Game-theoretic martingales and applications to model free financial mathematics

Nicolas Perkowski

Humboldt–Universität zu Berlin

April 22nd, 2016
LSE Risk & Stochastics Conference 2016

Based on joint work with M. Beiglböck (Vienna), A. Cox (Bath), M. Huesmann (Bonn), D. Prömel (Zurich)

Model free super-replication

- Aim: derive model free pricing hedging duality for exotic derivatives on underlying $(S_t)_{t \in [0,N]}$.
- Data: call prices $p(n, K)$ for payoff $(S_n - K)^+$ for $K \in \mathbb{R}$, $n = 1, \dots, N$.

Theorem (Beiglböck-Cox-Huesmann-P-Prömel '15)

For $G = \gamma(\mathfrak{t}(S), \langle S \rangle_1, \dots, \langle S \rangle_N)$ we have

$$\overline{p}(G) = \sup\{\mathbb{E}_{\mathbb{P}}[G] : \mathbb{P} \text{ martingale measure w/ } \mathbb{E}_{\mathbb{P}}[(S_n - K)^+] \equiv p(n, K)\}.$$

Here

- *$\overline{p}$ pathwise minimal superhedging price with dynamic trading in S, static in $(S_n - K)^+$;*
- *$\langle S \rangle$ quadratic variation;*
- *$\mathfrak{t}(S)$ time reparametrization such that $\langle \mathfrak{t}(S) \rangle_t = t$.*

Model free super-replication

- Aim: derive model free pricing hedging duality for exotic derivatives on underlying $(S_t)_{t\in[0,N]}$.
- Data: call prices $p(n,K)$ for payoff $(S_n-K)^+$ for $K\in\mathbb{R}$, $n=1,\dots,N$.

Theorem (Beiglböck-Cox-Huesmann-P-Prömel '15)

For $G=\gamma(\mathfrak{t}(S),\langle S\rangle_1,\dots,\langle S\rangle_N)$ we have

$$\overline{p}(G)=\sup\{\mathbb{E}_{\mathbb{P}}[G]:\mathbb{P}\text{ martingale measure w/ }\mathbb{E}_{\mathbb{P}}[(S_n-K)^+]\equiv p(n,K)\}.$$

Here

- $\overline{p}$ *pathwise minimal superhedging price with dynamic trading in S, static in $(S_n-K)^+$;*
- $\langle S\rangle$ *quadratic variation;*
- $\mathfrak{t}(S)$ *time reparametrization such that $\langle\mathfrak{t}(S)\rangle_t=t$.*

Outline

Pricing-hedging duality in classical theory

Classical approach:

- Stochastic model $S = (S_t)_{t \in [0,T]}$ for (discounted) asset price evolution.
- Conditions that S is reasonable model ("no arbitrage", (NFLVR)); Delbaen-Schachermayer '94: iff $\exists \mathbb{Q} \sim \mathbb{P}$ local martingale measure.
- Duality:

$$\inf\{\lambda : \exists H \text{ s.t. } \lambda + (H \cdot S)_T \geq G \text{ a.s.}\} = \sup\{\mathbb{E}_{\mathbb{Q}}[G] : \mathbb{Q} \sim \mathbb{P} \text{ LMM}\}.$$

Problem: Assumption that statistical evolution of S is known.

- Unlike in physics: no first principles, no controlled experiments to derive / verify model.
- Even if we assume (say) Black-Scholes model, then parameters not known $\Rightarrow$ model uncertainty.

Pricing-hedging duality in classical theory

Classical approach:

- Stochastic model $S = (S_t)_{t\in[0,T]}$ for (discounted) asset price evolution.
- Conditions that S is reasonable model ("no arbitrage", (NFLVR));
  Delbaen-Schachermayer '94: iff $\exists \mathbb{Q} \sim \mathbb{P}$ local martingale measure.
- Duality:

$$\inf\{\lambda : \exists H \text{ s.t. } \lambda + (H \cdot S)_T \geq G \text{ a.s.}\} = \sup\{\mathbb{E}_{\mathbb{Q}}[G] : \mathbb{Q} \sim \mathbb{P} \text{ LMM}\}.$$

Problem: Assumption that statistical evolution of S is known.

- Unlike in physics: no first principles, no controlled experiments to derive / verify model.
- Even if we assume (say) Black-Scholes model, then parameters not known $\Rightarrow$ model uncertainty.

Inverse point of view

- European calls liquidly traded, prices part of market data.
- Use them to calibrate Black-Scholes model (implied volatility). Problem: inconsistency.
- Dupire '94: Find unique martingale diffusion reproducing call prices. Get consistent prices for exotic derivatives.
- But why should prices come from martingale diffusions?

Inverse point of view

- European calls liquidly traded, prices part of market data.
- Use them to calibrate Black-Scholes model (implied volatility). Problem: inconsistency.
- Dupire '94: Find unique martingale diffusion reproducing call prices. Get consistent prices for exotic derivatives.
- But why should prices come from martingale diffusions?

The model free approach

- Initiated by Hobson '98. Call prices known for maturity = terminal time.
- Require superhedging for all continuous paths. Trading S and calls.
- Consider specific exotic derivative G and show pricing hedging-duality

$$\overline{p}(G) = \sup_{\mathbb{Q}\ \mathrm{MM(p)}} \mathbb{E}_{\mathbb{Q}}[G],$$

  where $\overline{p}$ = minimal superhedging price, MM(p) = martingale measure reproducing known call prices.
- $\geq$ is easy, $\leq$ shown by constructing optimal martingale measure via Skorokhod embedding.
- Applied for example to
  - lookback options (Hobson '98, $G = \max_{t\in[0,T]} S_t$);
  - forward start digital options (Hobson-Pedersen '02, $G = \mathbb{1}_{\{\max_{t\in[T_1,T_2]} S_t \geq B\}}$);
  - options on local time (Cox-Hobson-Obłój '08, $G = F(L^S_T)$);
  - on variance (e.g. Davis-Obłój-Raval '14, Cox-Wang '13, $G = F(\langle \log S\rangle_T)$);
  - and many more.

The model free approach

- Initiated by Hobson '98. Call prices known for maturity = terminal time.
- Require superhedging for all continuous paths. Trading S and calls.
- Consider specific exotic derivative G and show pricing hedging-duality

$$\overline{p}(G) = \sup_{\mathbb{Q}\ \mathrm{MM(p)}} \mathbb{E}_{\mathbb{Q}}[G],$$

  where $\overline{p}$ = minimal superhedging price, MM(p) = martingale measure reproducing known call prices.
- $\geq$ is easy, $\leq$ shown by constructing optimal martingale measure via Skorokhod embedding.
- Applied for example to
  - lookback options (Hobson '98, $G = \max_{t\in[0,T]} S_t$);
  - forward start digital options (Hobson-Pedersen '02, $G = \mathbb{1}_{\{\max_{t\in[T_1,T_2]} S_t \geq B\}}$);
  - options on local time (Cox-Hobson-Obłój '08, $G = F(L^S_T)$);
  - on variance (e.g. Davis-Obłój-Raval '14, Cox-Wang '13, $G = F(\langle \log S\rangle_T)$);
  - and many more.

The model free approach

- Initiated by Hobson '98. Call prices known for maturity = terminal time.
- Require superhedging for all continuous paths. Trading S and calls.
- Consider specific exotic derivative G and show pricing hedging-duality

$$\overline{p}(G) = \sup_{\mathbb{Q}\ \mathrm{MM(p)}} \mathbb{E}_{\mathbb{Q}}[G],$$

  where $\overline{p}$ = minimal superhedging price, MM(p) = martingale measure reproducing known call prices.
- $\geq$ is easy, $\leq$ shown by constructing optimal martingale measure via Skorokhod embedding.
- Applied for example to
  - lookback options (Hobson '98, $G = \max_{t\in[0,T]} S_t$);
  - forward start digital options (Hobson-Pedersen '02, $G = \mathbb{1}_{\{\max_{t\in[T_1,T_2]} S_t \geq B\}}$);
  - options on local time (Cox-Hobson-Obłój '08, $G = F(L_T^S)$);
  - on variance (e.g. Davis-Obłój-Raval '14, Cox-Wang '13, $G = F(\langle \log S\rangle_T)$);
  - and many more.

Skorokhod embedding approach: Example

Price digital barrier option $\mathbb{1}_{\{\tau_B \le T\}}$, where τ_B hitting time of $B > S_0$.

- Deterministic inequality for $K < B$:

$$\mathbb{1}_{\{\tau_B \le T\}} \le \frac{(S_T - K)^+}{B - K} + \frac{S_{\tau_B} - S_T}{B - K} \mathbb{1}_{\{\tau_B \le T\}}.$$

- So for $\overline{p}$ minimum superhedging price:

$$\overline{p}(\mathbb{1}_{\{\tau_B \le T\}}) \le \inf_{K<B} \frac{p((S_T - K)^+)}{B - K}.$$

- Skorokhod embedding: find martingale measure $\mathbb{P}$ with $\mathbb{E}_{\mathbb{P}}[(S_T - K)^+] = p((S_T - K)^+)$, $\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau_B \le T\}}] = \inf_{K<B} \frac{p((S_T-K)^+)}{B-K}$.
- But for MM(p) = martingale measures with correct call prices:

$$\sup_{\mathbb{Q}\ \mathrm{MM(p)}} \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{\tau_B \le T\}}] \le \overline{p}(\mathbb{1}_{\{\tau_B \le T\}}).$$

- In particular (Brown-Hobson-Rogers '01)

$$\overline{p}(\mathbb{1}_{\{\tau_B \le T\}}) = \sup_{\mathbb{Q}\ \mathrm{MM(p)}} \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{\tau_B \le T\}}] = \inf_{K<B} \frac{p((S_T - K)^+)}{B - K}.$$

Skorokhod embedding approach: Example

Price digital barrier option $\mathbb{1}_{\{\tau_B \le T\}}$, where τ_B hitting time of $B > S_0$.

- Deterministic inequality for $K < B$:

$$\mathbb{1}_{\{\tau_B \le T\}} \le \frac{(S_T - K)^+}{B - K} + \frac{S_{\tau_B} - S_T}{B - K} \mathbb{1}_{\{\tau_B \le T\}}.$$

- So for $\overline{p}$ minimum superhedging price:

$$\overline{p}(\mathbb{1}_{\{\tau_B \le T\}}) \le \inf_{K<B} \frac{p((S_T - K)^+)}{B - K}.$$

- Skorokhod embedding: find martingale measure $\mathbb{P}$ with $\mathbb{E}_{\mathbb{P}}[(S_T - K)^+] = p((S_T - K)^+)$, $\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\tau_B \le T\}}] = \inf_{K<B} \frac{p((S_T-K)^+)}{B-K}$.
- But for MM(p) = martingale measures with correct call prices:

$$\sup_{\mathbb{Q}\ \mathrm{MM(p)}} \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{\tau_B \le T\}}] \le \overline{p}(\mathbb{1}_{\{\tau_B \le T\}}).$$

- In particular (Brown-Hobson-Rogers '01)

$$\overline{p}(\mathbb{1}_{\{\tau_B \le T\}}) = \sup_{\mathbb{Q}\ \mathrm{MM(p)}} \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{\tau_B \le T\}}] = \inf_{K<B} \frac{p((S_T - K)^+)}{B - K}.$$

A general result

- Skorokhod approach gives more information than pricing hedging duality, e.g. explicit value.
- But needs new arguments for every new derivative.

Theorem (Dolinsky-Soner '13)

Assume call prices $p((S_T - K)^+)$ known $\forall K$ and G is Lipschitz in sup-norm. Then

$$\overline{p}(G) = \sup\{\mathbb{E}_{\mathbb{P}}[G] : \mathbb{P} \; MM \; w/ \; \mathbb{E}_{\mathbb{P}}[(S_T - K)^+] \equiv p((S_T - K)^+)\}.$$

- Proof based on martingale optimal transport.
- Dolinsky-Soner '15: Generalization to multivariate jump processes, multiple margins, partial information. Need uniform continuity in sup-norm.
- Exludes options on volatility; somewhat orthogonal to Skorokhod embedding approach.

A general result

- Skorokhod approach gives more information than pricing hedging duality, e.g. explicit value.
- But needs new arguments for every new derivative.

Theorem (Dolinsky-Soner '13)

Assume call prices $p((S_T - K)^+)$ known $\forall K$ and G is Lipschitz in sup-norm. Then

$$\overline{p}(G) = \sup\{\mathbb{E}_{\mathbb{P}}[G] : \mathbb{P} \text{ } MM \text{ } w/ \text{ } \mathbb{E}_{\mathbb{P}}[(S_T - K)^+] \equiv p((S_T - K)^+)\}.$$

- Proof based on martingale optimal transport.
- Dolinsky-Soner '15: Generalization to multivariate jump processes, multiple margins, partial information. Need uniform continuity in sup-norm.
- Exludes options on volatility; somewhat orthogonal to Skorokhod embedding approach.

A general result

- Skorokhod approach gives more information than pricing hedging duality, e.g. explicit value.
- But needs new arguments for every new derivative.

Theorem (Dolinsky-Soner '13)

Assume call prices $p((S_T - K)^+)$ known $\forall K$ and G is Lipschitz in sup-norm. Then

$$\overline{p}(G) = \sup\{\mathbb{E}_{\mathbb{P}}[G] : \mathbb{P} \text{ } MM \text{ } w/ \text{ } \mathbb{E}_{\mathbb{P}}[(S_T - K)^+] \equiv p((S_T - K)^+)\}.$$

- Proof based on martingale optimal transport.
- Dolinsky-Soner '15: Generalization to multivariate jump processes, multiple margins, partial information. Need uniform continuity in sup-norm.
- Exludes options on volatility; somewhat orthogonal to Skorokhod embedding approach.

Aims for the rest of the talk

- Show general pricing-hedging duality in Skorokhod approach, allowing options on volatility.
- Introduce game-theoretic martingales (important tool in proof).
- Combine game-theoretic martingales and Skorokhod embedding.

Outline

Vovk's game-theoretic approach

Set $\Omega := C([0,\infty),\mathbb{R})$, $S_t(\omega) = \omega(t)$. A simple strategy H consists of

1. stopping times $0 = \tau_0 < \tau_1 < \dots$ with $\lim_{n\to\infty} \tau_n(\omega) = \infty$ for all ω;
2. $\mathcal{F}_{\tau_n}$-measurable bounded functions $F_n : \Omega \to \mathbb{R}$.

Then for all ω, t

$$(H \cdot S)_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega)[S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)]$$

is well defined. H is λ-admissible if $(H \cdot S)_t(\omega) \geq -\lambda$ for all ω, t.

Definition (Vovk '09 / P-Prömel '15)

The outer measure $\overline{P}$ of $A \subseteq \Omega$ is

$$\overline{P}(A) := \inf\left\{\lambda : \exists (H^n)_n \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_t(\omega)) \geq \mathbb{1}_A(\omega) \forall \omega\right\}.$$

Game-theoretic martingales are the capital processes $\lambda + (H \cdot S)$, $H \in \mathcal{H}_\lambda$.

Vovk's game-theoretic approach

Set $\Omega := C([0,\infty),\mathbb{R})$, $S_t(\omega) = \omega(t)$. A simple strategy H consists of

1. stopping times $0 = \tau_0 < \tau_1 < \dots$ with $\lim_{n\to\infty} \tau_n(\omega) = \infty$ for all ω;
2. $\mathcal{F}_{\tau_n}$-measurable bounded functions $F_n : \Omega \to \mathbb{R}$.

Then for all ω, t

$$(H \cdot S)_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega)[S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)]$$

is well defined. H is λ-admissible if $(H \cdot S)_t(\omega) \geq -\lambda$ for all ω, t.

Definition (Vovk '09 / P-Prömel '15)

The outer measure $\overline{P}$ of $A \subseteq \Omega$ is

$$\overline{P}(A) := \inf\left\{\lambda : \exists (H^n)_n \subseteq \mathcal{H}_\lambda \text{ s.t. } \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_t(\omega)) \geq \mathbb{1}_A(\omega) \forall \omega\right\}.$$

Game-theoretic martingales are the capital processes $\lambda + (H \cdot S)$, $H \in \mathcal{H}_\lambda$.

Link with classical theory

Lemma (Vovk '12)

$$\sup_{\mathbb{P}\ MM} \mathbb{P}(A) \le \overline{P}(A), \qquad A \in \mathcal{F}_\infty.$$

- By scaling: $\overline{P}(A) = 0$ iff there are $(H^n) \subset \mathcal{H}_1$ with

$$\liminf_{n\to\infty}(1 + (H^n \cdot S)_\infty(\omega)) \ge \infty \cdot \mathbb{1}_A(\omega).$$

- Recall: Probability $\mathbb{P}$ on Ω satisfies (NA1) (= (NUPBR)) if

$$\{1 + (H \cdot S)_\infty : H \in \mathcal{H}_1\}$$

is bounded in probability.

Lemma (P-Prömel '15)

Let $A \in \mathcal{F}_\infty$. If $\overline{P}(A) = 0$, then $\mathbb{P}(A) = 0$ for all $\mathbb{P}$ with (NA1).

(NA1) is the minimal assumption any market model should fulfill!

(Platen's+Runggaldier's talks, Ankirchner '05, Karatzas-Kardaras '07, Ruf '13, Imkeller-P '15...)

Link with classical theory

Lemma (Vovk '12)

$$\sup_{\mathbb{P}\ MM} \mathbb{P}(A) \leq \overline{P}(A), \qquad A \in \mathcal{F}_\infty.$$

- By scaling: $\overline{P}(A) = 0$ iff there are $(H^n) \subset \mathcal{H}_1$ with
$$\liminf_{n\to\infty}(1 + (H^n \cdot S)_\infty(\omega)) \geq \infty \cdot \mathbb{1}_A(\omega).$$
- Recall: Probability $\mathbb{P}$ on Ω satisfies (NA1) (= (NUPBR)) if
$$\{1 + (H \cdot S)_\infty : H \in \mathcal{H}_1\}$$
is bounded in probability.

Lemma (P-Prömel '15)

Let $A \in \mathcal{F}_\infty$. If $\overline{P}(A) = 0$, then $\mathbb{P}(A) = 0$ for all $\mathbb{P}$ with (NA1).

(NA1) is the minimal assumption any market model should fulfill!

(Platen's+Runggaldier's talks, Ankirchner '05, Karatzas-Kardaras '07, Ruf '13, Imkeller-P '15...)

Link with classical theory

Lemma (Vovk '12)

$$\sup_{\mathbb{P}\ MM} \mathbb{P}(A) \leq \overline{P}(A), \qquad A \in \mathcal{F}_\infty.$$

- By scaling: $\overline{P}(A) = 0$ iff there are $(H^n) \subset \mathcal{H}_1$ with
$$\liminf_{n\to\infty}(1 + (H^n \cdot S)_\infty(\omega)) \geq \infty \cdot \mathbb{1}_A(\omega).$$
- Recall: Probability $\mathbb{P}$ on Ω satisfies (NA1) (= (NUPBR)) if
$$\{1 + (H \cdot S)_\infty : H \in \mathcal{H}_1\}$$
is bounded in probability.

Lemma (P-Prömel '15)

Let $A \in \mathcal{F}_\infty$. If $\overline{P}(A) = 0$, then $\mathbb{P}(A) = 0$ for all $\mathbb{P}$ with (NA1).

(NA1) is the minimal assumption any market model should fulfill!

(Platen's+Runggaldier's talks, Ankirchner '05, Karatzas-Kardaras '07, Ruf '13, Imkeller-P '15...)

Typical price paths

Property (P) holds for typical price paths if it is violated on a null set.

Observations due to Vovk:

- Typical price paths have a quadratic variation $\langle S \rangle$.
- Typical price paths have finite p-variation for $p > 2$.
- Typical price paths have no point of increase.

Observations due to P-Prömel:

- Typical price paths have an associated Itô rough path in the sense of Lyons.
- Typical price paths have a nice local time.
- We can construct an Itô integral for typical price paths.

Pathwise Dambis Dubins-Schwarz Theorem

Define a time-change operator by

$$\langle \mathfrak{t}(\omega) \rangle_t = t, \quad t \in [0, \infty).$$

Theorem (Vovk '12)

Let $\mathbb{W}$ *be the Wiener measure. For* $F \geq 0$ *measurable and* $c \in \mathbb{R}$

$$\overline{E}[(F \circ \mathfrak{t}) \mathbb{1}_{\{S_0 = c, \langle S \rangle_\infty = \infty\}}] = \int_\Omega F(c + \omega) \mathbb{W}(\mathrm{d}\omega).$$

Outline

Model-independent Pricing

Our (model-free) setting:

- $C_0([0,1],\mathbb{R})$ describes all possible price paths, all starting in 0.
- European calls with maturity 1 are frequently traded;
  $\Rightarrow$ marginal distribution of S at time 1 given by μ.
- Exotic derivative is functional $G\colon C_0([0,1],\mathbb{R}) \to \mathbb{R}$.

1. Pricing by martingale measures or primal maximization problem:

$$P := \sup_{\mathbb{P}\in \mathrm{MM}(\mu)} \mathbb{E}_{\mathbb{P}}[G],$$

where $\mathrm{MM}(\mu)$ are the martingale measure on $C_0([0,1],\mathbb{R})$ s.t. $S_1 \sim \mu$.

Model-independent Pricing

Our (model-free) setting:

- $C_0([0,1],\mathbb{R})$ describes all possible price paths, all starting in 0.
- European calls with maturity 1 are frequently traded;
  $\Rightarrow$ marginal distribution of S at time 1 given by μ.
- Exotic derivative is functional $G\colon C_0([0,1],\mathbb{R}) \to \mathbb{R}$.

1. Pricing by martingale measures or primal maximization problem:

$$P := \sup_{\mathbb{P}\in \mathrm{MM}(\mu)} \mathbb{E}_{\mathbb{P}}[G],$$

where $\mathrm{MM}(\mu)$ are the martingale measure on $C_0([0,1],\mathbb{R})$ s.t. $S_1 \sim \mu$.

Pathwise Super-replication

2. Pricing by super-replication:

- Simple strategies defined as in game-theoretic setting.
- $\mathcal{A}$ are "admissible" simple strategies.
- $\mathcal{E}$ are "European options available at price 0":

$$\mathcal{E} := \Big\{ \psi \in C(\mathbb{R}) \, : \, \int \psi(x)\mu(\mathrm{d}x) = 0 \Big\}.$$

$\Rightarrow$ Dual minimization problem:

$$D := \inf \left\{ p : \begin{array}{l} \exists (H^n) \subseteq \mathcal{A}, \psi \in \mathcal{E} \text{ s.t. } \forall \omega \\ p + \liminf_n (H^n \cdot S)_1(\omega) + \psi(S_1(\omega)) \geq G(\omega) \end{array} \right\}.$$

Pathwise super-replication Theorem

Consider (time-invariant) options

$$G(\omega) = \gamma(\mathfrak{t}(\omega), \langle\omega\rangle_1),$$

where $\mathfrak{t}(\omega)$ is Vovk's time-change operator, i.e. $\langle\mathfrak{t}(\omega)\rangle_t = t$.

Theorem (Beiglböck-Cox-Huesmann-P-Prömel '15)

Let γ be upper semi-continuous and bounded from above.

Then we have the duality relation $P = D$.

Related works:

- Dolinsky-Soner '14, Dolinsky-Soner '15, Hou-Obłój '15;
- Quasi-sure setting: many, but especially Guo-Tan-Touzi '15.

Sketch of the Proof

$P \leq D$ is easy:

- For $p > D$ find $(H^n) \subset \mathcal{A}$ and $\psi \in C(\mathbb{R})$ with $\int \psi(x)\mu(\mathrm{d}x) = 0$ s.t.

$$p + \liminf_n (H^n \cdot S)_1(\omega) + \psi(S_1(\omega)) \geq G(\omega).$$

- For martingale measure $\mathbb{P}$ with $S_1(Q) = \mu$ we have by Fatou's lemma

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[G] &\leq \mathbb{E}_{\mathbb{P}}[p + \liminf_n (H^n \cdot S)_1 + \psi(S_1)] \\ &\leq p + \liminf_n \mathbb{E}_{\mathbb{P}}[(H^n \cdot S)_1] + \mathbb{E}_{\mathbb{P}}[\psi(S_1)] \leq p. \end{aligned}$$

Sketch of the Proof

The inequality $P \geq D$:

Idea: Primal problem is linked to Skorokhod embedding problem (goes back to Hobson '98).

Problem (Skorokhod embedding problem (1961))

Given Brownian motion B and centered probability measure ν:

Find stopping time τ with $B_\tau \sim \nu$ and $B_{\cdot \wedge \tau}$ is u.i. martingale.

Observation: If $\mathbb{P}$ is a martingale measure, then by Dambis Dubins-Schwarz $(\mathfrak{t}(S)_{t \wedge \langle S \rangle_1})_{t \geq 0}$ is a stopped Brownian motion.

Lemma

The value P of the primal maximization problem is

$$P = P^* := \sup \{\mathbb{E}_{\mathbb{W}}[\gamma((B_s)_{s \leq \tau}, \tau)] : B_\tau \sim \mu, B_{\cdot \wedge \tau} \text{ is u.i. martingale}\}.$$

Recall: $G(\omega) = \gamma(\mathfrak{t}(\omega), \langle \omega \rangle_1)$, $\mathbb{W}$ = *Wiener measure.*

Sketch of the Proof

The inequality $P \geq D$:

Idea: Primal problem is linked to Skorokhod embedding problem (goes back to Hobson '98).

Problem (Skorokhod embedding problem (1961))

Given Brownian motion B and centered probability measure ν:

Find stopping time τ with $B_\tau \sim \nu$ and $B_{\cdot\wedge\tau}$ is u.i. martingale.

Observation: If $\mathbb{P}$ is a martingale measure, then by Dambis Dubins-Schwarz $(\mathfrak{t}(S)_{t\wedge\langle S\rangle_1})_{t\geq 0}$ is a stopped Brownian motion.

Lemma

The value P of the primal maximization problem is

$$P = P^* := \sup \{\mathbb{E}_{\mathbb{W}}[\gamma((B_s)_{s\leq\tau}, \tau)] : B_\tau \sim \mu, B_{\cdot\wedge\tau} \text{ is u.i. martingale}\} .$$

Recall: $G(\omega) = \gamma(\mathfrak{t}(\omega), \langle\omega\rangle_1)$, $\mathbb{W}$ = *Wiener measure.*

Sketch of the Proof

Theorem (Beiglböck-Cox-Huesmann '14)

Let γ be upper semi-continuous and bounded from above. Put

$$D^* := \inf\left\{ p : \begin{array}{l} \exists \psi \in \mathcal{E}, m \in C_b(\Omega) \textit{ s.t. } \mathbb{E}_{\mathbb{W}}[m] = 0 \textit{ and } \forall \omega, t \\ p + \mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_t](\omega) + \psi(B_t(\omega)) \geq \gamma(\omega, t) \end{array} \right\},$$

Then $P^* = D^*$.

- Apply $\mathfrak{t}$ on both sides:

$$D^* = \inf\left\{ p : \begin{array}{l} \exists \psi \in \mathcal{E}, m \in C_b(\Omega) \text{ s.t. } \mathbb{E}_{\mathbb{W}}[m] = 0 \text{ and } \forall \omega, t \\ p + \mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_{\langle \omega \rangle_t}](\mathfrak{t}(\omega)) + \psi(B_t(\omega)) \geq \gamma(\mathfrak{t}(\omega), \langle \omega \rangle_t) \end{array} \right\}.$$

- Apply Vovk's pathwise Dambis Dubins-Schwarz theorem to get a pathwise superhedge for $\mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_{\langle \omega \rangle_t}](\mathfrak{t}(\omega))$.

In particular

$$D \leq D^* = P^* = P.$$

Sketch of the Proof

Theorem (Beiglböck-Cox-Huesmann '14)

Let γ be upper semi-continuous and bounded from above. Put

$$D^* := \inf \left\{ p : \begin{array}{l} \exists \psi \in \mathcal{E}, m \in C_b(\Omega) \text{ s.t. } \mathbb{E}_{\mathbb{W}}[m] = 0 \text{ and } \forall \omega, t \\ p + \mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_t](\omega) + \psi(B_t(\omega)) \geq \gamma(\omega, t) \end{array} \right\},$$

Then $P^* = D^*$.

- Apply $\mathfrak{t}$ on both sides:

$$D^* = \inf \left\{ p : \begin{array}{l} \exists \psi \in \mathcal{E}, m \in C_b(\Omega) \text{ s.t. } \mathbb{E}_{\mathbb{W}}[m] = 0 \text{ and } \forall \omega, t \\ p + \mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_{\langle\omega\rangle_t}](\mathfrak{t}(\omega)) + \psi(B_t(\omega)) \geq \gamma(\mathfrak{t}(\omega), \langle\omega\rangle_t) \end{array} \right\}.$$

- Apply Vovk's pathwise Dambis Dubins-Schwarz theorem to get a pathwise superhedge for $\mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_{\langle\omega\rangle_t}](\mathfrak{t}(\omega))$.

In particular

$$D \leq D^* = P^* = P.$$

Generalizations, applications and limitations

Case of multiple margins works analogously and is included in paper. With bigger effort it would also be possible to treat:

- weaker regularity and boundedness assumptions on G;
- partial information for margins $1, \ldots, N-1$ (i.e. not all call prices known); but always need full information for N-th margin.

Examples:

- $G_1 = F_1(S_1, \ldots, S_N, \langle S\rangle_1, \ldots, \langle S\rangle_N)$;
- $G_2 = F_2(\max_{t\in[0,N]} S_t)$;
- $G_3 = F_3(\int_0^N f(S_s, \langle S\rangle_s)\mathrm{d}\langle S\rangle_s)$;
- $G_4 = F_4(G_1, G_2, G_3)$.

Non-example:

- Asian options $G = F(\int_0^N S_s \mathrm{d}s)$;
- ok: discretely monitored Asian options (replace integral by sum).

Biggest limitation:

- Cover essentially any derivative of interest in $d = 1$. But no chance for $d > 1$.

Generalizations, applications and limitations

Case of multiple margins works analogously and is included in paper. With bigger effort it would also be possible to treat:

- weaker regularity and boundedness assumptions on G;
- partial information for margins $1, \dots, N-1$ (i.e. not all call prices known); but always need full information for N-th margin.

Examples:

- $G_1 = F_1(S_1, \dots, S_N, \langle S \rangle_1, \dots, \langle S \rangle_N)$;
- $G_2 = F_2(\max_{t \in [0,N]} S_t)$;
- $G_3 = F_3(\int_0^N f(S_s, \langle S \rangle_s) \mathrm{d}\langle S \rangle_s)$;
- $G_4 = F_4(G_1, G_2, G_3)$.

Non-example:

- Asian options $G = F(\int_0^N S_s \mathrm{d}s)$;
- ok: discretely monitored Asian options (replace integral by sum).

Biggest limitation:

- Cover essentially any derivative of interest in $d = 1$. But no chance for $d > 1$.

Generalizations, applications and limitations

Case of multiple margins works analogously and is included in paper. With bigger effort it would also be possible to treat:

- weaker regularity and boundedness assumptions on G;
- partial information for margins $1, \dots, N-1$ (i.e. not all call prices known); but always need full information for N-th margin.

Examples:

- $G_1 = F_1(S_1, \dots, S_N, \langle S \rangle_1, \dots, \langle S \rangle_N)$;
- $G_2 = F_2(\max_{t \in [0,N]} S_t)$;
- $G_3 = F_3(\int_0^N f(S_s, \langle S \rangle_s) \mathrm{d} \langle S \rangle_s)$;
- $G_4 = F_4(G_1, G_2, G_3)$.

Non-example:

- Asian options $G = F(\int_0^N S_s \mathrm{d} s)$;
- ok: discretely monitored Asian options (replace integral by sum).

Biggest limitation:

- Cover essentially any derivative of interest in $d = 1$. But no chance for $d > 1$.

Conclusion

- Pricing-hedging duality among fundamental results in math finance.
- In model-free context: Skorokhod embedding approach of Hobson (case-by-case).
- Or martingale optimal transport approach of Dolinsky-Soner: systematic, but excludes options on volatility.
- Combining game-theoretic martingales and new results on Skorokhod embedding: get systematic approach that includes virtually all options of practical interest, but only in $d = 1$.

Thank you