $\mathbb{P}$-martingale $(\tilde{S}, Z) = (S, \frac{1}{Y})$ is an SCLMS.

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$$0 \leq V_T^{\text{liq},x}(\phi^0, \phi^1) \leq x + \int_0^T \phi_t^1 d\tilde{S}_t.$$

- As a result

$$\mathbb{E}^{\mathbb{Q}}[V_T^{\text{liq},x}(\phi^0, \phi^1)] \leq x < 1,$$

for any $\phi \in \mathcal{A}_x$, which is a contradiction to the fact that $V_T^{\text{liq},x}(\phi^{0,*}, \phi^{1,*}) \geq \mathbf{1}_{\{Y_T>0\}} = 1$, $\mathbb{P}$-a.s. (and hence $\mathbb{Q}$-a.s.).

NUPBR

Definition: We say S admits an Unbounded Profit with Bounded Risk (UPBR) with transaction cost λ if there exists a sequence of admissible portfolio $(\phi^{0,n}, \phi^{1,n})_{n\in\mathbb{N}}$ in $\mathcal{A}_1(\lambda)$ and the corresponding terminal liquidation value $(V_T^{\text{liq},1}(\phi^{0,n}, \phi^{1,n}))_{n\in\mathbb{N}}$ is unbounded in probability, i.e.,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left(V_T^{\text{liq},1}(\phi^{0,n}, \phi^{1,n}) \geq m\right) > 0.$$

If no such sequence exists, the market satisfies the NUPBR condition under the transaction cost $(\lambda_t)_{t\in[0,T]}$.

A new definition

$$\mathcal{A}_x^{\text{bd}}(\lambda) \triangleq \{(\phi^0, \phi^1) : |\phi_t^1| \leq M, \ \mathbb{P}\text{-a.s.}, \ t \in [0, T]$$
$$\text{for some } M > 0 \text{ where } (\phi^0, \phi^1) \in \mathcal{A}_x(\lambda)\}.$$

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Definition

We say that S satisfies No Local Arbitrage with Bounded Portfolios (NLABP) with the transaction cost λ if there exists a sequence of stopping times $\tau_n \nearrow T$ as $n \to \infty$ such that for each $n \in \mathbb{N}$, we can not find $(\phi^{0,n}, \phi^{1,n}) \in \mathcal{A}^{\mathrm{bd}}(\lambda)$ which satisfies

$$\begin{aligned} &\mathbb{P}\left(V_{\tau_n}^{\mathrm{liq},0}(\phi^{0,n}, \phi^{1,n}) \geq 0\right) = 1, \\ \text{and} \quad &\mathbb{P}\left(V_{\tau_n}^{\mathrm{liq},0}(\phi^{0,n}, \phi^{1,n}) > 0\right) > 0. \end{aligned} \tag{1}$$

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If $\tau_n = T$, then this would be the usual No-Arbitrage, which implies NLABP.

Robust versions

These conditions hold in the robust sense if there exists a strictly favorable market satisfying these conditions.

Main Result

Theorem

The following two assertions are equivalent.

(1) *There exists a SCLMS* $(\tilde{S}, Z)$ *for the market with transaction cost* λ*, i.e.,* $\mathcal{Z}^s_{loc}(\lambda) \neq \emptyset$.

(2) S *satisfies NUPBR and NLABP conditions with the transaction cost* λ *in the robust sense.*

Utility maximization

Theorem

Suppose that there exists some $x > 0$ such that $u(x) < +\infty$ (and hence for all $x > 0$). Consider the following three assertions:

(1) *S satisfies NUPBR and NLABP conditions with the transaction cost λ in the robust sense.*

(2) *For any initial wealth $x > 0$, there exists a unique optimal portfolio $(\phi^{0,*}, \phi^{1,*}) \in \mathcal{A}_x(\lambda)$, i.e., $V_T^{*,x} \in \mathcal{V}_x(\lambda)$ such that*

$$u(x) = \mathbb{E}[U(V_T^{*,x})].$$

(3) *S satisfies the NUPBR condition with the transaction cost λ.*

We have the following implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Sketch of the Proof (Theorem 1: (1) $\Rightarrow$ (2), NUPBR)

- Define $\xi_t = \inf_{s\in[0,t]} \left(\lambda_s S_s - |S_s - \tilde{S}_s|\right) > 0$ a.s. for all $t \in [0, T]$. Choose $S'_t \triangleq S_t$ for all $t \in [0, T]$ and $\lambda'_t \triangleq \lambda_t - \xi_t \frac{\lambda_t}{(1+\lambda_t)S_t}$. This market is strictly favorable.

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- it is enough to show that the smaller spread $[(1-\lambda'_t)S'_t, (1+\lambda'_t)S'_t]$ satisfies NUPBR with the transaction cost λ'. Verify that

$$V_t^{\text{liq},1}(\phi^0, \phi^1) \leq 1 + \int_0^t \phi_u^1 d\tilde{S}_u, \quad \forall t \in [0, T]$$

  for any admissible portfolio $\phi \in \mathcal{A}_1(S', \lambda')$.

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- Define $\xi_t = \inf_{s\in[0,t]}\left(\lambda_s S_s - |S_s - \tilde{S}_s|\right) > 0$ a.s. for all $t \in [0, T]$. Choose $S'_t \triangleq S_t$ for all $t \in [0, T]$ and $\lambda'_t \triangleq \lambda_t - \xi_t \frac{\lambda_t}{(1+\lambda_t)S_t}$. This market is strictly favorable.
- it is enough to show that the smaller spread $[(1-\lambda'_t)S'_t, (1+\lambda'_t)S'_t]$ satisfies NUPBR with the transaction cost λ'. Verify that

$$V_t^{\text{liq},1}(\phi^0, \phi^1) \leq 1 + \int_0^t \phi_u^1 d\tilde{S}_u, \quad \forall t \in [0, T]$$

  for any admissible portfolio $\phi \in \mathcal{A}_1(S', \lambda')$.
- Show that $(1 + \int_0^t \phi_u^1 d\tilde{S}_u)Z_t$ is a supermartingale, and therefore, $\mathbb{E}[V_T^{\text{liq},1} Z_T] \leq \mathbb{E}[X_T Z_T] \leq X_0 Z_0 = 1$.

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- Moreover, since $|\phi^1_t| \leq M$ a.s. for some $M > 0$, the stochastic integral $\int_0^{t\wedge\tau_n} \phi^1_u d\tilde{S}_u$ is a true martingale under $\mathbb{Q}$. Therefore, we can deduce that

$$\mathbb{E}^{\mathbb{Q}}[V^{\mathrm{liq},0}_{\tau_n}(\phi^0, \phi^1)] \leq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_n} \phi^1_u d\tilde{S}_u\Big] = 0.$$

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- This is a contradiction to $\mathbb{Q}(V^{\text{liq},0}_{\tau_n}(\phi^0, \phi^1) \ge 0) = 1$ and $\mathbb{Q}(V^{\text{liq},0}_{\tau_n}(\phi^0, \phi^1) > 0) > 0$ by the fact that $\mathbb{Q} \sim \mathbb{P}$ as well as (1).

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- Since $\mathcal{V}_x$ is not convex, we can instead consider its solid hull:

$$\mathcal{C}(1) = \{V \in \mathbb{L}^0_+ : V \leq V_T^{\mathrm{liq},1} \in \mathcal{V}_1(\lambda)\}. \tag{2}$$

Clearly $\mathcal{C}(x) = \{V \in \mathbb{L}^0_+ : V \leq V_T^{\mathrm{liq},x} \in \mathcal{V}_x(\lambda)\} = x\mathcal{C}(1)$ and $\mathcal{C}(x)$ is convex and solid.

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  - RNUPBR => $\{\|\phi\|_T^1 : (\phi^0, \phi^1) \in \mathcal{A}_1\}$ is bounded in probability.
  - Komlos' Lemma and the stability properties of the predictable Stieltjes integrals.

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- Define

$$\mathbf{C}^\tau(x) \triangleq \{V_\tau^{\text{liq},0}(\phi^0, \phi^1) : \text{for } (\phi^0, \phi^1) \in \mathcal{A}_x\}, \; \mathbf{C}^\tau = \bigcup_{x \geq 0} \mathbf{C}^\tau(x).$$

and

$$\mathbf{C}_M^\tau \triangleq \{V_\tau : V_\tau = V_\tau^{\text{liq},0}(\phi^0, \phi^1) \in \mathbf{C}^\tau \text{ and } |\phi_t^1| \leq M, \; \mathbb{P}\text{-a.s. for } t \text{ on } [\![0, \tau]\!]\}.$$

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- Cost to enter a position (ϕ^0,ϕ^1) at time t

$$V_t^{\mathrm{cost},x}(\phi^0,\phi^1) = V_t^{\mathrm{liq},x}(\phi^0,\phi^1) + 2\lambda_t S_t|\phi^1_t|, \quad t \in [0,T].$$

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Lemma

NLABP => For $M > 0$ large enough, for each $n \in \mathbb{N}$, we have that for any $x > 0$ and any x-admissible portfolio $(\phi^0,\phi^1) \in \mathcal{A}_x^{bd}$ with $V_{\tau_n}^{liq,0}(\phi^0,\phi^1) \in \mathbf{C}_M^{\tau_n}(x)$, we have

$$V_{\tau_n}^{liq,0}(\phi^0,\phi^1) \geq -a \ \text{ a.s.} \quad \Rightarrow \quad V_t^{cost,0}(\phi^0,\phi^1) \geq -a, \ \text{ a.s. } \forall t \leq \tau_n,$$

for any $0 < a < x$.

Sketch of the Proof (Theorem 1: (2) $\Rightarrow$ (1))

- Since S is locally bounded, we have a localizing sequence of stopping time ρ_n. Define

$$\bar{\tau}_n \triangleq \tau_n \wedge \rho_n, \quad \text{for } n \in \mathbb{N}.$$

Lemma

$\mathbf{C}_M^{\bar{\tau}_n}$ is Fatou closed, i.e., if there exists a sequence $(V^m)_{m\in\mathbb{N}}$ in $\mathbf{C}_M^{\bar{\tau}_n}$ such that V^m converges to some $\mathcal{F}_{\bar{\tau}_n}$-measurable random variable V, $\mathbb{P}$-a.s., and $V^m \geq -a$ for some $a > 0$, then we have $V \in \mathbf{C}_M^{\bar{\tau}_n}$.

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- Banach-Alaoglu theorem, Komlos Lemma, and one more localization to paste together a local martingale Z.

A useful Lemma

Let $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ be two cadlag bounded processes. The following conditions are equivalent:

(i) There exists a cadlag martingale $(M_t)_{t\in[0,T]}$ such that

$$X \le M \le Y, \quad \text{a.s.}.$$

(ii) For all stopping times σ, τ such that $0 \le \sigma \le \tau \le T$ a.s., we have

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \le Y_\sigma \quad \text{and} \quad \mathbb{E}[Y_\tau|\mathcal{F}_\sigma] \ge X_\sigma \quad \text{a.s.}.$$

Sketch of the Proof (Theorem 1: (2) $\Rightarrow$ (1))

- Since $V \in \mathbf{C}_M^{\bar{\tau}_n} \cap \mathbb{L}^\infty$, we have $\mathbb{E}^{\mathbb{Q}^n}[V] \leq 0$, using a judicious choice of portfolios we obtain for $\eta \leq \sigma \leq \hat{\tau}_n$

$$\mathbb{E}^{\mathbb{Q}^n}[S_\sigma(1+\lambda_\sigma)|\mathcal{F}_\rho] \geq S_\rho(1-\lambda_\rho)$$

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- By the previous lemma we obtain a martingale $\tilde{S}^n$ up to $\hat{\tau}_n$
- Paste $\tilde{S}^n$ as in Guasoni-Lepinette-Rasonyi. Easy to verify that $\tilde{S}Z$ is a local martingale.

Some Future Work

- Extension to multiple risky assets.
- In Theorem 2, can we show the equivalence (1) $\Leftrightarrow$ (2) ?
- No short selling constraints?