# Diversification and protection of liability holders

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- In modern solvency regimes acceptance sets ($\rightarrow$ risk measures) provide the mathematics of *capital adequacy tests*.
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- The primary objective of capital adequacy tests, as instruments of microprudential regulation, is the *protection of liability holders*.

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  - Expected Shortfall ($\mathrm{ES}$) in Swiss Solvency Test and Basel III
- The primary objective of capital adequacy tests, as instruments of microprudential regulation, is the *protection of liability holders*.
- In this presentation we explore the following questions:
  1. What makes an acceptance set be aligned with the "protection of liability holders"? $\rightarrow$ surplus invariance
  2. Which acceptance sets satisfy this requirement and are also able to capture diversification? $\rightarrow$ convexity

# Financial positions and acceptance sets

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$$L^0_+ = \{X \in L^0\,; X \geq 0 \ a.s.\} \quad \text{and} \quad L^0_- = \{X \in L^0\,; X \leq 0 \ a.s.\}$$

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**Remark.** Instead of $L^0$ we could also work in $L^p$ with $0 \leq p < \infty$, $L^\infty$ with the weak*-topology (but not with the norm topology), and Orlicz hearts.

# Acceptance sets defined

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- A non-empty, proper subset $\mathcal{A} \subsetneqq L^0$ is called an *acceptance set* (or *capital adequacy test*) if it is *monotone*, i.e.

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- Acceptance sets that are convex or coherent (convex cones) are important because acceptability is preserved by diversification
  - The $\mathrm{VaR}$-acceptance set is a non-convex cone and the $\mathrm{ES}$ acceptance set is coherent ($\rightarrow$ more on these acceptance sets later)!

# Surplus invariance

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$$X = S_X - D_X$$

where

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**Consideration.** Absence of surplus invariant implies the existence of $X \in \mathcal{A}$ and $Y \notin \mathcal{A}$ although $X$ is worse for liability holders than $Y$. What makes $X$ acceptable is that it is better than $Y$ for the owners!

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**Observation**

For an acceptance set $\mathcal{A} \subset L^0$, the following statements are equivalent:

- $\mathcal{A}$ is surplus invariant;
- $X \in \mathcal{A} \implies -D_X \in \mathcal{A}$;
- $\mathcal{A} = \mathcal{D} + L^0_+$ for some solid set $\mathcal{D} \subset L^0_-$. In this case,

$$\mathcal{D} = \mathcal{A} \cap L^0_- = \{-D_X\,; X \in \mathcal{A}\}.$$

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- The $\mathrm{VaR}_\alpha$ acceptance set at level $\alpha \in (0,1)$

$$\mathcal{A} = \{X \in L^0\,;\ \mathrm{VaR}_\alpha(X) \le 0\} = \{X \in L^0\,;\ \mathbb{P}(X < 0) \le \alpha\}$$

where $\mathrm{VaR}_\alpha(X)$ is the negative upper $\alpha$-quantile of $X$.

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- The *shortfall-risk* acceptance set at level $c > 0$

$$\mathcal{A} = \{X \in L^0\,;\ \mathbb{E}[\lambda(D_X)] \leq c\}$$

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- The *scenario-based* acceptance set

$$\mathcal{A} = \{X \in L^0\,;\ 1_A X \ge 0\}$$

  where $A \in \mathcal{F}$ is a given set of test scenarios.

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is **not surplus invariant**. Indeed, for every $X \in L^0$ we have

$$\begin{aligned}\mathrm{ES}_\alpha(X) &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta \\ &= \frac{1}{\alpha}\int_0^{\mathbb{P}(X<0)} \mathrm{VaR}_\beta(-D_X)d\beta + \frac{1}{\alpha}\int_{\mathbb{P}(X<0)}^\alpha \mathrm{VaR}_\beta(S_X)d\beta\ .\end{aligned}$$

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**Consequence.** $\mathrm{ES}_\alpha(X)$ depends on the surplus of $X$ whenever $\mathbb{P}(X < 0) < \alpha$ holds.

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Let $0 < \alpha < \beta < 1$ and define

$$\begin{aligned} \mathcal{D}_\alpha(\mathrm{VaR}) &= \{D_X \in L^0\,;\ X \text{ is acceptable under } \mathrm{VaR} \text{ at level } \alpha\} \\ \mathcal{D}_\beta(\mathrm{ES}) &= \{D_X \in L^0\,;\ X \text{ is acceptable under } \mathrm{ES} \text{ at level } \beta\}\,. \end{aligned}$$

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**Proposition**

$$\mathcal{D}_\alpha(\mathrm{VaR}) \subset \mathcal{D}_\beta(\mathrm{ES}) \quad \text{but} \quad \mathcal{D}_\beta(\mathrm{ES}) \not\subset \mathcal{D}_\alpha(\mathrm{VaR})\,.$$

In particular, for every $D_X \in \mathcal{D}_\alpha(\mathrm{VaR})$ and for every $m \geq 0$ there exists $D_Y \in \mathcal{D}_\beta(\mathrm{ES})$ such that

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(a) $D_Y \geq D_X$;

(b) $\mathrm{ES}_\beta(Y) \leq -m$.

**Consequence.** The Swiss Solvency Test ($\mathrm{ES}$ at $\beta = 1\%$) admits worse default profiles than Solvency II ($\mathrm{VaR}$ at $\alpha = 0.5\%$)!

# Dual representation for convex, surplus invariant acceptance sets

# Stability of surplus invariant acceptance sets

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Then, the following statements hold:

(a) $\mathcal{A} = \mathrm{cl}(\mathcal{A}^\infty)$.

(b) If $\mathcal{A}$ is convex, then $\mathcal{A}^\infty$ is weak*-closed.

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**Remark.** This result does not hold if $\mathcal{A}$ is not surplus-invariant.

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For $\mathcal{C} \subset L^\infty$ define its *(lower) support function w.r.t. the weak*-topology* $\sigma_\mathcal{C} : L^1 \to \mathbb{R} \cup \{-\infty\}$ by

$$\sigma_\mathcal{C}(Z) := \inf_{X \in \mathcal{C}} \mathbb{E}[ZX]\,.$$

The domain of $\sigma_\mathcal{C}$ is called the *barrier cone* $b(\mathcal{C})$.

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The domain of $\sigma_{\mathcal{C}}$ is called the *barrier cone* $b(\mathcal{C})$.

**Theorem (Hahn-Banach)**

Let $\mathcal{C}$ be a nonempty, weak*-closed, convex subset of $L^\infty$. Then

$$\mathcal{C} = \bigcap_{Z \in b(\mathcal{C})} \{X \in L^\infty \, ; \; \mathbb{E}[ZX] \geq \sigma_{\mathcal{C}}(Z)\} \, .$$

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If $\mathcal{A} \subset L^0$ is a surplus-invariant acceptance set, then $b(\mathcal{A}^\infty) \subset L^1_+$ and $\sigma_{\mathcal{A}^\infty}$ is a decreasing map. In particular,

$$\sigma_{\mathcal{A}^\infty}(Z) \leq 0 \quad \text{for all } Z \in L^1_+ \ .$$

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**Dual Representation Theorem**

Let $\mathcal{A} \subset L^0$ be a closed, convex, surplus-invariant acceptance set. Then,

$$\mathcal{A} = \bigcap_{Z \in b(\mathcal{A}^\infty) \cap L^\infty_+} \{X \in L^0 \,;\, -\mathbb{E}[D_X Z] \geq \sigma_{\mathcal{A}^\infty}(Z)\} \,.$$

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**Remark.**

- This is a dual representation result in the non locally-convex space $L^0$!
- Surplus invariance allows us to take $Z \in b(\mathcal{A}^\infty) \cap L^\infty_+$ instead of $L^1_+$ and to constrain $D_X$ instead of $X$!

# Sketch of the proof of the Dual Representation Theorem

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Step 1: The set $\mathcal{A}^\infty$ is weak*-closed in $L^\infty$, hence by the Hahn-Banach theorem cited earlier

$$\begin{aligned} \mathcal{A}^\infty &= \bigcap_{Z \in b(\mathcal{A}^\infty)} \{X \in L^\infty \,;\, \mathbb{E}[XZ] \geq \sigma_{\mathcal{A}^\infty}(Z)\} \\ &= \bigcap_{Z \in b(\mathcal{A}^\infty) \cap L^\infty_+} \{X \in L^\infty \,;\, -\mathbb{E}[D_X Z] \geq \sigma_{\mathcal{A}^\infty}(Z)\} \end{aligned}$$

where we were able to take $Z \in L^\infty_+$ instead of $L^1_+$ because $\sigma_{\mathcal{A}^\infty}$ is decreasing and to constrain $D_X$ instead of $X$ because $X \in \mathcal{A}$ if and only if $-D_X \in \mathcal{A}$.

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Step 2: By the Stability Lemma above

$$\mathcal{A} = \mathrm{cl}(\mathcal{A}^\infty) = \bigcap_{Z\in b(\mathcal{A}^\infty)\cap L^\infty_+} \{X \in L^0\,;\; -\mathbb{E}[D_X Z] \geq \sigma_{\mathcal{A}^\infty}(Z)\}$$

where we have used that the closure of $\mathcal{A}^\infty$ is defined by the same inequalities by monotone convergence.

# The structure of convex, surplus-invariant acceptance sets

# Coherent surplus-invariant acceptance sets

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**Structure Theorem (Coherent Case)**

A closed, coherent acceptance set $\mathcal{A} \subset L^0$ is surplus invariant if and only if there exists $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ such that

$$\mathcal{A} = L^0_+(A) \oplus L^0(A^c) = \{X \in L^0\,;\ 1_A X \geq 0\}\,.$$

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**Interpretation.** Under the above conditions, we have

$$\text{surplus invariance} \iff \begin{cases} \text{no default on } A \\ \text{no constraints on } A^c \end{cases}$$

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$$\mathcal{A} = L^0_+(A) \oplus (\mathcal{D} + L^0_+(B)) \oplus L^0(C)\,,$$

where $\mathcal{D}$ is a closed, convex, solid subset of $L^0_-(B)$ such that

$$\mathrm{VaR}_\alpha[X] \leq \mathrm{VaR}_\alpha[X^*] \quad \text{for every } X \in \mathcal{D} \text{ and every } \alpha \in (0,1)\,.$$

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Using an exhaustion argument à la Yan (1980) one proves that there exists a set $C \in \mathcal{F}$ with $\mathbb{P}(C) < 1$ such that

(a) $Z = 0$ a.s. on $C$ for every $Z \in b(\mathcal{A}^\infty)$,

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Then elements in $\mathcal{A}$ must be positive on $A$!

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# Sketch of the proof of the Structure Theorem (2/2)

Step 3: Set

$$B = (A \cup C)^c \in \mathcal{F} ,$$

and use (b) above to prove that $B$ is bounded in probability.

Step 4: Set

$$\mathcal{D} = \mathcal{A} \cap L^0_-(B) .$$

To conclude the proof use the equivalences

(a) $\mathcal{D}$ is bounded in probability
(b) $\mathcal{D}$ is stochastically bounded from below
(c) there exists $X^* \in L^0$ such that

$$\mathrm{VaR}_\alpha[X] \leq \mathrm{VaR}_\alpha[X^*] \quad \text{for every } X \in \mathcal{D} \text{ and every } \alpha \in (0,1) .$$

# Conclusion

# Concluding comments (1/2)

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  - if a position is unacceptable, then no position that is worse for liability holders should be acceptable
- From the two most widely used capital adequacy tests, those based on $\mathrm{VaR}$ are surplus invariant, but those based on $\mathrm{ES}$ are not
  - this does not invalidate the known problems with $\mathrm{VaR}$ (i.e. lack of convexity and tail blindness)!

# Concluding comments (2/2)

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- The only coherent, surplus-invariant tests are Scenario Tests $\{X\,; 1_A X \geq 0\}$.
  - If $A = \Omega$ no defaults are allowed
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- The set of convex, surplus-invariant tests is richer but still limited since the default options on the set where default is constrained must belong to a set that is bounded (in probability)
- Surplus invariance severely limits the choice of capital adequacy tests that give credit for diversification
  - when choosing an acceptability criterion a tradeoff must be made between competing, reasonable requirements

# References

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**Thank you for your attention!**