How does asymmetrical information create market incompleteness?

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Introduction (1/2)

What are complete/incomplete markets ?

- **In Economics**, incomplete markets are viewed as markets in which the number of Arrow-Debreu securities is less than the number of states of nature
- **In Finance**, complete markets are markets where any contingent claim is attainable, i.e. can be written by means of admissible strategy bases on existing assets in the market.

In any point of view, always a matter of number of assets available...

Introduction (2/2)

What are the possible sources of incompleteness ?

- First and most well-known answer : lack of assets.

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- But not only !
  Economc works also mention other sources of incompleteness : friction, ambiguity,...
- Less emphasized : lack of information

Aim of this study

- Highlight asymmetrical information as a source of incompleteness
- Isolate asymmetrical information effect on the incompleteness
- Build a model where all the incompleteness of the market is only due to a lack of information.
- Give market incompleteness a new perspective

Studied problem

- Actual differences between information available for owners, managers, and different kinds of investors.
- How do investors' informations influence their strategies when investing on markets ?
- Optimization point of view : already studied in litterature
- **Hedging point of view** : how hedging strategies are impacted by different informations available ?
- 2 different models : small investor / large influencial investor
- Second model will lead to an incomplete market model where the incompleteness is entirely due to the asymmetrical information

Outline

1. Hedging problems for small investors
   - Option hedging with asymmetrical information : insider trading
2. Influential trader model
   - Market price with an influential and informed trader
   - Complete market study
   - Incomplete market for the non informed trader
   - Measure of the lack of information
3. Conclusion

General hedging problem

- First model - small investor : does not influence asset prices
- 1 risky asset, whose price is driven by an extended BS model :

$$dP_t = P_t b(t, P_t)dt + P_t \sigma(t, P_t)dW_t$$

  standard hypotheses on σ and b to have a complete market.
- An agent wants to hedge against a contingent claim ξ in this market with maturity T : his wealth at time T is

$$X_T = \xi$$

.

General hedging problem written as a BSDE

- π_t investing strategy at time t. Standard self-financing hypothesis can be written as

$$dX_t = \underbrace{X_t r_t dt + \pi_t(b_t - r_t)dt}_{-f(t,X_t,Z_t)dt} + \underbrace{\pi_t \sigma_t}_{Z_t} dW_t$$

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- Integrating from t to T, the problem can be written as a BSDE :

$$X_t = \xi + \int_t^T f(s, X_s, Z_s)ds - \int_t^T Z_s dW_s$$

where $X_T = \xi$, $Z_s = \sigma_s \pi_s$
and the driver is $f(x, X_s, Z_s) = -X_s r_s + Z_s \sigma_s^{-1}(r_s - b_s)$.

Information Problem

- Insider trading : the trader possesses an additional information on future prices (r.v. $L \in \mathcal{F}_{T'}$))
  $\Longrightarrow$ **Initial Enlargement of the Brownian Filtration**
- Adding L to the initial filtration : $\mathcal{Y}_t = \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(L))$

Hypothesis ($\mathbf{H_3}$) (Jacod, Jeulin 1985)

There exists a probability Q equivalent to P under which $\mathcal{F}_t$ and $\sigma(L)$ are independent, $\forall t < T'$.

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- Financially : standard examples meet Hyp. ($\mathbf{H_3}$) (e.g. know the price at time T' of an upcoming takeover bid, having information on the evolution of prices in an interval...)
- Major math. problem : Representation property. Under ($\mathbf{H_3}$), result from Jacod and Shiryaev [JS03] provides needed Martingale Representation Theorem.

First results

Theorem ([EL05])

Under Hypothesis ($\mathbf{H_3}$)*, and standard Lipschitz and Q-integrability hypotheses on driver* f*, there exists a* **unique solution** *of the BSDE*

$$X_t = \xi + \int_t^T f(s, X_s, Z_s)ds - \int_t^T Z_s dW_s$$

in the enlarged space $(\Omega, \mathcal{Y}, Q)$.

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Which financially implies

Proposition ([EL05])

The insider trader has a unique solution of the hedging problem, which is **the same** *as the solution in the non informed trader hedging problem.*

Extension to BSDE with random terminal time

Same results are obtained in the case of BSDE with random terminal time (Joint work with M. Royer-Carenzi) (hedging horizon is a random time)

Theorem ([ELRC10])

Under Hypothesis $(\mathbf{H_3})$, *and standard Lipschitz hypotheses and integrability conditions on driver* f, *if* τ *is a stopping time, there exists a* **unique solution** *of the BSDE*

$$X_t = \xi + \int_t^{T \wedge \tau} f(s, X_s, Z_s)ds - \int_t^{T \wedge \tau} Z_s dW_s$$

in the enlarged space $(\Omega, \mathcal{Y}, Q)$.

The consequences and financial interpretation are the same for hedging of options with random terminal horizon (American-style options, Lookback options, ...).

First Conclusion

- In a **small investor** framework, if the market is complete without the information, no additional hedging strategy created when accessing to an additional information.
- Reason : small investor / additional information = not realistic enough
- New model : the informed trader has influence on asset prices $\Rightarrow$ **Large of Influential investor**

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Market price with an influential and informed trader

- Risky asset price is supposed to be driven by

$$dP_t = b(t, P_t, X_t, \pi_t)dt + \sigma(t, P_t, X_t, \pi_t)dW_t$$

- **Additional information** : initial enlargement of filtration with L satisfying Hypothesis $(\mathbf{H_3})$.
- **Influence hypothesis** : the informed trader may influence asset prices dynamics :
  - **Large investor** : his wealth X may influence drift b and volatility σ of price dynamics.
  - **Influential investor** : his investment strategy π may influence drift b and volatility σ of price dynamics.

Influential Problem

- Fundamental Property as in previous section : Martingale Representation Theorem under $(\mathcal{Y}, Q)$
  $\Longrightarrow$ **Complete market for informed trader** [EL11].
- Difference with previous model : a non informed agent investing on the market has the information $\mathcal{F}^P$ filtration generated by prices, who satisfies :

$$\mathcal{F} \subset \mathcal{F}^P \subset \mathcal{Y}$$

  No martingale representation Theorem under $\mathcal{F}^P$,
  $\Longrightarrow$ **Incomplete market for non informed trader** [EL13].
- Link between completeness of market and information availability

Outline

Solving FBSDE under initially enlarged filtration

- The problem leads to a coupling between the forward equation of prices and the backward equation of wealth

$$\begin{cases} P_t = P_0 + \int_0^t b(s, P_s, X_s, Z_s)ds + \int_0^t \sigma(s, P_s, X_s, Z_s)dW_s \\ X_t = \xi - \int_t^T f(s, P_s, X_s, Z_s)ds - \int_t^T Z_s, dW_s \end{cases}$$

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- Under Lipschitz, linear growth and integrability hypotheses on b, σ, f and ξ (cf Pardoux-Tang [PT99]), and additional integrability hypotheses under probability Q, existence and uniqueness can be proved

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- 3 cases where results are obtained
  - *Weak influence* : b and σ weakly depend on X and Z,
  - *The agent wants to hedge a finite value a.s.* : ξ does not depend on price P,
  - *The portfolio does not influence volatility of prices* : σ is independent of Z.

Results

Proposition

Under hypothesis $(\mathbf{H_3})$, *for any functions* f, b, σ *satisfying previous hypotheses and one of the 3 influence cases, the FBSDE in the enlarged space* $(\Omega, \mathcal{Y}, Q)$ *has a unique solution.*

- Financial signification : the influent agent has a **unique admissible hedging strategy**.
- The strategy is adapted to $\mathcal{Y}$ enlarged filtration.
- **Next step** : Comparison with the strategy of a non informed trader investing on this market ?

Outline

Incompleteness of the non informed trader market

- A non informed agent investing on the influenced market has the information $\tilde{\mathcal{F}}$ filtration generated by prices, who satisfies :

$$\mathcal{F} \subset \tilde{\mathcal{F}} \subset \mathcal{Y}$$

  No Martingale Representation Theorem under $\tilde{\mathcal{F}}$,
  $\Longrightarrow$ **Incomplete market for the non informed trader**
- No general solution of the hedging BSDE under filtration $\tilde{\mathcal{F}}$.
- Different Methods need to be used :
  - Kunita-Watanabe Decomposition
  - Quadratic Hedging in incomplete markets (Follmer-Schweizer [FS91])

Kunita-Watanabe Decomposition

- $\mathcal{Q}$ (resp. $\mathcal{Q}_N$) the set of $\mathcal{Y}$ (resp. $\tilde{\mathcal{F}}$)-martingale measures (risk-neutral probabilities for the insider (resp. non insider))

Definition

If $N, M \in \mathcal{M}^2(\mathcal{G}, \mathcal{P})$ then the unique Kunita-Watanabe decomposition of N w.r.t. M is :

$$N_t = N_0 + \int_0^t \theta_u dM_u + L_t, \;\; \mathcal{P} - p.s.$$

where $\theta \in L^2(M), L \in \mathcal{M}_0^2(\mathcal{G}, \mathcal{P})$ is orthogonal to M in $(\mathcal{G}, \mathcal{P})$.

- Let $\tilde{Q} \in \mathcal{Q}$, and $\xi \in \mathcal{L}^2(\Omega, \tilde{\mathcal{F}}, \tilde{Q})$. The martingale representation Theorem under filtration $\mathcal{Y}$ gives

$$\xi = E_{\tilde{Q}}(\xi|\sigma(L)) + \int_0^T \phi_s^L dP_s$$

Kunita-Watanabe Decomposition (2)

- Under any $\tilde{Q} \in \mathcal{Q}_N$, we have the K-W decomposition of ξ w.r.t. $(\tilde{\mathcal{F}}, \tilde{Q})$ and the martingale P :

$$V_t := E_{\tilde{Q}}(\xi|\tilde{\mathcal{F}}_t) = E_{\tilde{Q}}(\xi) + \int_0^t \phi_s^{\tilde{Q}} dP_s + L_t^{\tilde{Q}}$$

And from Follmer and Schweizer, $\phi_s^{\tilde{Q}} = \frac{d<V,P>_s}{d<P>_s}$

Proposition (Filtering Result)

Under any $\tilde{Q} \in \mathcal{Q}$, the integrand of this decomposition may be written

$$\phi_s^{\tilde{Q}} = E_{\tilde{Q}}\left(\phi_s^L|\tilde{\mathcal{F}}_s\right)$$

Kunita-Watanabe Decomposition (3)

Proof :

- $L_t = E_{\tilde{Q}}(\xi|\tilde{F}_t) - E_{\tilde{Q}}(\xi) - \int_0^t E_{\tilde{Q}}(\phi_s^L|\tilde{\mathcal{F}}_s)dP_s$ is a $(\tilde{\mathcal{F}}, \tilde{Q})$-martingale, and $E_{\tilde{Q}}(L_t) = 0$
- Prove that L_t is orthogonal to the stable space generated by P_t.
- $L_t = E_{\tilde{Q}}\left[\int_0^T \phi_s^L dP_s|\tilde{\mathcal{F}}_t\right] - \int_0^t E_{\tilde{Q}}(\phi_s^L|\tilde{\mathcal{F}}_s)dP_s + N_t$
  where $N_t = E_{\tilde{Q}}\left(E_{\tilde{Q}}(\xi|\sigma(L))|\tilde{\mathcal{F}}_t\right) - E_{\tilde{Q}}(\xi)$.
  From a filtering lemma from Pardoux (1989), $L_t = N_t$.
- We can write $E_{\tilde{Q}}(\xi|\sigma(L)) = f(L)$ where f is a borelian function, and we show that $Q^* = \frac{f(L)}{E_{\tilde{Q}}(f(L))}\tilde{Q} \in \mathcal{Q}$
- We deduce for any θ $\tilde{\mathcal{F}}$-adapted bounded process,

$$E_{\tilde{Q}}\left[N_t \int_0^t \theta_s dP_s\right] = E_{\tilde{Q}}\left[f(L)\int_0^t \theta_s dP_s\right] = E_{\tilde{Q}}(f(L))E_{Q^*}\left[\int_0^t \theta_s dP_s\right] = 0$$

  N so L is orthogonal to P.
- So it is the unique Kunita-Watanabe expected decomposition.

Clark-Ocone Formula

- An expression of ϕ_s^L may be derived from the Malliavin derivative of ξ.

Proposition

If $\xi \in L^2(\Omega, \mathcal{Y}, Q)$ *and if* $\forall x, \xi(., x) \in \mathbb{D}^{1,2}$, *then*

$$\phi_s^L = (\sigma_s^L)^{-1} \left(E_Q \left[D_s(\xi(., x)) | \mathcal{F}_t \right] |_{x=L} \right)$$

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Expression of the quadratic residual risk

- From the problem introduced by Schweizer on the quadratic risk, we study the variance of $L_T^{\tilde{Q}}$, and we obtain an expression of the **quadratic residual risk which measures the risk taken by an agent who doesn't know information L**.
- For a probability measure $\tilde{Q} \in \mathcal{Q}_N \setminus \mathcal{Q}$,

$$Var_{\tilde{Q}}\left(L_T^{\tilde{Q}}\right) = E_{\tilde{Q}}\left((\xi - E_{\tilde{Q}}(\xi))^2\right) - E_{\tilde{Q}}\left(\left(\int_0^T \phi_s^{\tilde{Q}} dP_s\right)^2\right)$$

- For a probability measure $\mathbb{Q}^* \in \mathcal{Q}$, we obtain a more simple expression

$$Var_{\mathbb{Q}^*}(L_T^{\mathbb{Q}^*}) = E_{\mathbb{Q}^*}\left(E_{\mathbb{Q}^*}\left[E_{\mathbb{Q}^*}(\xi|\sigma(L))|\tilde{\mathcal{F}}_T\right]^2\right) - E_{\mathbb{Q}^*}(\xi)^2$$

Expression of the quadratic residual risk (2)

- A minimum risk exists, it corresponds to the minimum risk linked to un-information : the agent does not have all the information driving the market.
  $\Rightarrow$ **Measure of the lack of information.**
- The hedging strategy of the non informed agent that minimizes the risk under a given equivalent martingale measure, is the conditional expectation of the insider hedging strategy, given $\mathcal{F}^P$ the information of the non informed agent.
- But measuring a risk under a probability measure is arbitrary. The evaluated risk is the risk of the model under the chosen risk-neutral information, and not the intern risk of the model : the historical probability measure is not taken into account.

Influence model example

- Example of influence model that satisfies our hypotheses :

$$
\begin{aligned}
f(s,p,x,z) &= xr + \sigma'^{-1}(b'-r)z \\
b(t,x,p,\pi) &= p\left(b_0 + \frac{b_1}{(1+p)(1+\pi^2)}\right) \\
\sigma(t,x,p) &= p\left(\sigma^0 \mathbf{1}_{[0,\eta[}(t) + \sigma^1 \mathbf{1}_{[\eta,T]}(t)\right), \quad \eta \in [0, T+\epsilon] \\
\xi &= (P_T - K)_+
\end{aligned}
$$

- Strong initial information : $L = \eta$ jump time of volatility.
- Lipschitz, linear growth.
- Integrability : all coefficients are zero whenever p, x, z are zero.
- 3rd influence case : σ independent of z.

Conclusion

- More difficult/interesting when no representation property exist, when the **asymmetry makes the market incomplete**.
- Interesting point of view for "non-standard" markets (insurance, reinsurance), where asymmetrical information is more important than on standard financial markets.
- Further research / questions : study of the incompleteness of the market, and link between $\mathcal{F}^P$ and the lack of information.
  - Which part of the information is transferred to the market ?
  - What kind of information is sufficient to complete this incomplete market, due to the lack of information ?
  - What is the best risk-neutral probability choice ?
  - How to measure the intern risk of lack of information ? And relate it to a measure of the incompleteness ?

Thank you for your attention !

S. ANKIRCHNER, C. BLANCHET-SCALLIET, and A. EYRAUD-LOISEL, *Credit risk premia and quadratic bsdes with a single jump*, IJTAF **13** (2010), no. 07, 1103–1129.

C. BLANCHET-SCALLIET, A. EYRAUD-LOISEL, and M. ROYER-CARENZI, *Bsde with uncertain horizon and hedging of defaultable contingent claims*, BFAt **10** (Dec. 2010), no. 20, 85–100.

A. EYRAUD-LOISEL, *Backward stochastic differential equations with enlarged filtration. option hedging of an insider trader in a financial market with jumps*, SPA **115** (2005), no. 11, 1745–1763.

______, *Option hedging by an influent informed investor*, Applied Stochastic Models for Business and Industry **27** (2011), 707–722.

______, *Quadratic hedging in an incomplete market derived by an influential informed investor*, Stochastics **85** (2013), no. 3, 412–430.

A. EYRAUD-LOISEL and M. ROYER-CARENZI, *BSDE with random terminal time under enlarged filtration. american-style options hedging by an insider*, ROSE **2** (2010), no. 18, 149–171.

H. FÖLLMER and M. SCHWEIZER, *Hedging of contingent claims under incomplete information*, Applied stochastic analysis, Stochastics Monogr., vol. 5, New York, 1991, pp. 389–414.

J. JACOD and A. N. SHIRYAEV, *Limit theorems for stochastic processes*, second ed., Principles of Mathematical Sciences, vol. 288, Springer-Verlag, Berlin, 2003.

E. PARDOUX and S. TANG, *Forward-backward stochastic differential equations and quasilinear parabolic PDEs*, Probability Theory and Related Fields **114** (1999), no. 2, 123–150.