Quantile-based Risk Sharing

Paul Embrechts

RiskLab, Department of Mathematics, ETH Zurich

Senior SFI Professor

www.math.ethz.ch/~embrechts/

Based on joint work with Haiyan Liu (Waterloo) and Ruodu Wang (Waterloo)

Outline

Risk Measures

Risk measures

A (regulatory) risk measure calculates the amount of regulatory capital of a financial institution taking a risk (random loss) X in a fixed period.

A risk measure is a functional $\rho : \mathcal{X} \to [-\infty, \infty]$.

- $\mathcal{X}$: a convex cone of random variables, e.g. $\mathcal{X} = L^\infty$ or L^1 in an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $X \in \mathcal{X}$: loss/profit
- F_X: cdf of X

Risk Measures

Question

What is a good risk measure to use?

- Regulator's and firms' perspectives can be different or even conflicting
  - taxpayers versus shareholders
  - systemic risk in an economy versus risk of a single firm
  - firm: manages its risk to reduce regulatory capital
  - regulator: maintain enough capital for the whole economy

Value-at-Risk and Expected Shortfall

Value-at-Risk (VaR) at level $\alpha \in [0,1)$

$\mathrm{VaR}_\alpha : L^0 \to (-\infty, \infty]$,

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(1-\alpha) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge 1-\alpha\}.$$

Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $\beta \in [0,1]$

$\mathrm{ES}_\beta : L^1 \to (-\infty, \infty]$,

$$\mathrm{ES}_\beta(X) = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_\alpha(X)\mathrm{d}\alpha, \quad \beta \in (0,1).$$

$$\mathrm{ES}_0(X) = \mathrm{VaR}_0(X); \quad \mathrm{ES}_1(X) = \mathbb{E}[X].$$

Remark: small α convention in this talk ...

Risk Sharing

General setup

- n agents sharing a total risk (or asset) $X \in \mathcal{X}$
- $\rho_1, \ldots, \rho_n$: underlying risk measures

Target: for $X \in \mathcal{X}$,

$$\text{minimize} \quad \sum_{i=1}^{n} \rho_i(X_i) \quad \text{subject to } X_1 + \cdots + X_n = X, \qquad (1)$$

and find an optimal allocation of X: a solution to (1) (if it exists)

- For monetary risk measures:
  optimal allocations $\Leftrightarrow$ Pareto-optimal allocations
- We consider arbitrary allocations

Risk Sharing

Some interpretations

- Regulatory capital reduction within a single firm
- Regulatory capital reduction for a group of firms
- Insurance-reinsurance contracts and risk-transfer
- Risk redistribution among agents

Allocations and Inf-convolution

The set of allocations of $X \in \mathcal{X}$:

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i = X \right\}.$$

The inf-convolution of n risk measures is a risk measure $\square_{i=1}^{n} \rho_i$ mapping $\mathcal{X}$ to $[-\infty, \infty]$:

$$\mathop{\square}_{i=1}^{n} \rho_i(X) = \inf \left\{ \sum_{i=1}^{n} \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}.$$

An optimal allocation $(X_1^*, \dots, X_n^*)$ of X:

$$\sum_{i=1}^{n} \rho_i(X_i^*) = \mathop{\square}_{i=1}^{n} \rho_i(X).$$

Risk Sharing

Some classic references in the mathematical finance literature (mainly focusing on convex risk measures)

- Barrieu-El Karoui (2005)
- Jouini-Schachermayer-Touzi (2008)
- Filipovic-Svindland (2008)
- Delbaen (2012)
- Rüschendorf (2013)

Some questions

Some questions on risk sharing:

- Explicit forms of optimal risk allocation?
- What property does an optimal allocation have (such as comonotonicity)?
- Is an optimal allocation robust? How does it react to model uncertainty?

More importantly, how do the answers to the above questions vary with respect to different underlying risk measures?

- In particular, implications for regulation, VaR and ES?

Progress of the Talk

Range-Value-at-Risk (RVaR)

Extended range for the parameter in VaR: $\alpha \in \mathbb{R}_+ := [0, \infty)$,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge 1 - \alpha\}, \; X \in L^0.$$

That is to add $\mathrm{VaR}_\alpha(X) = -\infty$ for $\alpha \ge 1, \; X \in L^0$.

Two-parameter family of risk measures, RVaR: $\alpha, \beta \in \mathbb{R}_+$,

$$\mathrm{RVaR}_{\alpha,\beta}(X) := \begin{cases} \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma & \beta > 0, \\ \mathrm{VaR}_\alpha(X) & \beta = 0, \end{cases} \quad X \in L^1.$$

Practical values of (α, β) are $\alpha, \beta \ge 0$ and $\alpha + \beta \le 1$.

- $\mathrm{RVaR}_{\alpha,\beta}(X) = -\infty$ for $\alpha + \beta > 1$, $X \in L^1$.

Range-Value-at-Risk (RVaR)

For $\alpha, \beta > 0$ and $\alpha + \beta < 1$,

- $\mathrm{RVaR}_{\alpha,\beta}$ is a distortion risk measure (monetary, comonotonic additive, positive homogeneous, ...)
- $\mathrm{RVaR}_{\alpha,\beta}$ is robust (continuous wrt weak convergence)
  - VaR_α and ES_β are not continuous wrt weak convergence (VaR_α is "almost continuous")
  - See Cont-Deguest-Scandolo (2010), Krätschmer-Schied-Zähle (2014) and Embrechts-Wang-Wang (2015) on robustness issues for risk measures

RVaR bridges the gap between VaR and ES: For $X \in L^1$,

- $\mathrm{VaR}_\alpha(X) = \mathrm{RVaR}_{\alpha,0}(X) = \lim_{\beta \to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X), \ \alpha \in \mathbb{R}_+$.
- $\mathrm{ES}_\beta(X) = \mathrm{RVaR}_{0,\beta}(X) = \lim_{\alpha \to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X), \ \beta \in [0,1)$.

Range-Value-at-Risk (RVaR)

- RVaR is an average quantile
- Take $\mathcal{X} = L^1$ (the common domain) in this talk
- The set of risk measures

$$\mathcal{G} = \left\{ \mathrm{RVaR}_{\alpha,\beta} : (\alpha,\beta) \in \mathbb{R}_+^2 \right\}$$

can be arranged into three categories:

  - ES: $\alpha = 0$
  - true VaR: $\beta = 0$, $\alpha > 0$
  - true RVaR: $\beta > 0$, $\alpha > 0$

Quantile inequalities

Theorem 1

For any $X_1, \ldots, X_n \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$, we have

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i).$$

- $\bigvee_{i=1}^n \beta_i = \max\{\beta_1, \ldots, \beta_n\}$.
- RVaR satisfies a special form of subadditivity ($+$ and $\vee$ can both be viewed as additive operations on $\mathbb{R}$).

Quantile inequalities

Corollary: taking $\beta_1 = \cdots = \beta_n = 0$,

$$\mathrm{VaR}_{\sum_{i=1}^n \alpha_i}\left(\sum_{i=1}^n X_i\right) \leq \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i).$$

(Actually valid for $\mathcal{X} = L^0$)

Corollary: taking $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_n = \beta$,

$$\mathrm{ES}_{\beta}\left(\sum_{i=1}^n X_i\right) \leq \sum_{i=1}^n \mathrm{ES}_{\beta}(X_i).$$

(Classic subadditivity of ES)

Solution to the Risk Sharing Problem

Theorem 2

For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$ and $X \in \mathcal{X}$, we have

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\sum_{i=1}^{n}\alpha_i, \bigvee_{i=1}^{n}\beta_i}(X).$$

Proof of Theorem:

- "$\leq$": by construction
- "$\geq$": by the previous RVaR inequality

Remark:

- $(\mathcal{G}, \square)$ is a commutative monoid (semi-group), isomorphic to the monoid $(\mathbb{R}_+^2, (+, \vee))$.

Solution to the Risk Sharing Problem

$U_X \in L^\infty$: $U_X \sim \mathrm{U}[0,1]$ such that $F_X^{-1}(U_X) = X$.

Optimal allocation

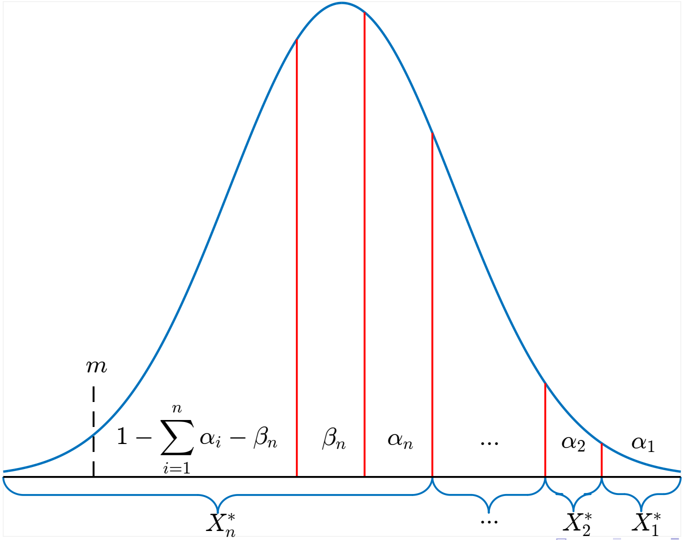
In the previous theorem, if $p := \sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_n$, an optimal allocation $(X_1^*, \dots, X_n^*)$ of X is given by

$$X_i^* = (X - m)\,\mathrm{I}_{\{1-\sum_{k=1}^{i}\alpha_k < U_X \le 1-\sum_{k=1}^{i-1}\alpha_k\}}, \quad i = 1, \dots, n-1, \tag{2}$$

$$X_n^* = (X - m)\,\mathrm{I}_{\{U_X \le 1-\sum_{k=1}^{n-1}\alpha_k\}} + m, \tag{3}$$

where $m \in (-\infty, \mathrm{VaR}_p(X)]$ is a constant.

Solution to the Risk Sharing Problem



Case of VaR

Corollary 3

For $\alpha_1, \dots, \alpha_n \geq 0$ and $X \in \mathcal{X}$, we have

$$\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i}(X) = \mathrm{VaR}_{\sum_{i=1}^{n} \alpha_i}(X).$$

Moreover, if $p := \sum_{i=1}^{n} \alpha_i < 1$, an optimal allocation of X is given by (2)-(3) where $m \in (-\infty, \mathrm{VaR}_p(X)]$.

- The case of ES: trivial optimal allocation (all risk allocated to one with the largest β_i) since ES is subadditive

Progress of the Talk

Properties of Optimal Allocations

In the following:

- Underlying risk measures are $\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n}$, where $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in [0,1)$, $0 < \alpha_i + \beta_i < 1$, $i = 1, \dots, n$ (practical values).
- Assume $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$ so that an optimal allocation exists.
- $X \in \mathcal{X}$ is doubly continuous if both its cdf and inverse cdf are continuous.

Extra Sources of Randomness

For continuously distributed X, U_X and $(X_1^*, \dots, X_n^*)$ are adapted to $\sigma(X)$ (functions of X).

- $\mathbb{F}_n$: the set of sharing principles $(f_1, \dots, f_n)$ where each $f_i : \mathbb{R} \to \mathbb{R}$ has at most finitely many discontinuous points, $f_i(X) \in \mathcal{X}$ for $X \in \mathcal{X}$, $i = 1, \dots, n$, and $f_1(x) + \cdots + f_n(x) = x$ for all $x \in \mathbb{R}$.
- Example as in (2)-(3):

$$X_i^* = f_i(X) = (X - m)\,\mathrm{I}_{\{F_X^{-1}(1-\sum_{k=1}^{i}\alpha_k) < X \le F_X^{-1}(1-\sum_{k=1}^{i-1}\alpha_k)\}}$$

for $i = 1, \dots, n-1$, and

$$X_n^* = f_n(X) = (X - m)\,\mathrm{I}_{\{X \le F_X^{-1}(1-\sum_{k=1}^{n-1}\alpha_k)\}} + m.$$

Extra Sources of Randomness

For true VaR and true RVaR: an optimal allocation of non-continuously distributed X may not be determined by the total loss X (e.g. gambling behaviour).

- Example: to share a constant 0 using two VaRs with $\alpha_1 + \alpha_2 > 1$, some $X_1 = -X_2$ lead to $\mathrm{VaR}_{\alpha_1}(X_1) + \mathrm{VaR}_{\alpha_2}(X_2) < 0$.

Comonotonicity

Theorem 4

For a continuously distributed $X \in \mathcal{X}$, there exists a comonotonic optimal allocation of X if and only if there exists $i = 1, \ldots, n$, such that for all $j = 1, \ldots, n$, $j \neq i$, $\alpha_j = 0$ and $\beta_i \geq \beta_j$.

Remarks:

- To have a comonotonic optimal allocation, at most one true RVaR or true VaR (the one with the largest β_i) is allowed.
- The comonotonic optimal allocation can be chosen as $X_i^* = X$ and $X_j^* = 0$, $j = 1, \ldots, n$, $j \neq i$.
- If $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_n$ (case of ES), any comonotonic allocation is optimal.

Robustness

How does one know about a risk?

- Typically through a model/distribution: simulation, parametric models, expert opinion, ...
- Information asymmetry, model misspecification, data sparsity, random errors ...
- Model uncertainty!

Robustness

Definition 5

For given n risk measures $\rho_1, \dots, \rho_n$ on $\mathcal{X}$ and $X \in \mathcal{X}$, an allocation $(f_1(X), \dots, f_n(X)) \in \mathbb{A}_n(X)$ with $(f_1, \dots, f_n) \in \mathbb{F}_n$ is L^1-robust if $\sum_{i=1}^n (\rho_i \circ f_i)$ is continuous with respect to L^1-norm.

- Robustness $\Rightarrow$ small model misspecification does not ruin the optimality of a sharing principle
- L^1 can be replaced by other metrics

Robustness

Theorem 6

For a doubly continuous $X \in \mathcal{X}$, there exists an L^1-robust optimal allocation of X if and only if $\beta_1, \ldots, \beta_n > 0$.

Remarks:

- To have an L^1-robust optimal allocation, no true VaR is allowed.
- $(X_1^*, \ldots, X_n^*)$ in (2)-(3) is
  - L^1-robust if $\beta_1, \ldots, \beta_n > 0$ (true RVaR and ES),
  - Lévy-metric-robust if $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n > 0$ (true RVaR),
  - not L^∞-robust if $\beta_i = 0$ for some i (at least one true VaR).

Progress of the Talk

Implications for Regulatory Capital

Some issues with VaR as the regulatory risk measure, assuming arbitrary risk sharing is allowed in the market:

- A firm has incentives to split its risk: regulatory arbitrage
- Sharing among firms is not comonotonic: moral hazard
- Sharing among firms may be reliant on an extra source of randomness outside the market: gambling behaviour
- Sharing is not robust: insolvency under model uncertainty
- Total regulatory capital after sharing is much smaller than $\mathrm{VaR}_\alpha(X)$: insufficient capital for the whole economy

Implications for Regulatory Capital

Some partial solutions

- Regulate against (particular forms of) risk sharing
  - risk sharing does not reduce the total risk in the economy, but reduces the total regulatory capital
- Introduce costs for splitting the business of a firm
- Use ES

Conclusions

Mathematical results

- Optimal risk sharing problem for RVaRs
- RVaR inequalities
- Inf-convolution of VaRs/RVaRs
- Robustness of optimal allocations

VaR should taken with extra caution as a regulatory risk measure

- Discussion paper: Embrechts-Puccetti-... (2014 Risks)

References

Barrieu, P. and El Karoui, N. (2005). Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics*, **9**, 269–298.

Cont, R., Deguest, R., and Scandolo, G. (2010). Robustness and sensitivity analysis of risk measurement procedures. *Quantitative Finance*, **10**(6), 593–606.

Delbaen, F. (2012). *Monetary utility functions*. Osaka University Press, Osaka.

Embrechts, P., Puccetti, G., Rüschendorf, L., Wang, R. and Beleraj, A. (2014). An academic response to Basel 3.5. *Risks*, **2**(1), 25-48.

Embrechts, P., Wang, B. and Wang, R. (2015). Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance and Stochastics*, **19**(4), 763–790.

Filipovic, D. and Svindland, G. (2008). Optimal capital and risk allocations for law- and cash-invariant convex functions. *Finance and Stochastics*, **12**, 423–439.

Jouini, E., Schachermayer, W., and Touzi, N. (2008). Optimal risk sharing for law invariant monetary utility functions. *Mathematical Finance*, **18**(2), 269–292.

Krätschmer, V., Schied, A. and Zähle, H. (2014). Comparative and quantitiative robustness for law-invariant risk measures. *Finance and Stochastics*, **18**(2), 271–295.

Rüschendorf, L. (2013). *Mathematical risk analysis. Dependence, risk bounds, optimal allocations and portfolios*. Springer, Heidelberg.

Thank You

Thanks you for your kind attendance

The manuscript can be downloaded at

`http://ssrn.com/abstract=2744142`