Optimal arbitrage and portfolio optimization for market models satisfying NUPBR but not NFLVR

Wolfgang Runggaldier

(based on joint work with *N.H.Chau and P.Tankov*)

Dipartimento di Matematica, Università di Padova

2016 Risk and Stochastics Conference

LSE, April 21-22.

Motivation

- The classical no-arbitrage condition of NFLVR is often too strong and so various weaker notions of no-arbitrage have been introduced more recently and their impact on pricing, hedging and portfolio optimization have been studied (see a survey in Fontana'13).

  → A first purpose here is then to derive an approach to construct models that do not satisfy NFLVR but some weaker form of no-arbitrage.

Motivation

- Under absence of NFLVR, classical arbitrage is possible and so a major purpose here is to study optimal arbitrage.
- Another purpose is portfolio optimization and the classical duality relationship also under absence of an ELMM.

Outline

- Recalling some preliminary notions.
- Optimal arbitrage (*a general approach*).
- Models based on filtration enlargement
  i) *Optimal arbitrage;*
  ii) *Portfolio optimization.*

Weaker notions of no-arbitrage

No increasing profit
(no Immediate arbitrage opportunity)
$\Uparrow$
No unbounded profit with bounded risk (NUPBR)
(no arbitrage of the 1st kind)
$\Uparrow$
No Free Lunch with Vanishing Risk (NFLVR)

$\rightarrow$ Under some assumptions, the first two are robust w.r.to *change of numeraire, absolutely continuous measure change, change of reference filtration*.

$\rightarrow$ This does not necessarily hold with NFLVR and implies also that NFLVR cannot be verified on the basis of the semimartingale characteristics of the discounted price processes in the market.

Weaker notions of no-arbitrage

- It was shown (Karatzas, Kardaras ’07) that NUPBR is the minimal condition to meaningfully solve portfolio optimization problems: without NUPBR one has either no solution or infinitely many (*more to this effect later)*

  $\rightarrow$ *There is thus interest in situations that are intermediate between NFLVR and NUPBR.*

Portfolios

- In order to recall the definitions, in particular that of NUPBR, which involve the portfolio value process, consider a market $((\Omega, \mathcal{F}, (\mathcal{F}_t), P), S)$ with $S = (S_t) = (S_t^1, \cdots, S_t^d)$ the already discounted prices of d risky assets supposed to be general non-negative semimartingales.
- Given a self-financing, predictable strategy $H = (H_t)$, let

$$V^{x,H} = (V_t^{x,H}) = x + (H \cdot S)_t = x + \int_0^t H_u dS_u$$

be the value process corresponding to H with $V_0^{x,H} = x$.

Portfolios

Definition *(admissible strategy)*

i) An $S-$integrable, predictable H is α–*admissible* if $H_0 = 0$ and $V_t^{0,H} \geq -\alpha,\ t \in [0, T]$ *a.s.*

$\rightarrow$ Denote its set by $\mathcal{A}_\alpha$.

- H is *admissible* if it is admissible for some $\alpha > 0$.

$\rightarrow$ Denote its set by $\mathcal{A}$.

Definitions

Definition *(NUPBR)* There is No Unbounded Profit With Bounded Risk (NUPBR) if the set

$$\mathcal{K}_1 = \left\{ V_T^{0,H} \mid H = (H_t) \text{ is a 1–admissible strategy for } S \right\}$$

is bounded in L^0, that is, if

$$\lim_{c \uparrow \infty} \sup_{W \in \mathcal{K}_1} P(W > c) = 0$$

Definitions

Definition *(NA1)* An $\mathcal{F}_T-$measurable random variable ξ is called an Arbitrage of the First Kind if $P(\xi \geq 0) = 1$, $P(\xi > 0) > 0$, and for all $x > 0$ there exists an admissible strategy H such that $V_T^{x,H} \geq \xi$. We shall say that the market admits No Arbitrage of the First Kind (NA1), if there is no arbitrage of the first kind in the market.

$\rightarrow$ NA1 and NUPBR can be shown to be equivalent.

Further definitions

Definition *(Classical arbitrage strategy)* An admissible H is an arbitrage strategy if $P(V_T^{0,H} \geq 0) = 1$ and $P(V_T^{0,H} > 0) > 0$. It is a strong arbitrage if $P(V_T^{0,H} > 0) = 1$.

$\rightarrow$ We denote by NA absence of classical arbitrage.

Further definitions

Classical arbitrage can be divided into:

- Scalable arbitrage: take $H \in \mathcal{A}_0$ and define $H^n := nH, n = 1, 2, \cdots$
  - $\rightarrow$ $H^n \in \mathcal{A}_0$, $n(H \cdot S)_t \geq 0$ *(admissibility)*, H^n generates an unbounded profit as $n \rightarrow \infty$.
- Unscalable arbitrage: take $H \in \mathcal{A}_x$ with $x > 0$
  - $\rightarrow$ $n(H \cdot S)_t \geq -nx$ a.s. for all $t \in [0, T]$ so that $H^n = nH \in \mathcal{A}_{nx}$. Again, H^n generates an unbounded profit as $n \rightarrow \infty$, but the investor also faces an unbounded risk.

Further definitions

- Below, when mentioning classical arbitrage, we shall always mean unscalable arbitrage as it is the only one possible under NUPBR.
  - $\rightarrow$ Scalable arbitrages not only are not possible under NUPBR, they are not even meaningful economically.

Further definitions

The following relation is known to hold

$$NUPBR + NA = NFLVR$$

In the quest of models that are intermediate between NFLVR and NUPBR one may thus search for models, for which NUPBR holds , but (unscalable) NA does not hold.

$\rightarrow$ In the intermediate region between NFLVR and NUPBR there exists thus the possibility for classical (unscalable) arbitrage.

- Arbitrages may be exploited in various ways, one might thus be interested in finding optimal arbitrage

Optimal arbitrage (via superhedging)

- Define

$$U(T) := \sup\left\{c > 0 \mid \exists H \in \mathcal{A}_1 \text{ with } V_T^{1,H} \geq c,\ P-\text{a.s.}\right\}$$

$\rightarrow$ *it is the maximum amount that one can realize at* T *by an* $1-$*admissible strategy when starting from* 1.

Definition: $U(T)$ is called an optimal arbitrage profit if $U(T) > 1$.

Optimal arbitrage

- One possibility to obtain optimal arbitrage is based on superhedging.

Given an $\mathcal{F}_T-$measurable claim G, let

$$SP_+(G) := \inf\left\{x \geq 0 \mid \exists H,\ x\text{−admissible},\ V_T^{x,H} \geq G,\ P-\text{a.s.}\right\}$$

it is the superhedging price of G. (The $+$ stands to indicate that, here, $V_t^{x,H} \geq 0$).

→ In [Chau,Tankov '14] it is shown that $U(T) = (SP_+(1))^{-1}$ and this relates optimal arbitrage to superhedging

→ If $SP_+(1) < 1$ then there is optimal arbitrage and the superhedging strategy of 1 realizes the optimal arbitrage. *(It can be shown that NUPBR $\Rightarrow$ $SP_+(1) > 0$).*

Optimal arbitrage

Always in [Chau,Tankov '14] this approach via superhedging is implemented as follows.

- Start from a market model that satisfies NFLVR and consider a martingale measure Q. The idea is then to make a non-equivalent change of measure to obtain a P, under which NUPBR holds, but NFLVR fails.
  - $\rightarrow$ The superhedging price under P (*where arbitrage should be possible, i.e. $SP_+(1) < 1$*) can be obtained in terms of the superhedging price under Q (*where NFLVR holds and so it is simpler to obtain*).

Optimal arbitrage

To show this in more detail, consider a market with a single risky asset S_t (*locally bounded semimartingale*) that satisfies NFLVR (*martingale measure Q*).

- Let $Y_t \geq 0$ be a cadlag $Q-$martingale with $Y_0 = 1$ and such that, if it goes to zero, it goes there smoothly. More precisely, for $\tau := \inf\{t \geq 0 \mid Y_t = 0\}$, one has

$$Q(\{\tau \leq T\}) \cap \{Y(\tau-) > 0\}) = 0$$

- Define $dP/dQ_{|\mathcal{F}_t} = Y_{t \wedge \tau}$ so that one has

$$P\{\tau \leq T\} = E^Q \left\{ Y_{T \wedge \tau} \mathbf{1}_{\{\tau \leq T\}} \right\} = 0$$

Optimal arbitrage

In [Chau,Tankov '14] it is shown that the (P, S) market satisfies NUPBR (*proved by showing that* $\exists$ *an ESMD*) and, for any $G \in \mathcal{F}_T$,

$$SP_+^P(G) = SP_+^Q\left(G\mathbf{1}_{\{Y_T>0\}}\right)$$

so that, putting $G = 1$,

$$SP_+^P(1) = SP_+^Q\left(\mathbf{1}_{\{Y_T>0\}}\right)$$

$\rightarrow$ If the latter expression is < 1, then we have optimal arbitrage under P and the arbitrage strategy is the one that in Q superreplicates $\mathbf{1}_{\{Y_T>0\}}$.

Models satisfying NUPBR but not NFLVR

- How can one find models satisfying NUPBR but not NFLVR?
- How can one verify whether NUPBR holds?

Insider information

In the quest of market models that fall between NFLVR and NUPBR and for which thus classical arbitrage (*optimal arbitrage*) is possible, a natural choice are models with asymmetric information, in particular insider information.

- $\rightarrow$ Insider information can be modeled by filtration enlargement: *initial, successive, progressive enlargement*.
- $\rightarrow$ Here we follow Chau,R.,Tankov'15.

Insider information

We consider here just the case of initial enlargement starting from a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with filtration $\mathcal{F} = (\mathcal{F}_t)$

- The $(\mathcal{F}, P)$-market of regular agents is supposed to satisfy NFLVR implying that the set $ELMM(\mathcal{F}, P)$ of equivalent local martingale measures is not empty.
- The insider is assumed to possess from the beginning additional information about the value of some $\mathcal{F}_T-$measurable r.v. G

$\rightarrow$ Starting from $\mathcal{F} = (\mathcal{F}_t)$ one can then consider the enlarged filtration $\mathcal{G} = (\mathcal{G}_t)$ with

$$\mathcal{G}_t = \cap_{\varepsilon > 0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma(G) \right)$$

Preliminaries

In the given context it is important to have a criterion which ensures that an $\mathcal{F}-$local martingale remains a $\mathcal{G}-$semimartingale (*semimartingales are important in financial modeling*)

→ **Jacod's condition***(non equivalent version):* Letting ν_t denote the regular conditional distribution of G given $\mathcal{F}_t$ and ν be the law of G, the mentioned condition holds if

$$\nu_t \ll \nu, \quad P-a.s. \text{ for } t < T$$

- The actual Jacod condition is $\nu_t \sim \nu$ but this would imply NFLVR for the $(\mathcal{G}, P)-$market and thus exclude arbitrage possibilities there. Also $t = T$ is excluded to allow for continuous G.

Preliminaries

To proceed we need an assumption concerning the density process $p_t^x(\omega)$ of ν_t with respect to ν that satisfies

$$\nu_t(dx) = p_t^x\,\nu(dx) \quad (p_t^x = P\{G \in dx \mid \mathcal{F}_t\}/P\{G \in dx\})$$

→ The assumption concerns the fact that p_t^x cannot jump to zero. Formally: if

$$\tau^x := \inf\{t \geq 0 \mid p_{t-}^x = 0 \text{ or } p_t^x = 0\} \wedge T$$

then $P\{\tau^x < T,\ p_{\tau-}^x > 0\} = 0$.

Preliminaries

G may be discrete or continuous. Here, for simplicity, we concentrate mainly on the discrete case $G \in \{g_1, \cdots, g_n\}$, namely when the insider is initially informed that G takes a specific value g_i. (*Later we shall briefly mention also the case when G is not purely atomic*).

- In the discrete case the insider can update his belief with a measure change $P \rightarrow Q^i$ thereby dismissing all scenarios not contained in $\{G = g_i\}$.
- The measure Q^i satisfies

$$\frac{dQ^i}{dP}_{|\mathcal{F}_t} = \frac{P\{G = g_i \mid \mathcal{F}_t\}}{P\{G = g_i\}} := p_t^{g_i}$$

  It gives total mass to $\{G = g_i\}$ and is absolutely continuous but not equivalent to P.

NUPBR

Theorem (NUPBR): For a discrete G and under the assumption that p_t^x cannot jump to zero we have that the $(\mathcal{G}, P)-$market satisfies NUPBR.

- The statement is proved by contradiction using the fact that NUPBR can be shown to hold for the $(\mathcal{F}, Q^i)-$market.
- Acciaio et al.'14 give sufficient conditions for NUPBR to hold in the $(\mathcal{G}, P)-$market by constructing a martingale deflator under $\mathcal{G}$. However they work on an ∞ horizon and their approach cannot be adapted to our finite horizon case.
- The Theorem shows that the $(\mathcal{G}, P)-$market satisfies NUPBR; it does not exclude that it satisfies also NFLVR. *This depends on the possibility of classical arbitrage in the various specific cases.*

Optimal arbitrage via Superhedging

Optimal arbitrage was seen to be related to super hedging. We therefore consider first superhedging starting from a more specific definition of a superhedging price. The filtration may be either $\mathcal{F}$ or $\mathcal{G}$ and we let $\mathcal{H} \in \{\mathcal{F}, \mathcal{G}\}$.

Superhedging

Definition: An $\mathcal{H}_0-$measurable r.v. $x_*^H(f)$ is called superhedging price, with respect to $\mathcal{H}$, of a given claim $f \geq 0$, if there exists an $\mathcal{H}-$predictable self financing strategy H such that

$$\begin{cases} x_*^H(f) + (H \cdot S)_t \geq 0 \quad P - a.s., \forall t \in [0, T] \\ x_*^H(f) + (H \cdot S)_T \geq f, \quad P - a.s. \end{cases}$$

and, if any $x \in \mathcal{H}_0$ satisfies these conditions, then $x_*^H(f) \leq x$, $P - a.s.$.

Superhedging

Theorem (SH): For a discrete G and under the assumption that p_t^x cannot jump to zero we have, in addition to the fact that the $(\mathcal{G}, P)-$market satisfies NUPBR, the following.

i) The superhedging price for a claim $f \geq 0$ in the $(\mathcal{G}, P)-$market is given by

$$x_*^G(f) = \sum_{i=1}^{n} x_*^F(f\mathbf{1}_{\{G=g_i\}})\mathbf{1}_{\{G=g_i\}}$$

ii) The associated hedging strategy is $H^{F,i}\mathbf{1}_{\{G=g_i\}}$ where $H^{F,i}$ is the super hedging strategy for $f\mathbf{1}_{\{G=g_i\}}$ in the $(\mathcal{F}, P)-$market, i.e.

$$x_*^F(f\mathbf{1}_{\{G=g_i\}})\mathbf{1}_{\{G=g_i\}} + \left(H^{F,i}\mathbf{1}_{\{G=g_i\}} \cdot S\right)_T \geq f\mathbf{1}_{\{G=g_i\}}, \quad P - a.s.$$

$\rightarrow$ Next is a more specific definition of optimal arbitrage.

Optimal arbitrage

Definition: There is optimal arbitrage in the $(\mathcal{H}, P)-$market if $x_*^H(1) \leq 1$ and $P\{x_*^H(1) < 1\} > 0$. If $x_*^H(1) < 1$, $P - a.s.$ then the optimal arbitrage is said to be strong.

- It can be shown that NUPBR implies $x_*^H(1) > 0$ *(see also* $SP_+(1) > 0$ *before)*.
- We shall write $x_*^F(f)$ or $x_*^G(f)$ to distinguish the case when $\mathcal{H} = \mathcal{F}$ or $\mathcal{H} = \mathcal{G}$ respectively.
- $x_*^G(f)$ is random since $\mathcal{G}_0$ is non trivial but $x_*^G(f)$ is constant on each event $\{G = g_i\}$.

Optimal arbitrage

As a Corollary to the Theorem (SH) we have

If there is an index i such that $x_*^F(\mathbf{1}_{\{G=g_i\}}) < 1$, then the insider has optimal arbitrage on the event $\{G = g_i\}$. If $x_*^G(1) < 1$ then the insider has strong optimal arbitrage.

$\longrightarrow$ If an arbitrage exists, NFLVR cannot hold.

Utility maximization

In addition to the issue of optimal arbitrage, we shall address also the following issues:

a) What is the more precise relationship between absence of arbitrage, in particular NUPBR, and portfolio optimization?
b) Can portfolio optimization, in particular expected log-utility maximization, in the enlarged market be achieved by an analog of classical duality also under absence of an ELMM?

Utility maximization

- Starting from a) above recall that, for a given utility function $U(\cdot)$, the portfolio optimization problem is defined as

$$u(x) := \sup_{H \in \mathcal{A}_x} E\left\{U(V_T^{x,H})\right\}$$

- It is shown in Karatzas, Kardaras'07 that, if NUPBR fails, then $u(x) = +\infty$ for all $x > 0$ or the problem has infinitely many solutions.
  - $\rightarrow$ If there exists $x > 0$ for which $u(x) < +\infty$, then NUPBR holds.

Utility maximization

A dominant role in portfolio optimization is plaid by the log-utility $u(v) = \log v$.

$\rightarrow$ A criterion allowing to show that NUPBR holds is thus to show that the log-utility maximization leads to a finite value.

$\rightarrow$ However, NUPBR does not imply that log-utility is finite.

- We shall now concentrate on log-utility maximization to address both points a) and b) above.
- A major result here is that, in a duality approach for expected log-utility maximization, an extra term appears that involves the entropy of the additional information G.

Utility maximization

For an initial enlargement by a discrete r.v. $G \in \{g_1, \cdots, g_n\}$ one has in fact

$$\begin{aligned} \sup_{H \in \mathcal{A}_1^G} E^P \left\{ \log V_T^{1,H} \right\} &= -\sum_{i=1}^n P\{G = g_i\} \log P\{G = g_i\} \\ &+ \sum_{i=1}^n \inf_{Z \in ELMM(\mathcal{F},P)} E^P \left\{ \mathbf{1}_{\{G=g_i\}} \log \frac{1}{Z_T} \right\} \end{aligned}$$

- The first component on the right always contributes to the profit of the insider and does not depend on the structure of the $(\mathcal{F}, P)-$market. The second component can be seen as the value of G with respect to the $(\mathcal{F}, P)-$market *(viewed not from the insider)*.

Utility maximization

- If the log-utility for the insider is finite, either because the two components are finite or because the 2nd component compensates the 1st one, then NUPBR holds.

$\rightarrow$ The duality approach leads to the value of the expected log-utility of the insider, it does in general not also give the optimal strategy.

$\rightarrow$ If the log-utility is finite, it gives however a new criterion to verify whether NUPBR holds *(meaningful especially for non-atomic G)*.

Utility maximization

- → Examples with computation of optimal arbitrages and log-utility maximization, for the case of continuous and discrete market models enlarged by a discrete G, can be found in [Chau, thesis].
- → More can be said if G is not purely atomic and various results for this latter case can be studied by a limiting procedure from the discrete case.

Non-atomic G

If G is not purely atomic and the set $ELMM(\mathcal{F}, P)$ of martingale densities for the $(\mathcal{F}, P)$ market of regular agents is uniformly integrable namely, for any $\varepsilon > 0$, $\exists\, K > 0$ such that

$$\sup_{Z \in ELMM(\mathcal{F},P)} E^P \left\{ Z_T \mathbf{1}_{\{Z_T > K\}} \right\} \leq \varepsilon$$

then it can be shown that NUPBR fails *(non uniform integrability helps compensating possible explosive profits coming from the additional information)*

Non-atomic G

- If the $(\mathcal{F}, P)$ market is complete then, for G not purely atomic, NUPBR cannot hold.
- In Chau (thesis) an incomplete market example is presented *(one asset, two independent driving Poisson)* where $G = S_T$ is not purely atomic and the set of martingale densities for the $(\mathcal{F}, P)$ market is uniformly integrable.

  $\rightarrow$ It gives an example of enlargement where NUPBR does not hold.

Non-atomic G

For the case of an incomplete market with G non-atomic one can apply the new criterion for verifying NUPBR based on the finiteness of the expected log-utility for the insider.

Non-atomic G

Always in Chau (thesis) another example is presented of an incomplete market *(one asset, one Wiener and two independent driving Poisson)* where, again, $G = S_T$ is not purely atomic but this time the expected log-utility is finite so that NUPBR holds. On the other hand, arbitrage is shown to be possible via a buy-and-hold strategy if the insider knows that $S_T > 1$ *(starting from* $S_0 = 1$*)*.

$\rightarrow$ This example ensures that the question of NUPBR and optimal arbitrage makes sense.

Thank you for your attention