Benchmark Approach to Finance

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Pl. & Heath (2006, 2010) A Benchmark Approach to Quantitative Finance. Springer

Challenges

- Low or negative interest rates
- Too low return on investment for long dated bonds (pensions, green bonds, life insurance...)
- Financial planning practice not supported by classical theory

  $\Rightarrow$ Current classical pricing and hedging too restrictive

Best Performing Portfolio as Benchmark

Criterion: **Highest value in the long run!**

- path wise
- no expectation
- prefer more for less
- ranking independent from denomination
- difficult to manipulate

Long Term Growth Rate

$$g^{\pi} = \limsup_{t \to \infty} \frac{1}{t} \ln\left(\frac{S_t^{\pi}}{S_0^{\pi}}\right) \to \max$$

e.g. Pl. & Heath (2010), Karatzas & Kardaras (2007)

$\Rightarrow$

$$\boldsymbol{g^{\pi} \leq g^{\pi *}}$$

$\boldsymbol{S^*}$ **numéraire portfolio** (NP) (Long 1990)

Existence of NP

$$\Leftrightarrow$$

Supermartingale property:

$$\hat{S}_t^{\pi} = \frac{S_t^{\pi}}{S_t^{*}} \geq E\left(\hat{S}_s^{\pi} | \mathcal{F}_t\right)$$

$$0 \leq t \leq s < \infty$$

$$\hat{S}_t^{\pi} \geq 0$$

Key property of financial market !

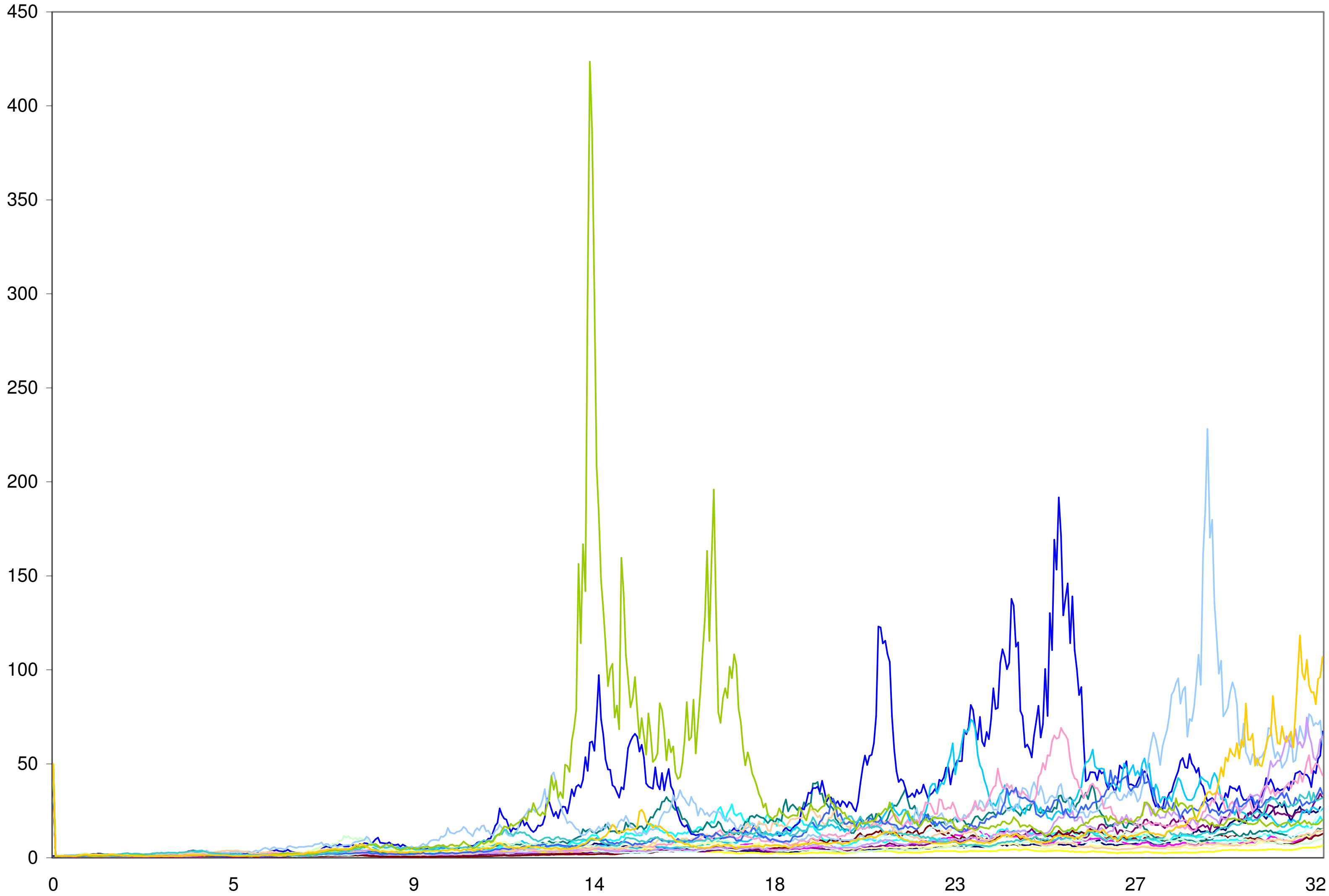

Figure 1: Primary security accounts under the MMM

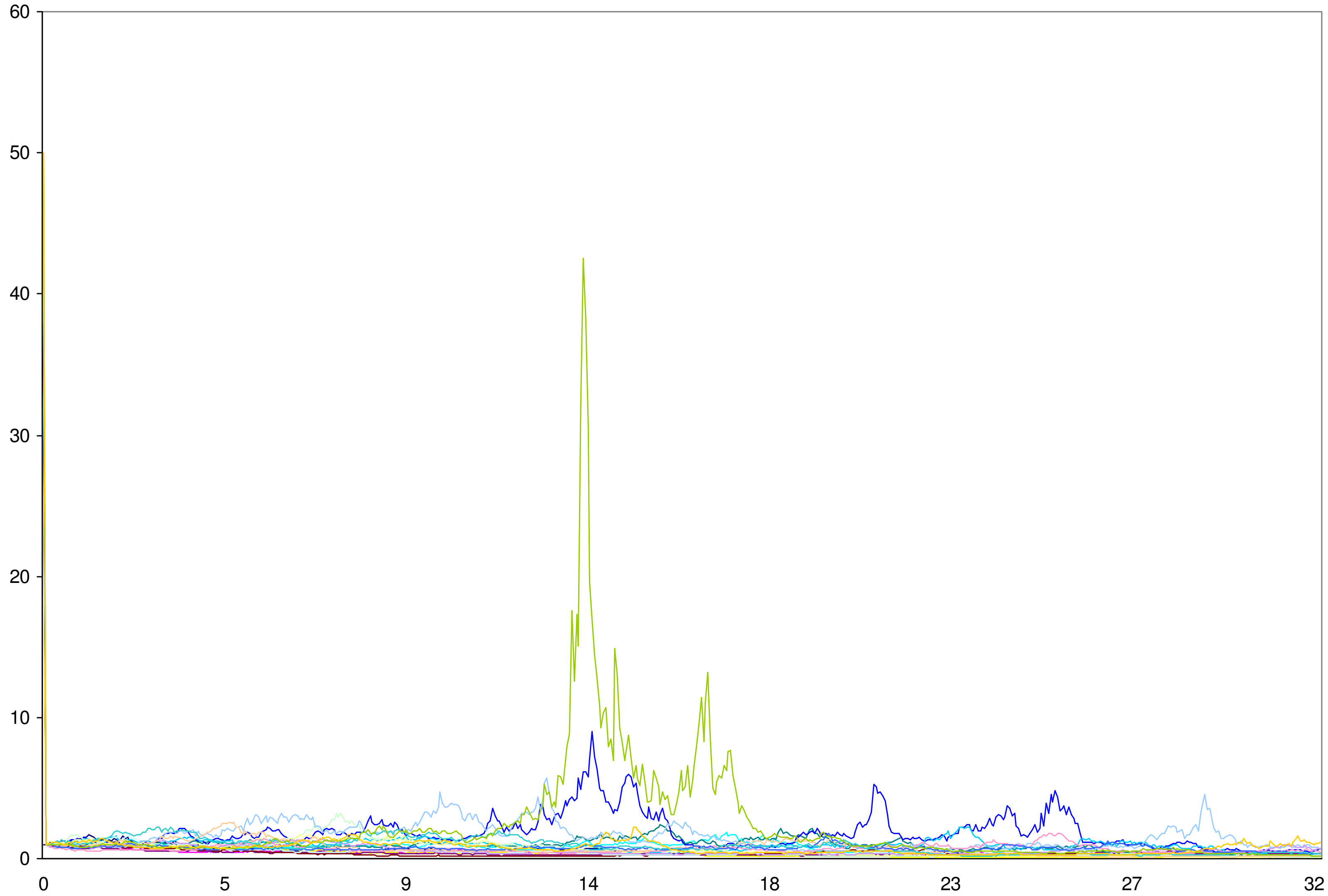

Figure 2: Benchmarked primary security accounts

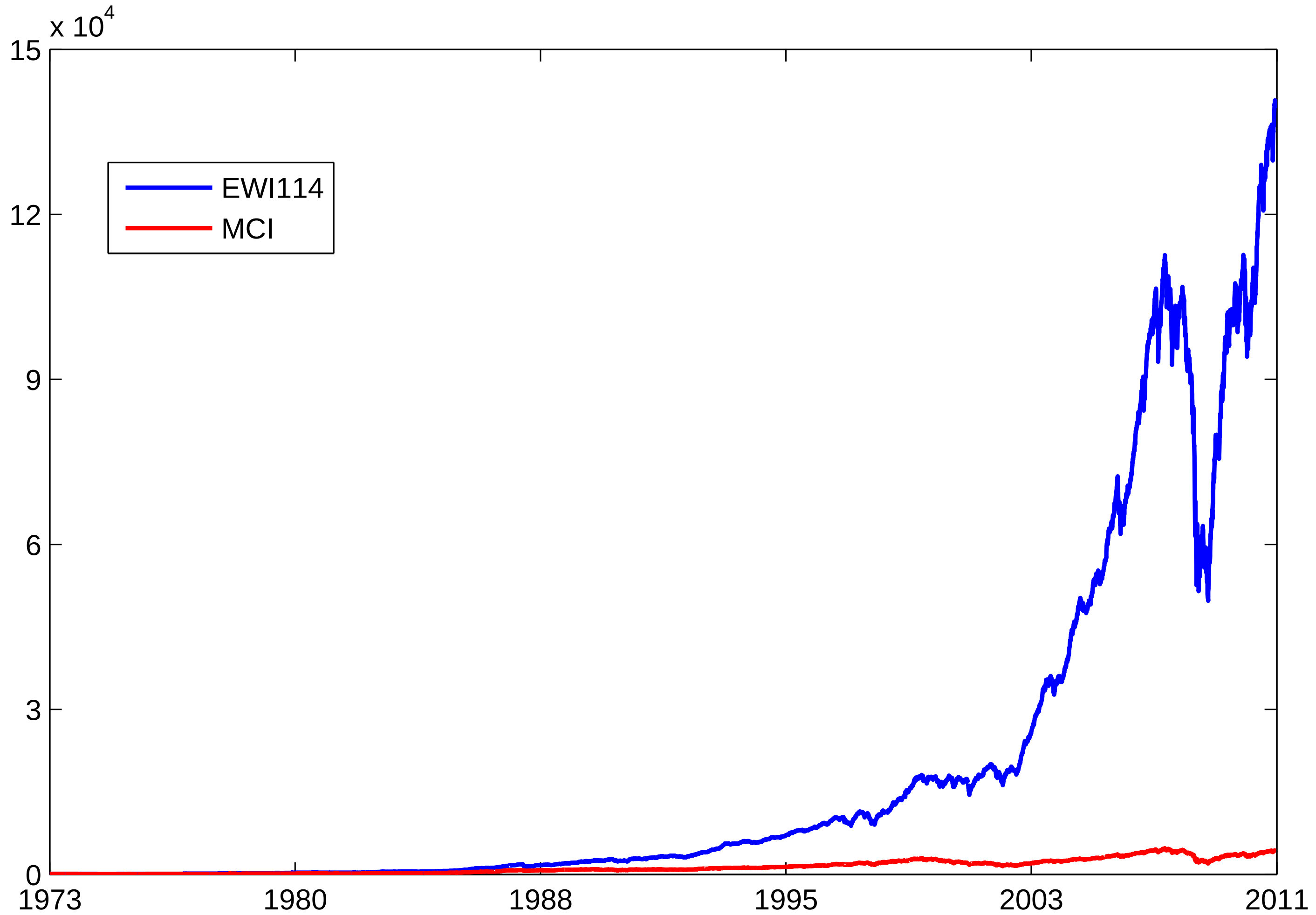

Figure 3: EWI141 and MCI

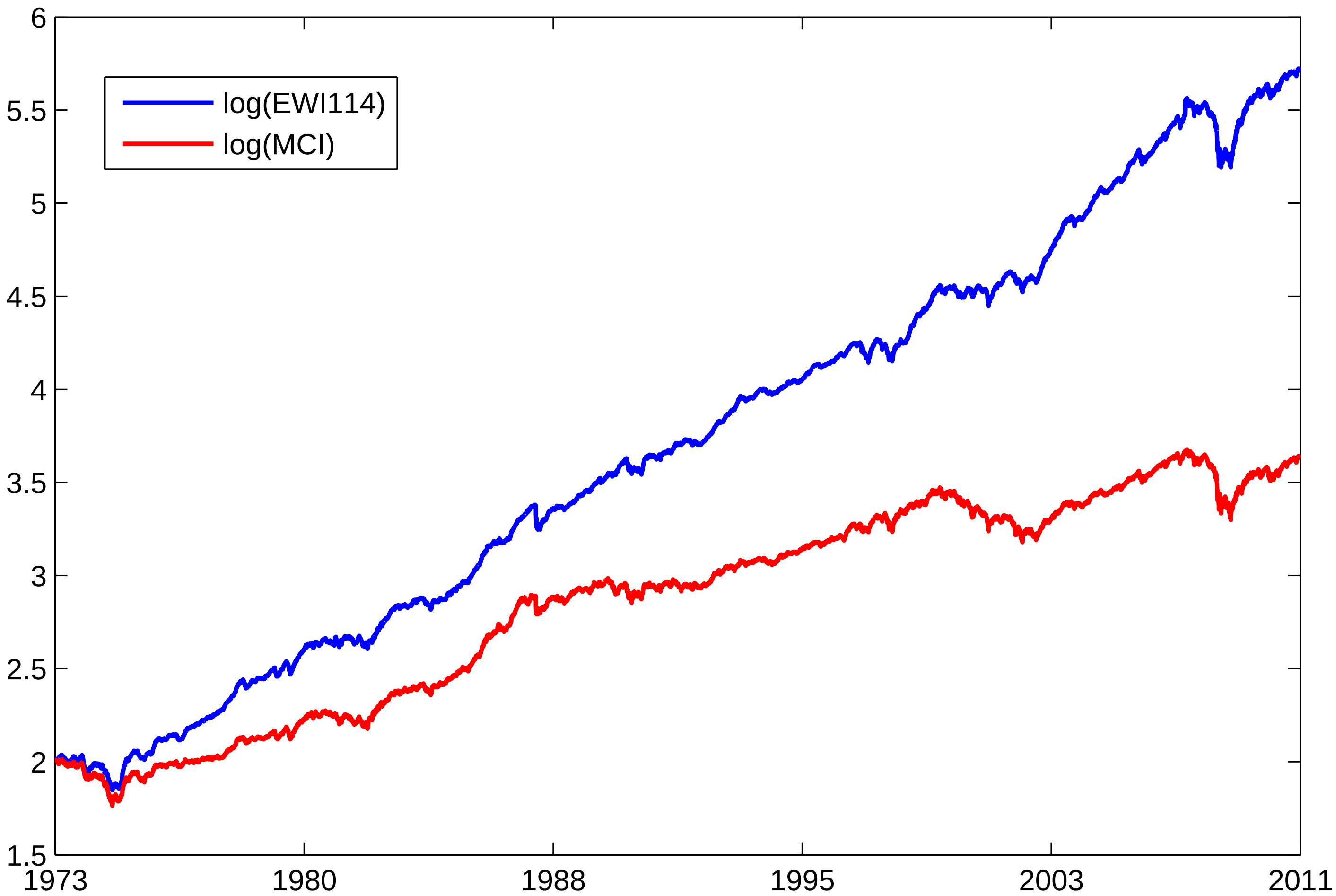

Figure 4: logarithms of EWI114 and MCI

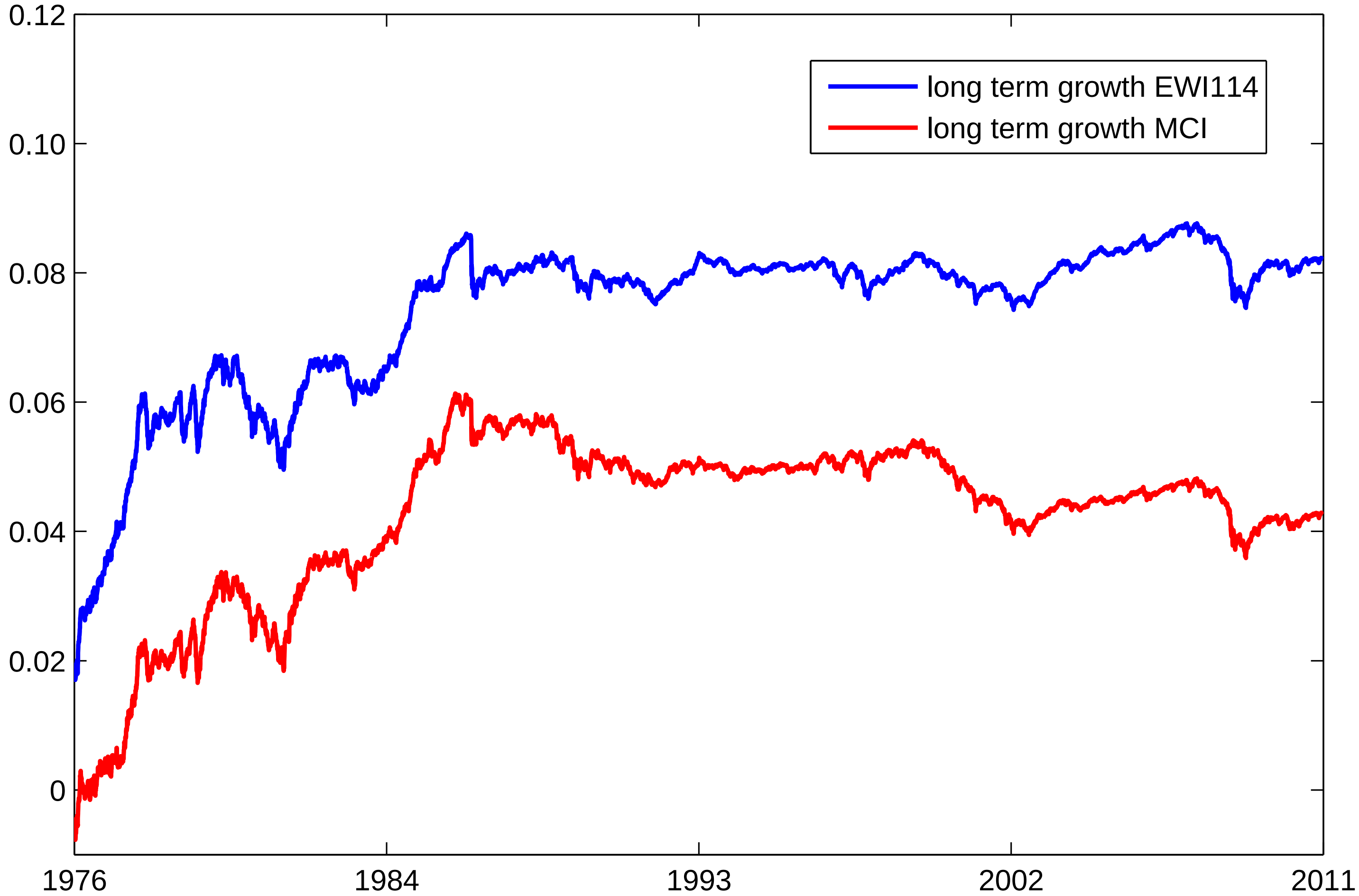

Figure 5: Long term growth of EWI114 and MCI

Performance Measure

- estimated long term growth rate

$$g_t^\pi = \frac{1}{t} \ln \left(\frac{S_t^\pi}{S_0^\pi} \right)$$

- difficult to manipulate
- allows systematic search for better proxies of NP
- well-diversified portfolios, proxies of respective NP
- driven by the same non-diversifiable uncertainty
- factors
- smart beta
- ...

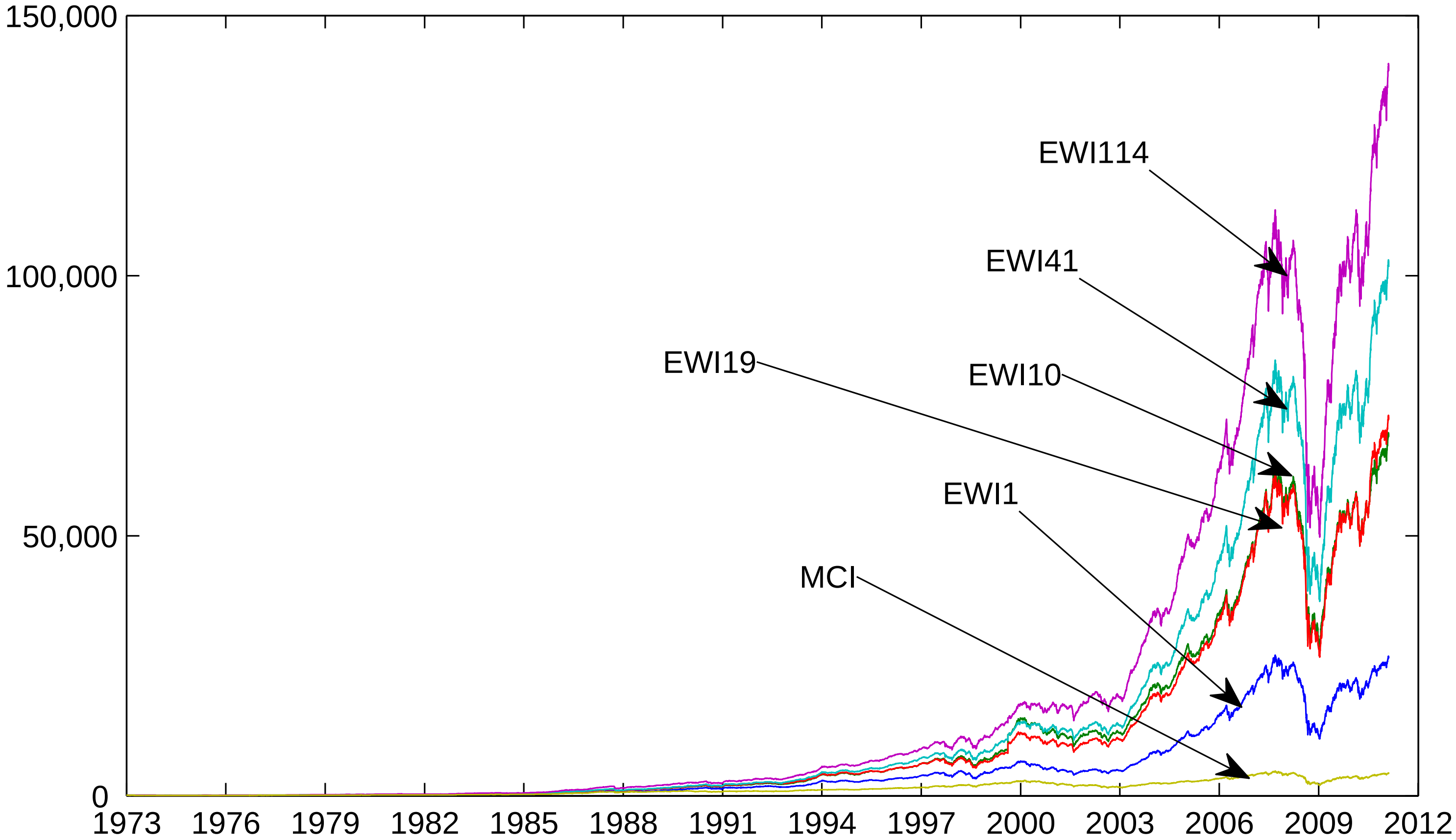

Figure 6: The MCI and five equi-weighted indices: EWI1 (market), EWI10 (industry), EWI19 (supersector), EWI41 (sector), EWI114 (subsector).

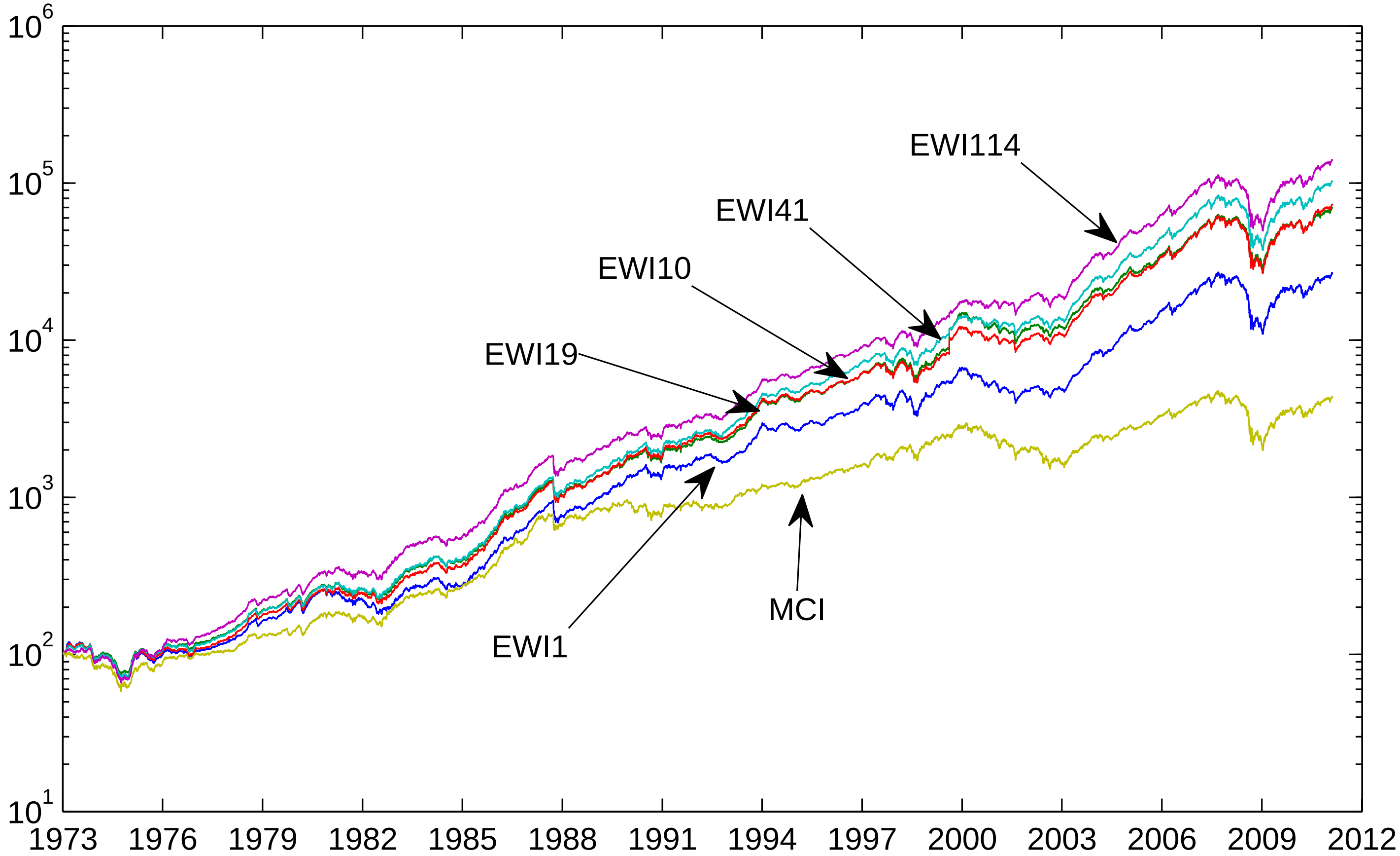

Figure 7: The MCI and five equi-weighted indices in log-scale: EWI1 (market), EWI10 (industry), EWI19 (supersector), EWI41 (sector), EWI114 (subsector).

Approximation of the NP

Pl. & Rendek (2012)

- Naive Diversification Theorem:
  generalized law of large numbers for benchmarked returns

$$\hat{S}_t^{EWI} = \frac{S_t^{EWI}}{S_t^*} \approx 1 \Rightarrow S_t^* \approx S_t^{EWI}$$

- Hierarchical equal value weighting provides excellent proxy without modelling or estimation
- Better proxies of NP
- MCI different to NP

Transaction cost	0	5	40	80	200	240
Reallocation terms	1					
Long term growth rate	0.1919	0.1901	0.1774	0.1529	0.1184	0.1049
Annualised average return	0.1979	0.1961	0.1834	0.1689	0.1254	0.1109
Annualised volatility	0.1135	0.1135	0.1135	0.1135	0.1134	0.1134
Sharpe ratio	1.4205	1.4046	1.2930	1.1654	0.7822	0.6544
Reallocation terms	2					
Long term growth rate	0.1889	0.1878	0.1799	0.1710	0.1440	0.1351
Annualised average return	0.1949	0.1938	0.1859	0.1770	0.1500	0.1411
Annualised volatility	0.1134	0.1134	0.1134	0.1134	0.1135	0.1136
Sharpe ratio	1.3955	1.3856	1.3163	1.2369	0.9987	0.9193
Reallocation terms	4					
Long term growth rate	0.1861	0.1852	0.1790	0.1720	0.1508	0.1437
Annualised average return	0.1921	0.1912	0.1850	0.1780	0.1568	0.1497
Annualised volatility	0.1135	0.1135	0.1134	0.1134	0.1134	0.1135
Sharpe ratio	1.3699	1.3622	1.3080	1.2459	1.0591	0.9967

Economically Meaningful Arbitrage

- When estimated long term growth rate $g_t^{\pi^*}$ explodes for finite time

  $\Leftrightarrow$ NP does not exist.

  $\Leftrightarrow$ economically meaningful arbitrage (EMA)

  $\Rightarrow$ model not viable

- **Existence of NP**:
  simple and intuitive no-arbitrage condition of Benchmark Approach
  $\triangleq$ NUPBR of Karatzas & Kardaras (2007)

- NFLVR $\triangleq$ restrictive classical assumptions
  $\triangleq$ risk neutral pricing; Delbaen & Schachermayer (1998)

- Beyond NFLVR: Loewenstein & Willard (2000), Pl. (2002), Fernholz (2002), Fernholz et al. (2005), Karatzas & Kardaras (2007), Galesso & Runggaldier (2010)

Continuous Financial Market

n-dimensional Brownian motion

$$W_t = (W_t^1, \ldots, W_t^n)^\top,$$

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Primary security accounts

$$S_t = (S_t^1, \dots, S_t^m)^\top,$$

$$\frac{dS_t}{S_t} = a_t dt + b_t \cdot dW_t.$$

Self-financing portfolio S_t^{π}

- ***Weights***

$$\pi_t = (\pi_t^1, \ldots, \pi_t^m)^\top$$

$$\pi_t^T \cdot \mathbf{1} = 1$$

$$\frac{dS_t^{\pi}}{S_t^{\pi}} = \pi_t^\top \cdot \frac{dS_t}{S_t} = \pi_t^\top \cdot a_t dt + \pi_t^\top \cdot b_t \cdot dW_t$$

Characterising NP

- Filipović & Pl. (2009) $\Rightarrow$ If NP exists, then for $\boldsymbol{S_0^* = 1}$, NP value unique

$$\frac{\boldsymbol{dS_t^*}}{\boldsymbol{S_t^*}} = \boldsymbol{\lambda_t^* dt + \theta_t^\top \cdot (\theta_t dt + dW_t)},$$

where

$$\boldsymbol{\theta_t = b_t^\top \cdot \pi_t^*},$$

with $\boldsymbol{\pi_t^*}$ and $\boldsymbol{\lambda_t^*}$ solving

$$\begin{pmatrix} \boldsymbol{b_t \cdot b_t^\top} & \boldsymbol{1} \\ \boldsymbol{1^\top} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi_t^*} \\ \boldsymbol{\lambda_t^*} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a_t} \\ \boldsymbol{1} \end{pmatrix}$$

- no risk premium puzzle

Filipović & Pl. (2009) $\Rightarrow$

$$\frac{dS_t^{\pi}}{S_t^{\pi}} = \lambda_t^* dt + \pi_t^{\top} \cdot b_t(\theta_t dt + dW_t)$$

$\Rightarrow$

- Benchmarked portfolio

  $$\hat{S}_t^{\pi} = \frac{S_t^{\pi}}{S_t^*}$$

  $$\frac{d\hat{S}_t^{\pi}}{\hat{S}_t^{\pi}} = \left(\pi_t^{\top} b_t - \theta_t^{\top}\right) dW_t$$

  $\Rightarrow \hat{S}^{\pi}$ driftless, **local martingale**.

- jump-diffusion market; Pl. & Heath (2010)

- semimartingale market: Christensen & Pl. (2005), Christensen & Larsen (2007), Karatzas & Kardaras (2007)

Assumption: $\hat{S}^{\pi}$ local martingale.

$\Rightarrow$ **Efficient Market Property:**

Market is efficient in the sense that benchmarked security prices are over infinitesimal time their best forecast; fully reflecting all available information:

$$\hat{S}_t^{\pi} = \lim_{\varepsilon \downarrow 0} E\left(\hat{S}_{t+\varepsilon}^{\pi} | \mathcal{F}_t\right)$$

- No preferences or representative agents.
- Answers Fama (1970) "Efficient Market Hypothesis"

Capital Asset Pricing Model

- Sharpe (1964), Lintner (1965), Mossin (1966), Merton (1973)
- Is there a generalized CAPM?

- market with locally riskless *savings account*

$$B_t = \exp\left\{\int_0^t r_s d_s\right\}.$$

Filipović & Pl. (2009) $\Rightarrow \lambda_t^* = r_t$

$$\frac{dS_t^*}{S_t^*} = r_t dt + \theta_t^\top \cdot (\theta_t dt + dW_t)$$

$S_0^* = 1$.

- *Market price of risk* vector

$$\theta_t = b_t^\top \cdot \pi_t^*,$$

- $S_t^\pi = \hat{S}_t^\pi S_t^* \Rightarrow$

$$\frac{dS_t^\pi}{S_t^\pi} = r_t dt + \pi_t^\top \cdot b_t(\theta_t dt + dW_t)$$

- $\Rightarrow$ **risk premium**

$$q_t = \pi_t^\top \cdot b_t \cdot \theta_t$$

Definition: *Normalised covariance* of returns of S_t and V_t

$$\mathrm{Cov}(S, V)_t = \lim_{\varepsilon \downarrow 0} E\left(\left(\frac{S_{t+\varepsilon}}{S_t} - 1\right)\left(\frac{V_{t+\varepsilon}}{V_t} - 1\right) | \mathcal{F}_t\right)$$

Definition: ***NP-efficient*** portfolio $\boldsymbol{V_t^f}$ invests fraction $\boldsymbol{f_t}$ in NP:

$$\frac{dV_t^f}{V_t^f} = r_t dt + f_t \theta_t^\top (\theta_t dt + dW_t).$$

- expected utility maximizing portfolios; Pl. & Heath (2010).
- maximum drawdown constrained portfolios; Kardaras et al. (2014)
- zero coupon bonds, annuities; Pl. & Bruti-Liberati (2010).

Definition: For NP-efficient portfolio V^f and portfolio S^π

systematic risk parameter beta:

$$\beta_t^{f,\pi} = \frac{\mathbf{Cov}(S^\pi, V^f)_t}{\mathbf{Cov}(V^f, V^f)_t}$$

⇒ ***Generalised Capital Asset Pricing Formula***

$$\beta_t^{f,\pi} = \frac{\pi_t^\top b_t \theta_t}{f_t \theta_t^\top \theta_t} = \frac{q_t^\pi}{q_t^f}$$

⇒ **risk premium**:

$$q_t^\pi = \beta_t^{f,\pi} q_t^f = \frac{\mathrm{Cov}(S^\pi, V^f)}{\mathrm{Cov}(V^f, V^f)} q_t^f$$

- holds when NP has no jumps
- holds for derivatives
- is a local statement

NP-efficient portfolio replacing the market portfolio

- **No** representative agents
- **No** preferences
- Empirically, market portfolio is not NP-efficient, smart beta strategies, Pl. & Rendek (2012)

- Fatou's lemma

  $\Rightarrow$ Nonnegative local martingales are supermartingales

  $\Rightarrow$ Supermartingale Property confirmed

- Supermartingale property

  $\Rightarrow$ current benchmarked value

  can be significantly larger

  than its expected benchmarked future value

$$\hat{S}_t^{\pi_{\hat{H}_T}} \geq E\left(\hat{H}_T | \mathcal{F}_t\right)$$

- Benchmark Approach allows **several price processes** delivering the same contingent claim.
- **No** Law of One Price

- Minimal supermartingale is martingale; Du & Pl. (2014)

  $\Rightarrow$

  Real world pricing formula

$$\hat{S}_t^{\pi_{\hat{H}_T}} = E\left(\hat{H}_T | \mathcal{F}_t\right)$$

  yields **minimal** possible price

- Law of Minimal Price

- Competitive, liquid market $\Rightarrow$ real world price

Pricing of New Contingent Claims

Assumption:

Efficient market property holds for all price processes.

$\Rightarrow$ benchmarked price processes are local martingales.

$\Rightarrow$ Generalise Capital Asset Pricing Formula

- $\Rightarrow$ NP does not explode
- **several** price processes usually exist for delivering a given benchmarked contingent claim $\hat{\boldsymbol{H}}_{\boldsymbol{T}}$

  e.g. formally obtained risk neutral price, real word price, ...

  $\Rightarrow$ no economically meaningful arbitrage (EMA)

Example: $\hat{\boldsymbol{H}}_{\boldsymbol{T}} = \frac{\boldsymbol{1}}{\boldsymbol{S}^*_{\boldsymbol{T}}}$ - benchmarked zero coupon bond payoff

assume deterministic short rate $\boldsymbol{r_t}$

- Minimal market model; Pl. & Heath (2010)
- S&P 500 as NP

- **real world pricing formula**:

$$\hat{P}(t,T) = E(\hat{H}_T | \mathcal{F}_t)$$

  - martingale, minimal possible price,

- formally obtained **risk neutral price**:

$$\hat{P}^*(t,T) = \frac{B_t}{B_T S_t^*}$$

  - strict supermartingale, higher price

$$\hat{P}^*(t,T) \geq \hat{P}(t,T)$$

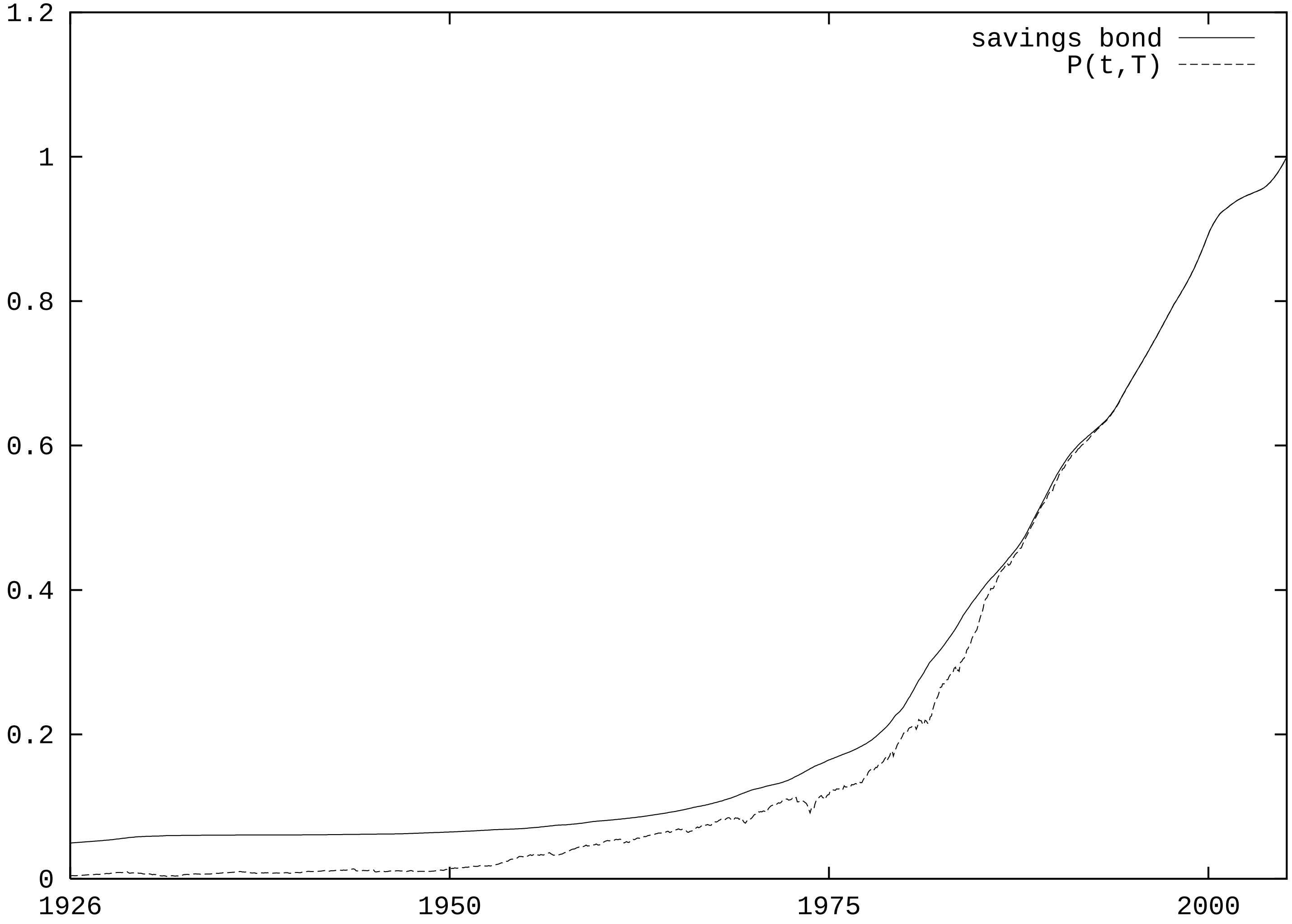

Figure 8: Zero coupon bond and savings bond

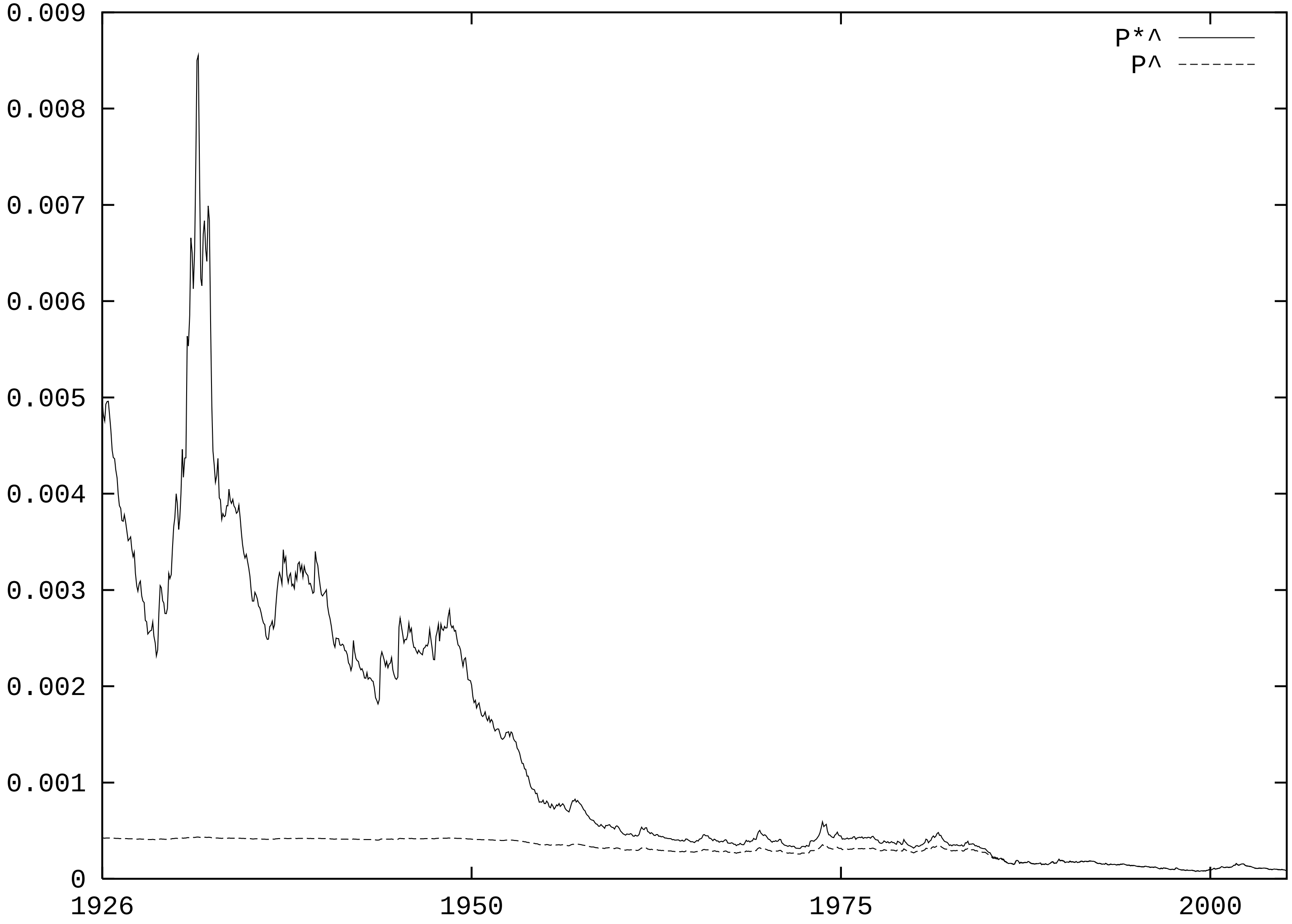

Figure 9: Benchmarked zero coupon bond

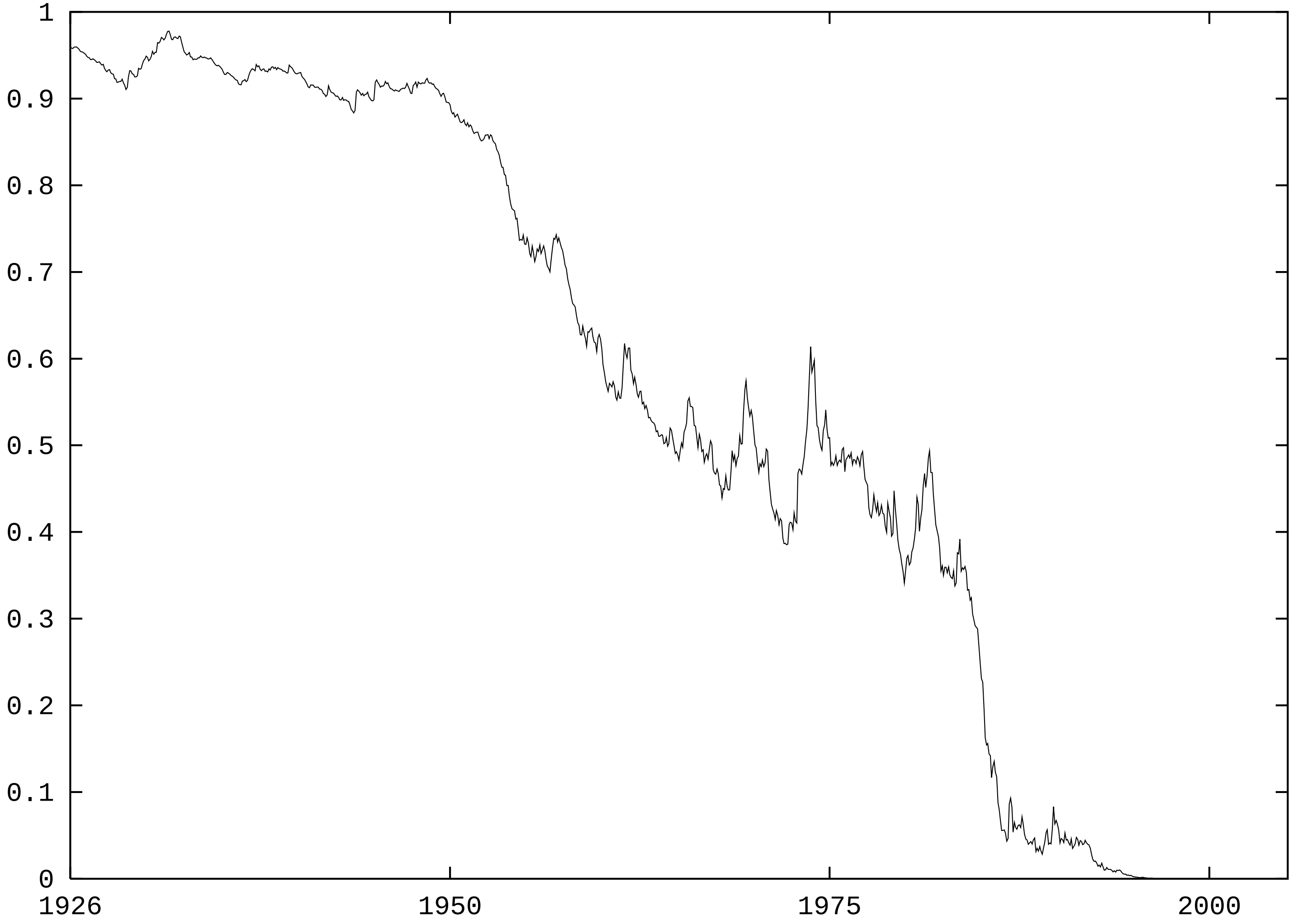

Figure 10: Ratio in the NP

- makes **financial planning** rigorous
- higher **return on investment**
- less expensive **annuities**
- higher **pensions**
- attractive **green bonds**

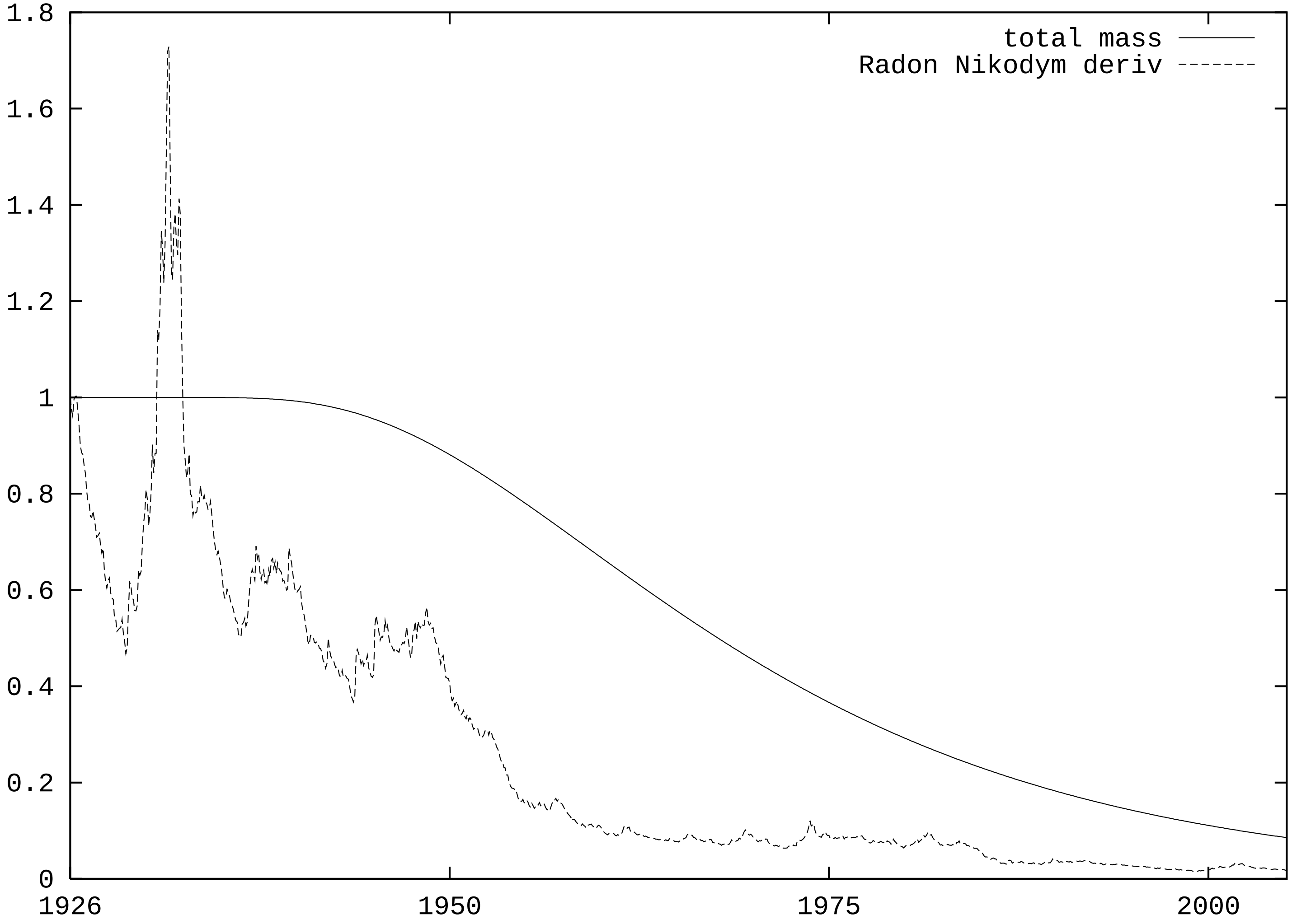

Figure 11: Radon-Nikodym derivative and total mass of candidate risk neutral measure

- **discounted NP**

$$d\frac{S_t^*}{B_t} = d\varphi_t + \sqrt{\frac{S_t^*}{B_t}}\,dW(\varphi_t)$$

Time transformed squared Bessel process of dimension four!

- **transformed time**

$$\varphi_t = \left\langle \sqrt{\frac{S_\cdot^*}{B_\cdot}} \right\rangle_t$$

Stylized Minimal Market Model

Pl. (2001, 2002, 2006), Pl. & Rendek (2012)

- assume **discounted NP drift** as

$$\alpha_t = \alpha \, \exp\{\eta \, t\} = \theta_t^2 \, \frac{S_t^*}{B_t}$$

- initial parameter $\alpha > 0$

- net growth rate η

- transformed time $\varphi_t = \frac{\alpha}{4\eta} \left(\exp\{\eta t\} - 1 \right)$

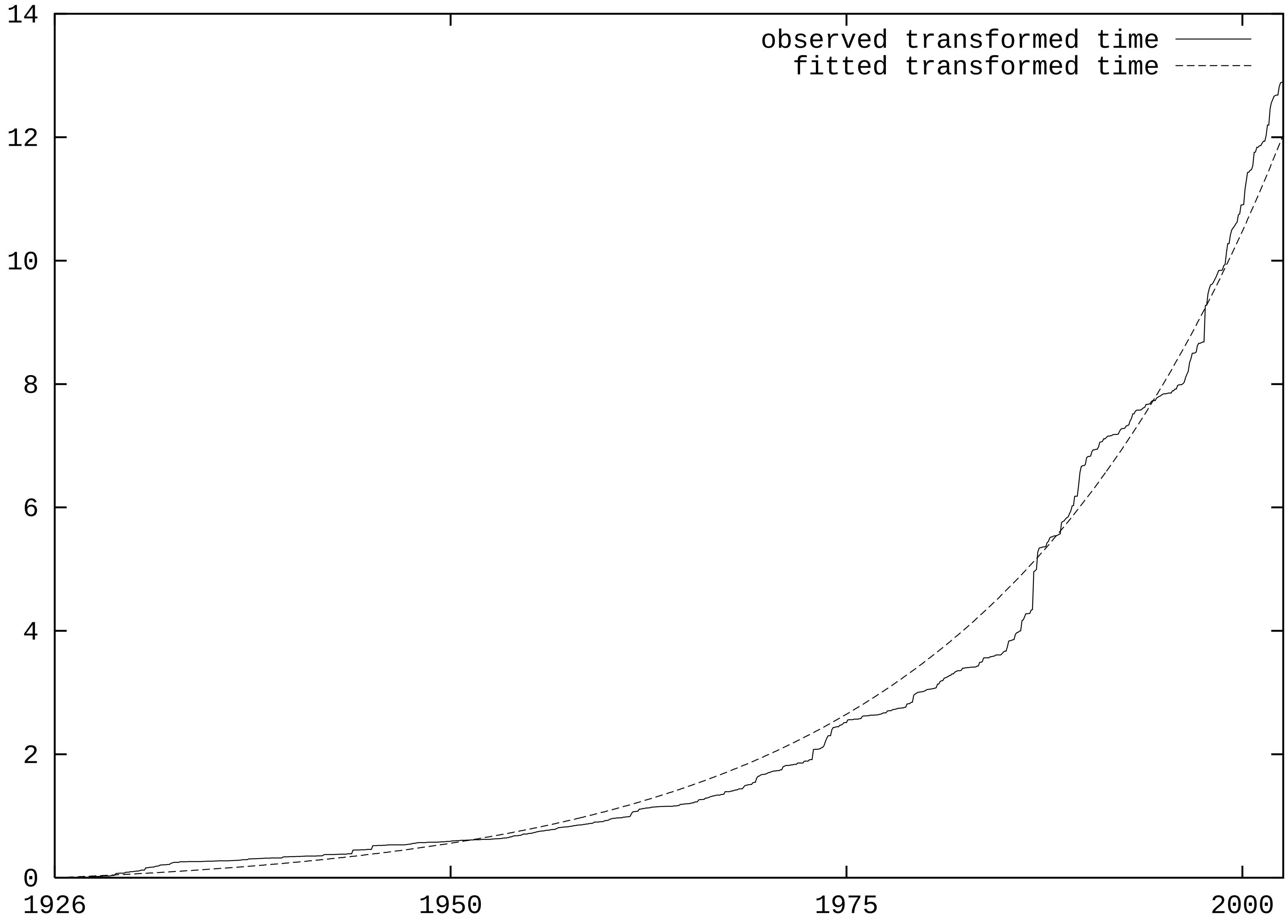

Figure 12: Fitted and observed transformed time

$\Rightarrow$ **volatility**:

$$\sigma_t = \sqrt{\frac{B_t \alpha_t}{S_t^*}}$$

- scaling parameter $\alpha = 0.043$
- net growth rate $\eta = 0.0528$

- **zero coupon bond**

$$P(t,T) = S_t^* \, E_t\left(\frac{1}{S_T^*}\right) = P_T^*(t)\, E_t\left(\frac{S_t^* B_T}{S_T^* B_t}\right)$$

with **savings bond**

$$P_T^*(t) = \frac{B_t}{B_T} = \exp\left\{-\int_t^T r_s\, ds\right\}$$

$$P(t,T) = P_T^*(t)\left(1 - \exp\left\{-\frac{S_t^*/B_t}{2\left(\varphi_T - \varphi_t\right)}\right\}\right) < P_T^*(t)$$

for $t \in [0, T)$, Pl. (2002)

$P(0,T) = 0.004$

$P_T^*(0) = 0.049$

$\frac{P(0,T)}{P_T^*(0)} \approx 0.085$

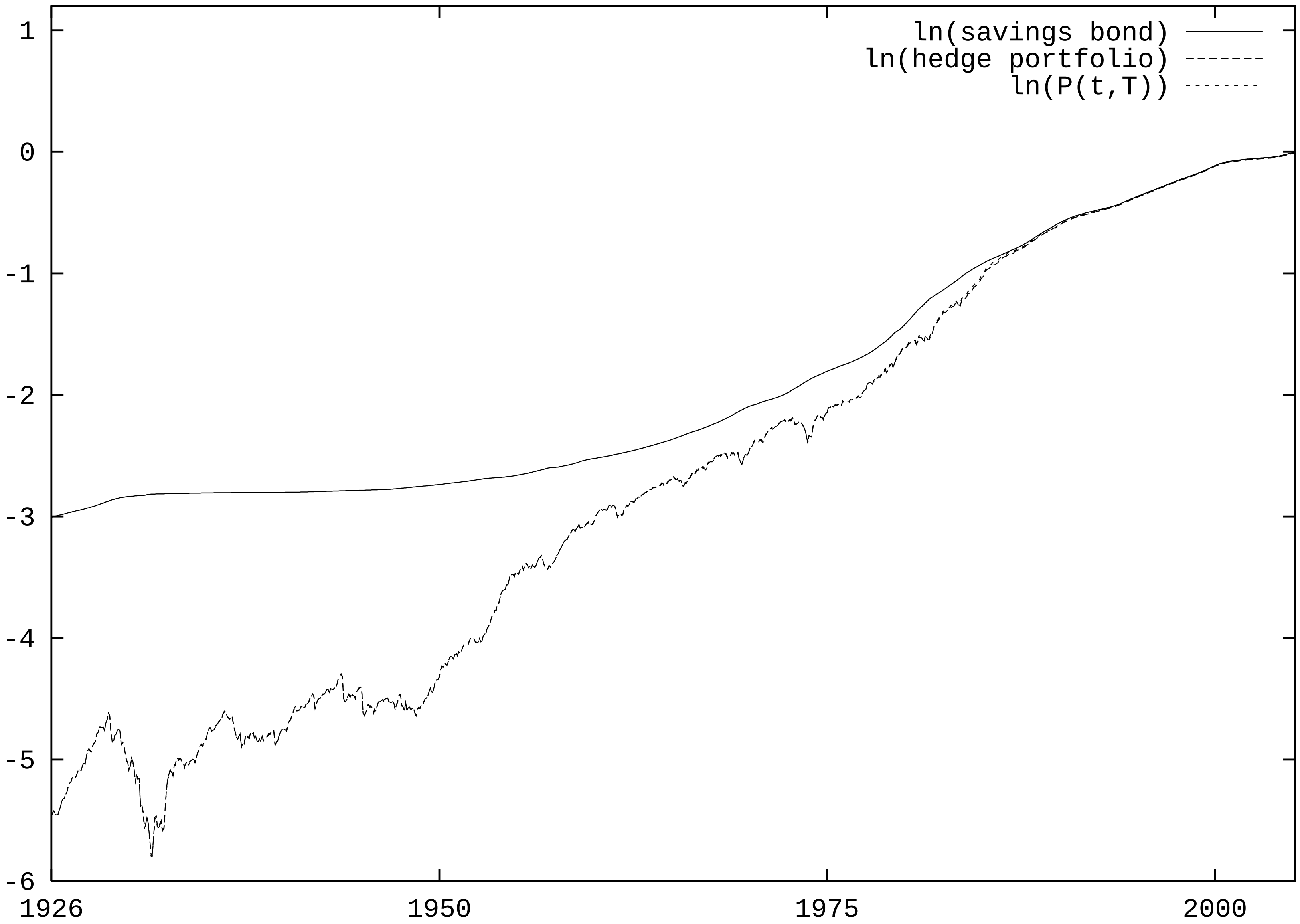

Figure 13: ln from Zero coupon bond, hedge portfolio and savings bond

Hedge Simulation

- **delta** units in the NP

$$\delta^*(t) = \frac{\partial P(t,T)}{\partial S_t^*} = \exp\left\{-\int_0^T r_s \, ds\right\} \exp\left\{-\frac{\bar{S}_t^*}{2(\varphi_T - \varphi_t)}\right\} \frac{1}{2(\varphi_T - \varphi_t)}$$

- Fergusson & Pl. (2014)

Generalized MMM

Pl. & Rendek (2012)

- time t becomes random market time τ_t

$$d\tau_t = \frac{1}{X_t} dt$$

$$dX_t = (\gamma - \varepsilon X_t) dt + \sqrt{\gamma X_t}\, dW_t$$

Recovering Real World Density

Barkhagen et al. (2016),

- Ross Recovery, Ross (2015)
- S&P 500 option data
- real world pricing formula

$$\bar{C}_{T,K}(0, \bar{S}_0^*) = E\left(\Lambda_T \bar{C}_{T,K}(T, \bar{S}_T^*) | \mathcal{F}_0\right)$$

$$\Lambda_t = (\bar{S}_t^*)^{-1} = \frac{B_t}{S_t^*}$$

$$E(\Lambda_t | \mathcal{F}_0) \leq \Lambda_0 = 1$$

- real world probability

$$p(0, \bar{S}_0^*; T, \bar{S}_T^*) = \Lambda_T^{-1} q(0, \bar{S}_0^*; T, \bar{S}_T^*)$$

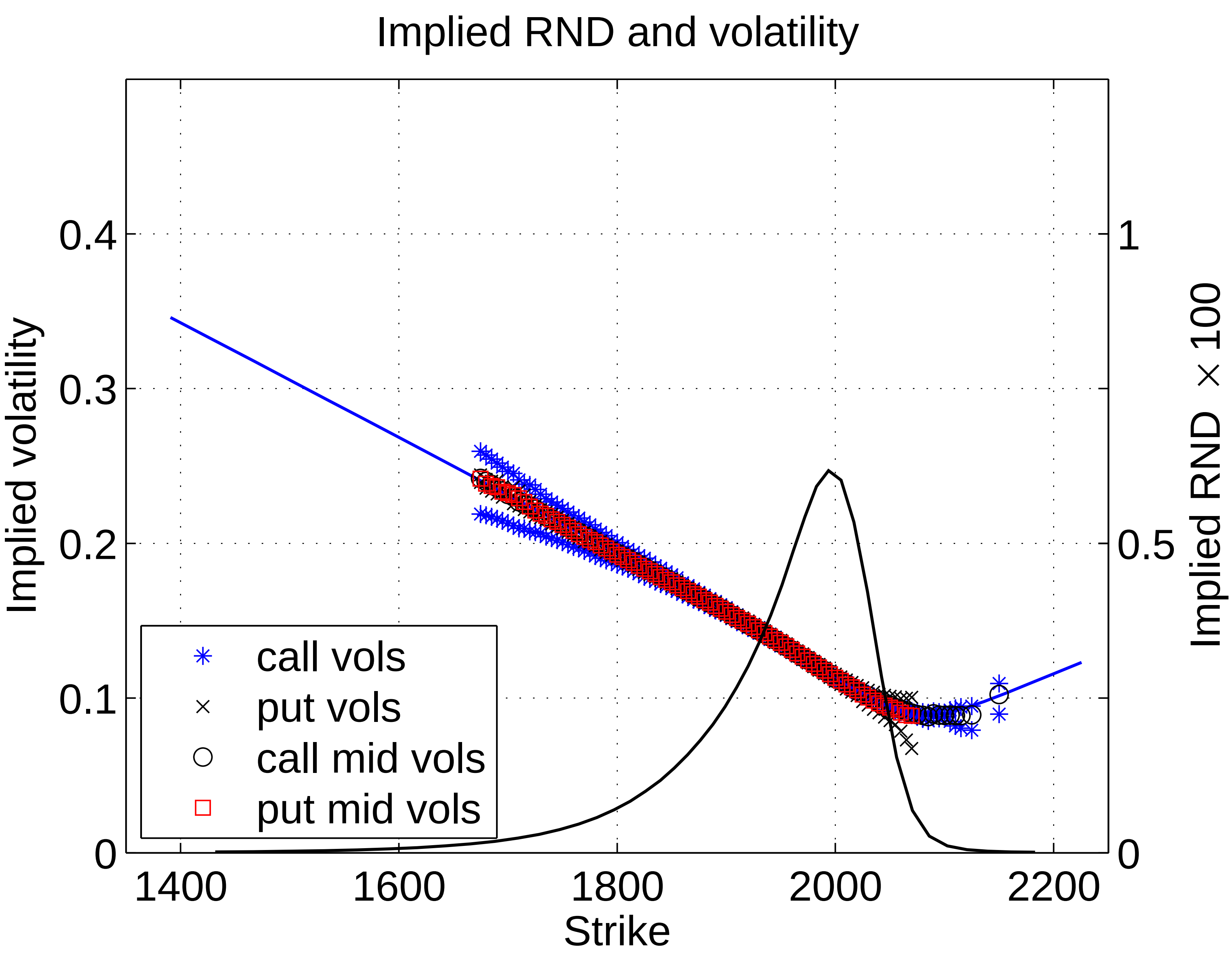
Implied RND and volatility
Implied volatility
Implied RND × 100
Strike
0
0.1
0.2
0.3
0.4
0
0.5
1
1400
1600
1800
2000
2200
call vols
put vols
call mid vols
put mid vols

Figure 14

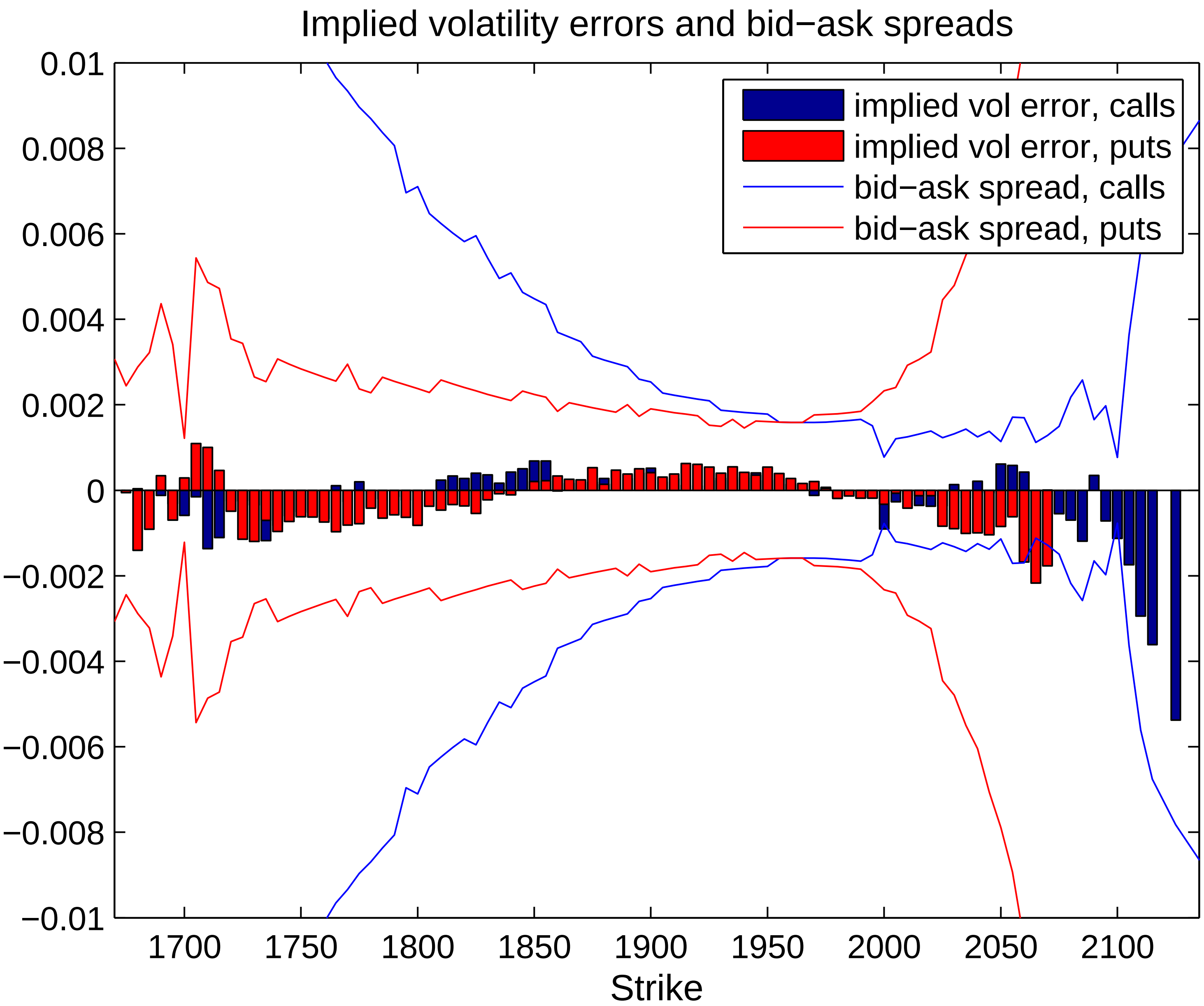
Implied volatility errors and bid−ask spreads
implied vol error, calls
implied vol error, puts
bid−ask spread, calls
bid−ask spread, puts
0.01
0.008
0.006
0.004
0.002
0
−0.002
−0.004
−0.006
−0.008
−0.01
1700
1750
1800
1850
1900
1950
2000
2050
2100
Strike

Figure 15

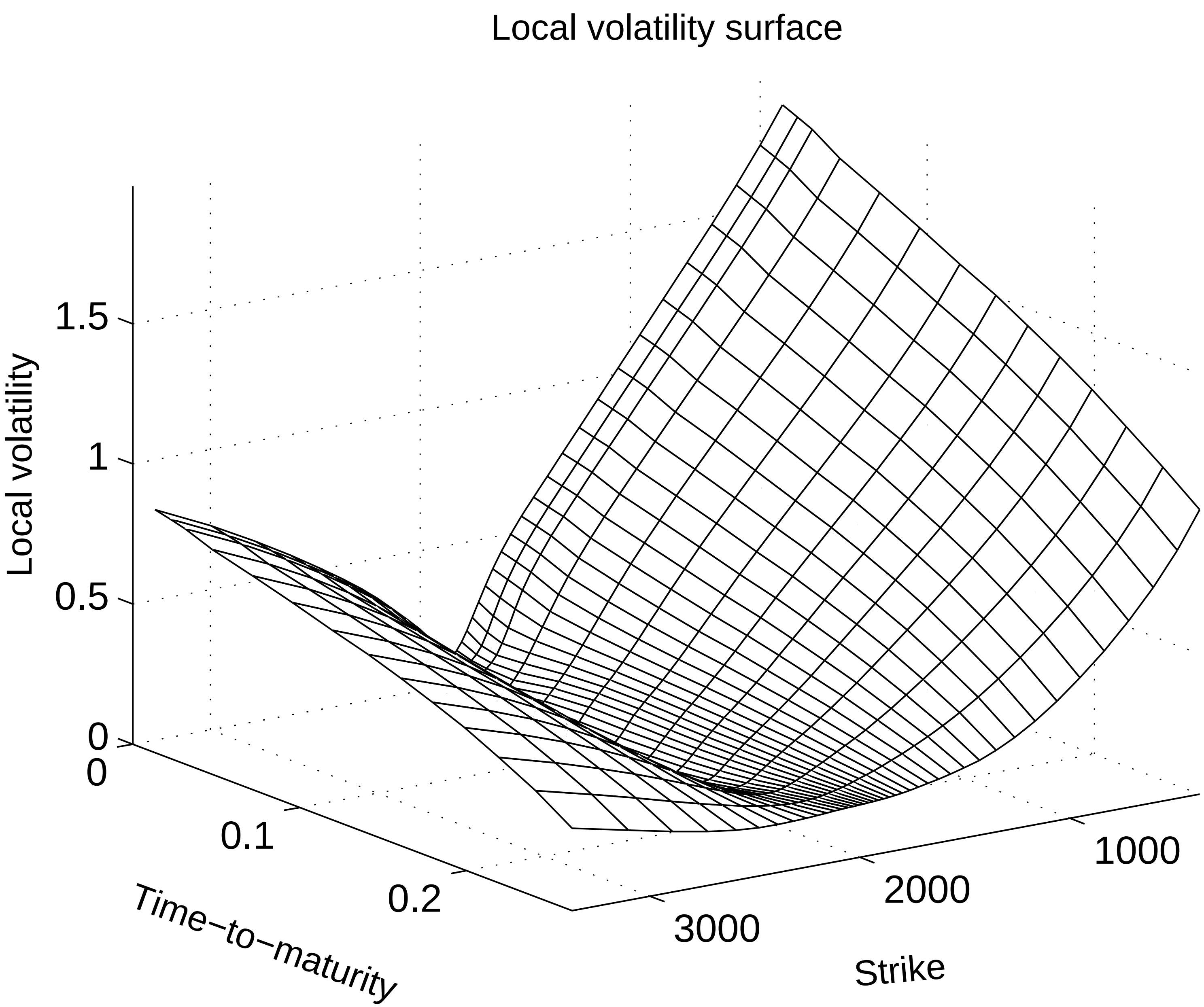
Local volatility surface
Local volatility
1.5
1
0.5
0
0
0.1
0.2
Time−to−maturity
1000
2000
3000
Strike

Figure 16

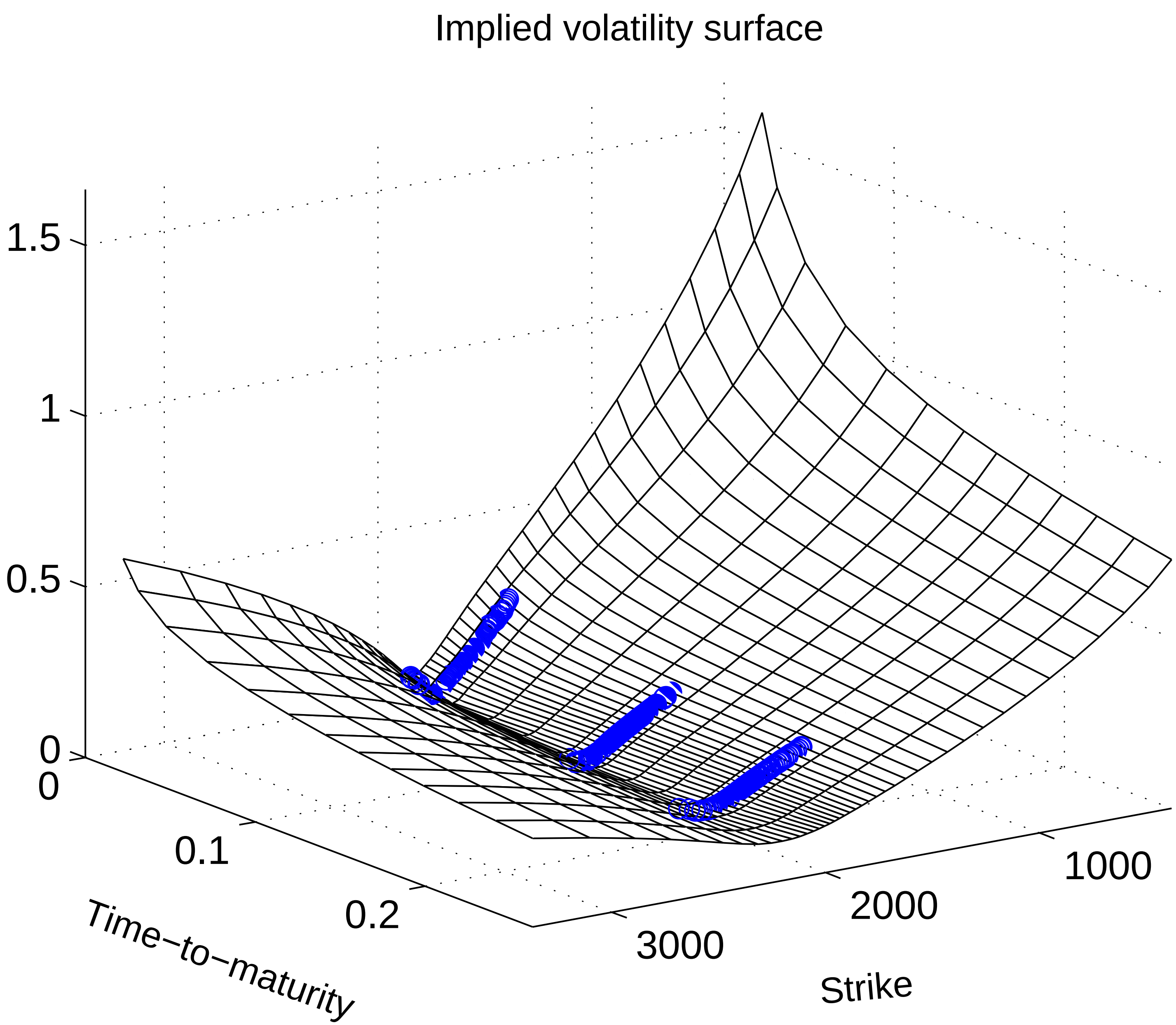
Implied volatility surface
1.5
1
0.5
0
0
0.1
0.2
Time−to−maturity
3000
2000
1000
Strike

Figure 17

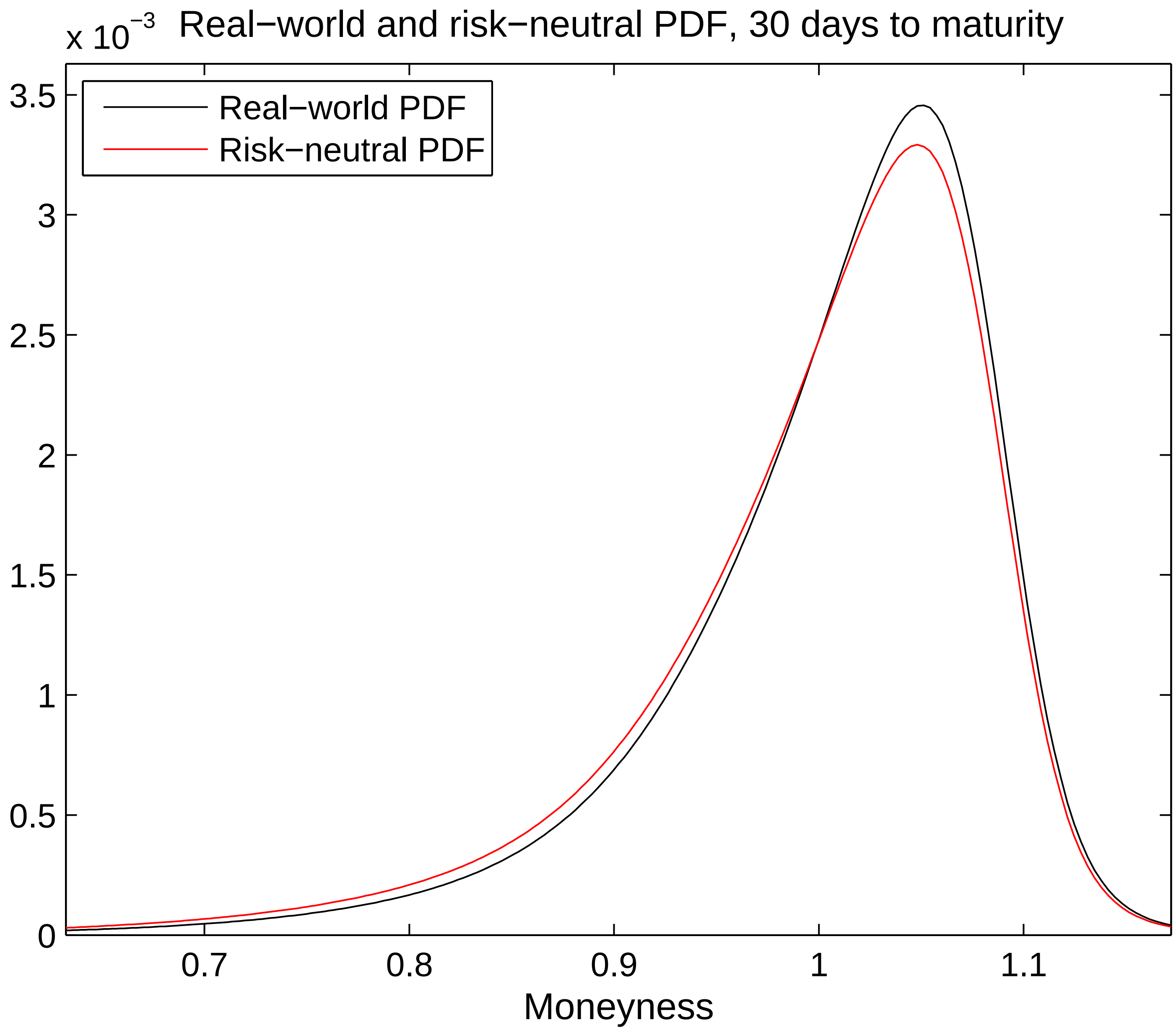
x 10^{-3} Real−world and risk−neutral PDF, 30 days to maturity
Real−world PDF
Risk−neutral PDF
3.5
3
2.5
2
1.5
1
0.5
0
0.7
0.8
0.9
1
1.1
Moneyness

Figure 18

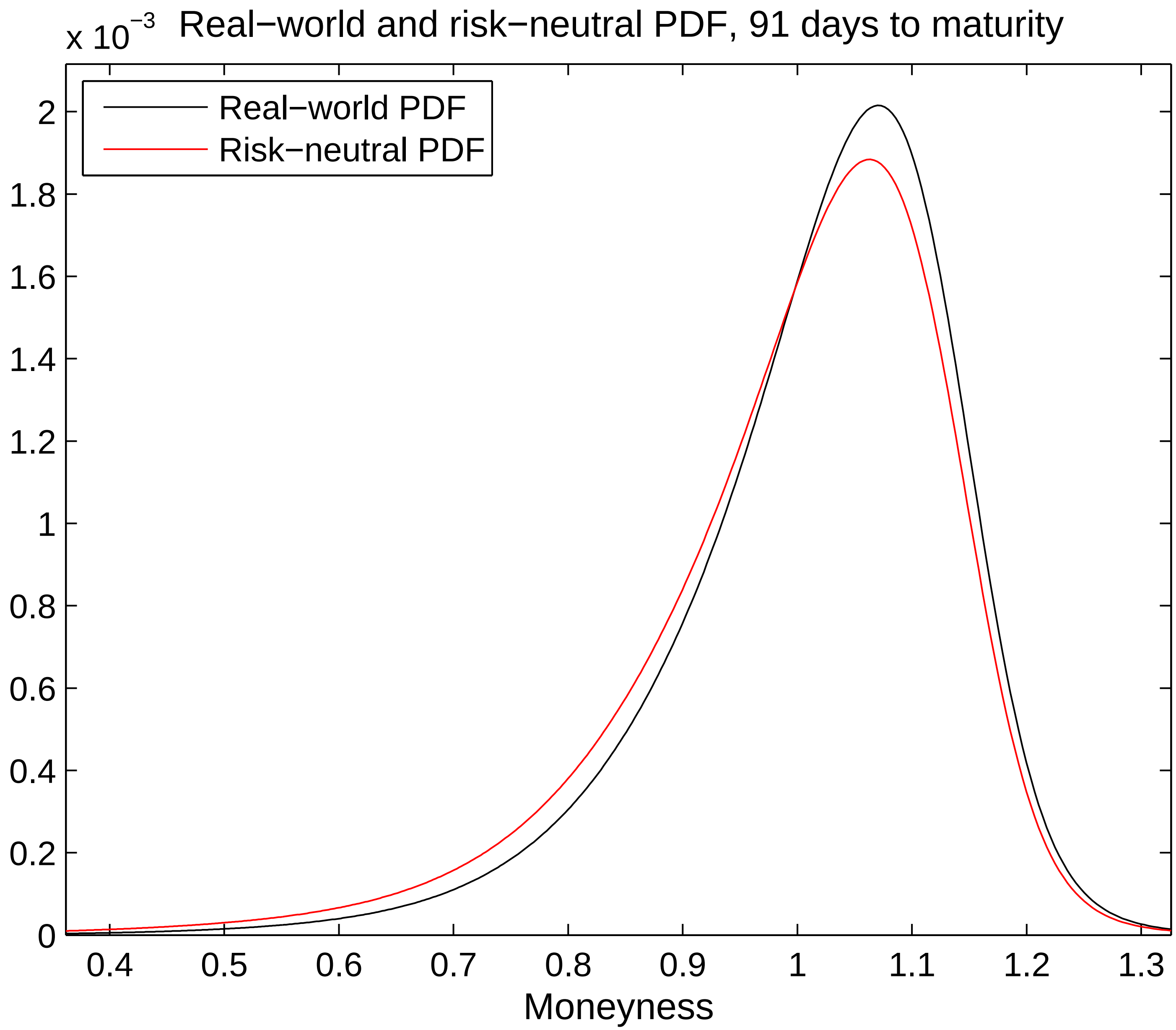
x 10^−3 Real−world and risk−neutral PDF, 91 days to maturity
Real−world PDF
Risk−neutral PDF
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
Moneyness

Figure 19

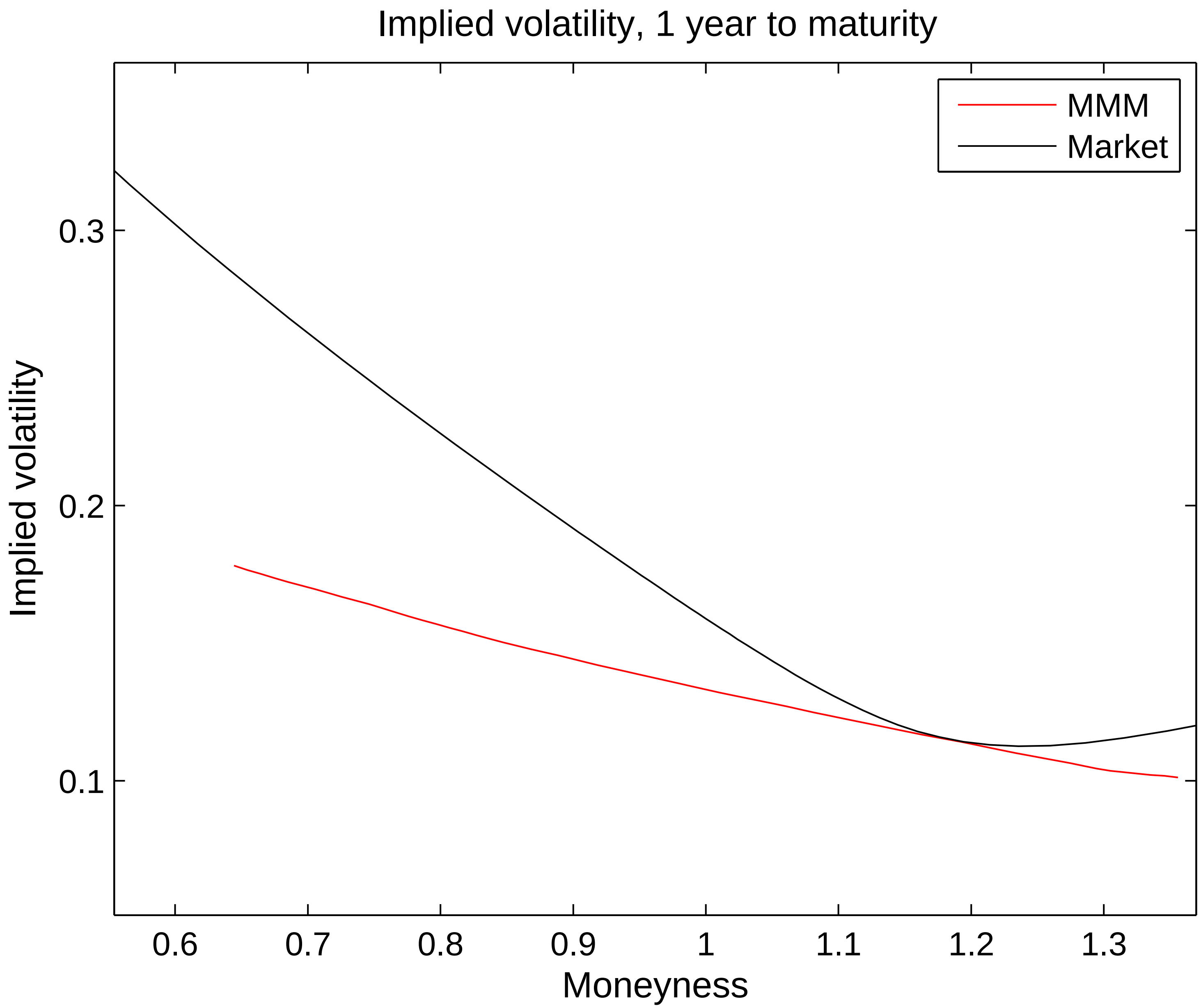
Implied volatility, 1 year to maturity
MMM
Market
0.3
0.2
0.1
Implied volatility
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
Moneyness

Figure 20

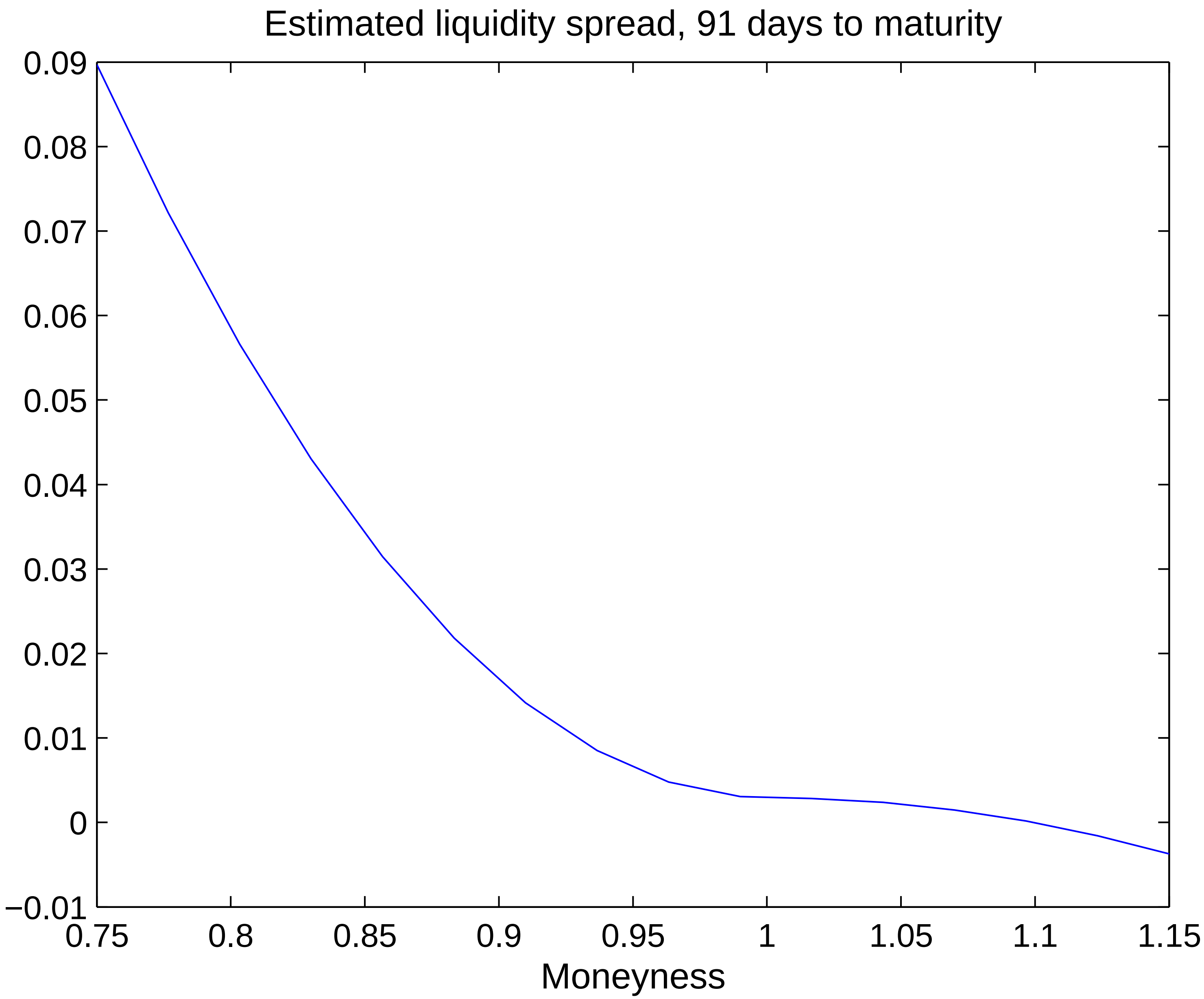
Estimated liquidity spread, 91 days to maturity
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
−0.01
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
Moneyness

Figure 21

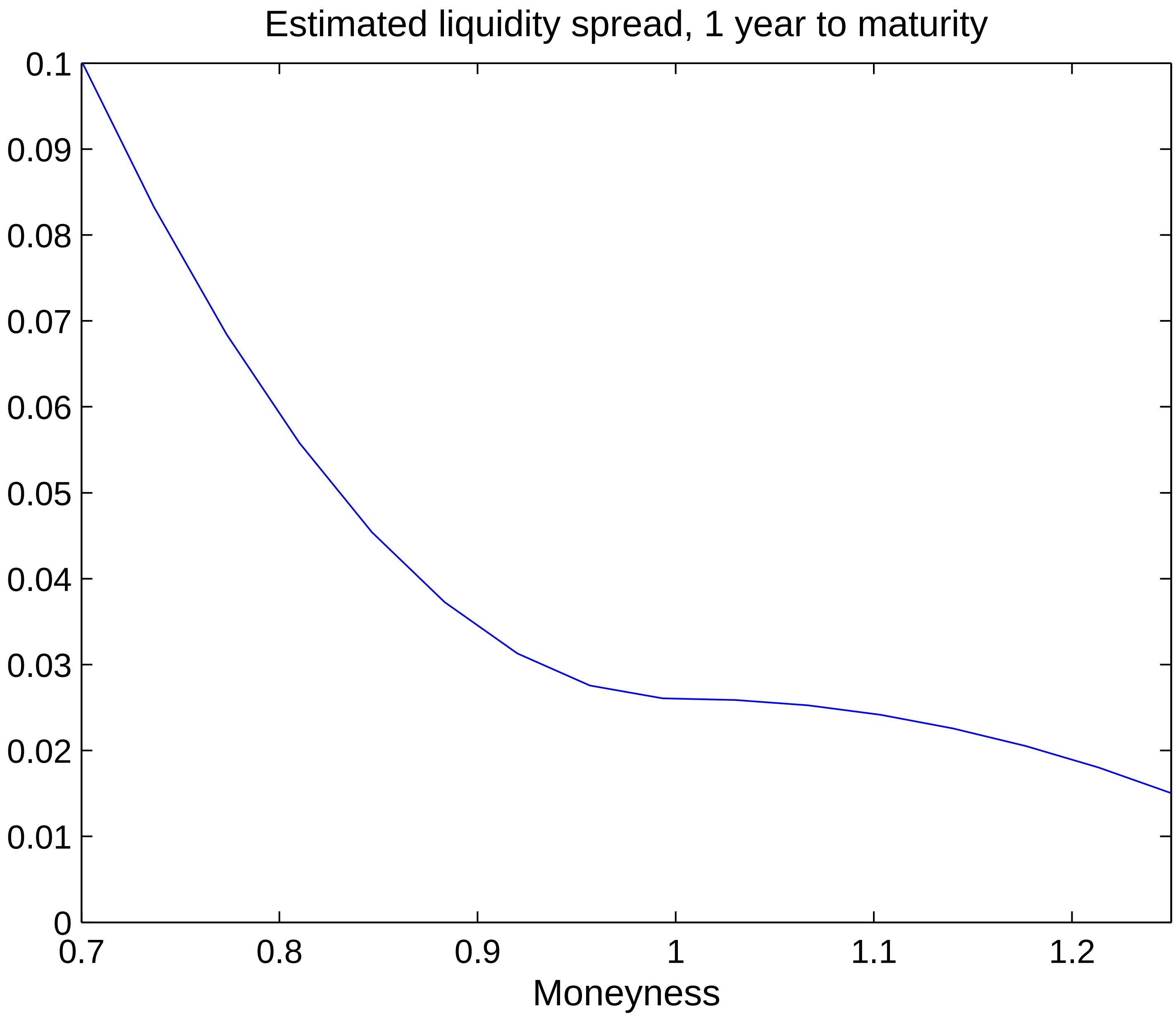
Estimated liquidity spread, 1 year to maturity
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.7
0.8
0.9
1
1.1
1.2
Moneyness

Figure 22

Drawdown Constrained Portfolios

- running maximum

$$\boldsymbol{X_t^* = \sup_{u \in [0,t]} X_u}$$

- drawdown

$$\boldsymbol{X_t^* - X_t}$$

Figure 23: logarithm of discounted S&P500 and its running maximum

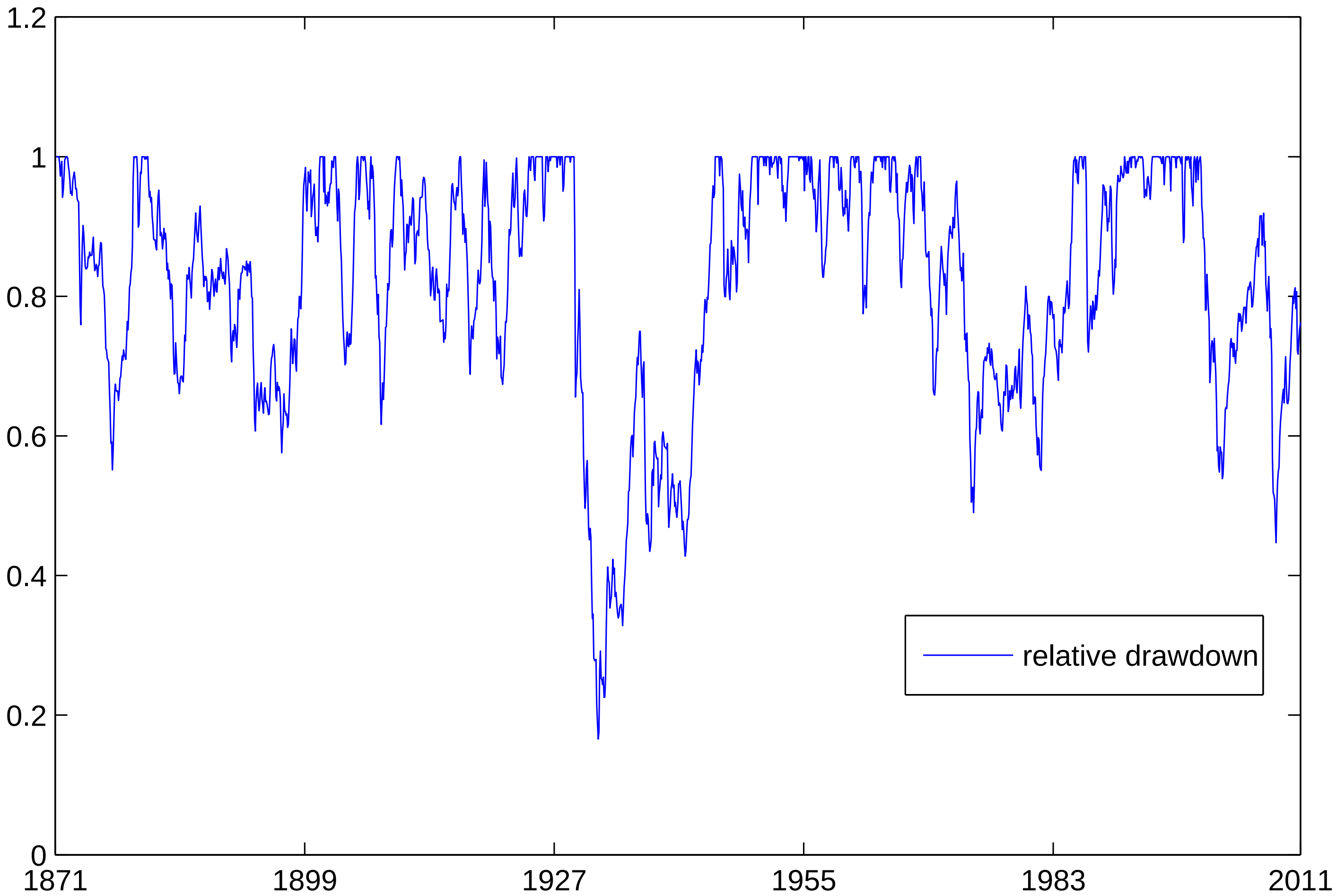

Figure 24: Relative drawdown of discounted S&P500

- maximum drawdown constrained portfolio

$$\alpha \in [0, 1)$$

$$\begin{aligned} {}^{\alpha}X_t =& \alpha (X_t^*)^{1-\alpha} + (1-\alpha)(X_t^*)^{-\alpha} X_t \\ \in& [\alpha X_t^*,\ (X_t^*)^{1-\alpha}] \end{aligned}$$

Kardaras et al. (2014):

$$^{\alpha}g^{\pi*} = (1-\alpha) g^{\pi*}$$

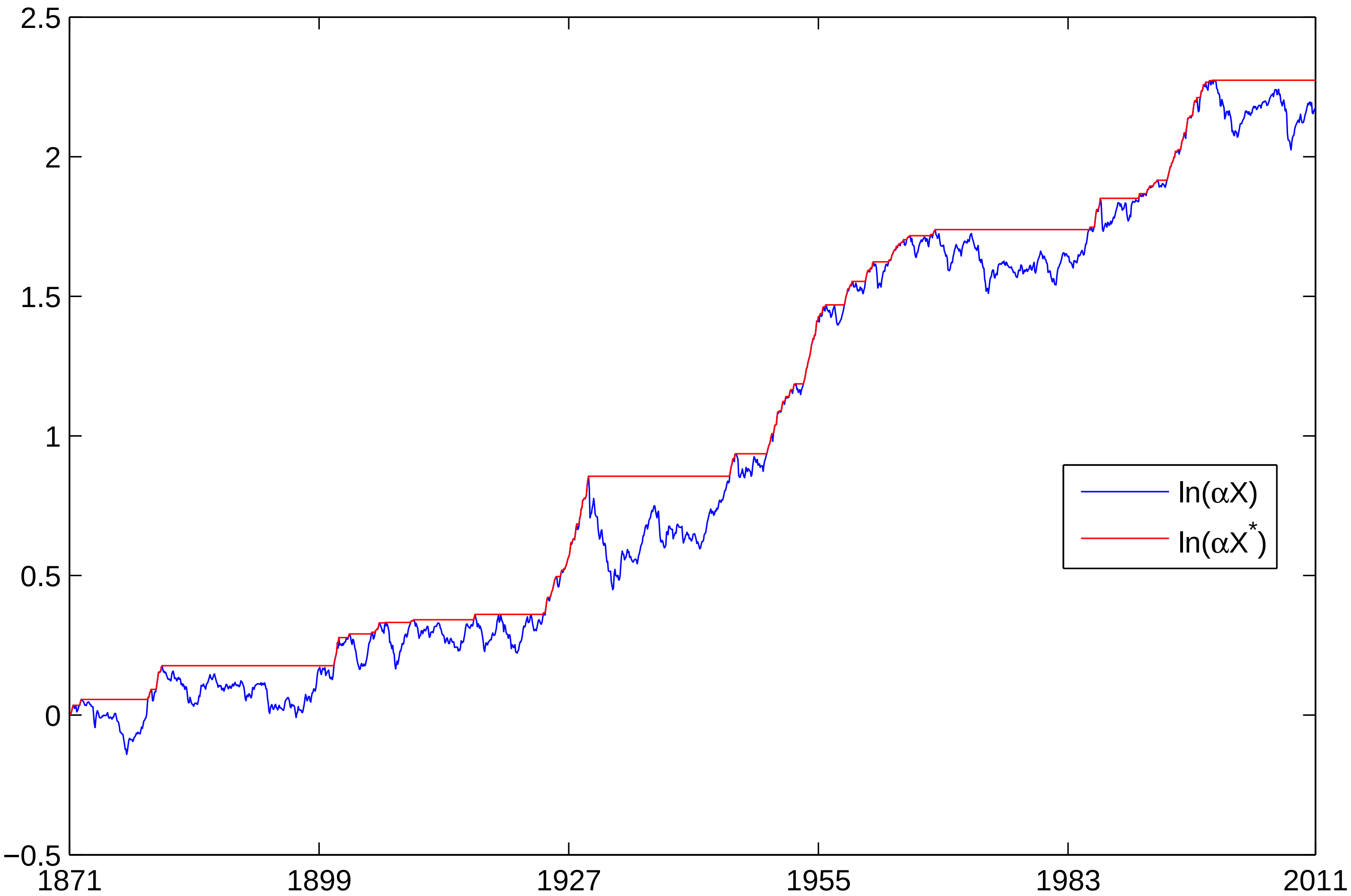

Figure 25: Logarithm of maximum drawdown constrained portfolio, $\alpha = 0.6$

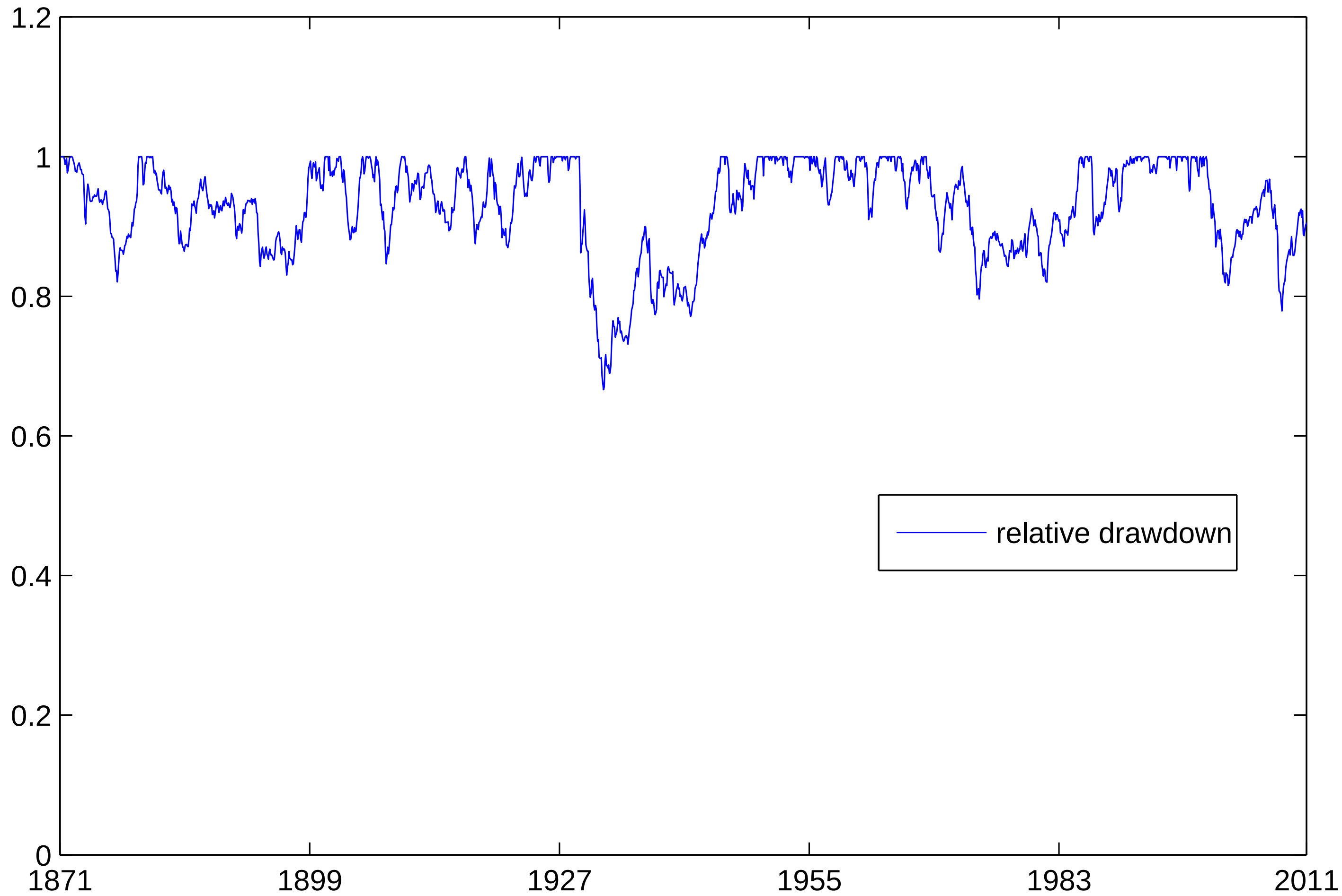

Figure 26: Relative drawdown for drawdown constrained portfolio, $\boldsymbol{\alpha = 0.6}$

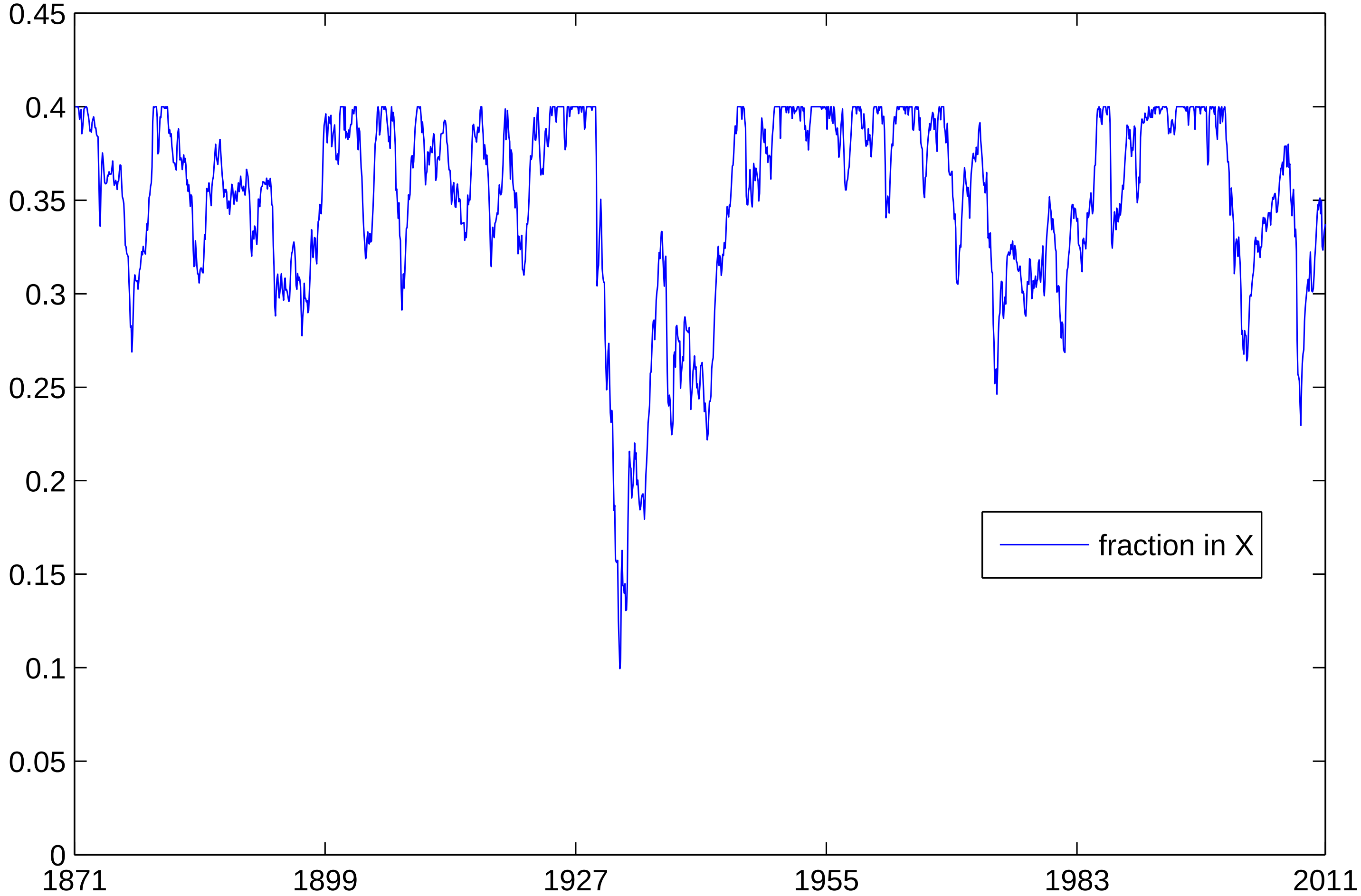

Figure 27: Fraction of wealth in S&P500 for $\alpha = 0.6$

- **Kardaras et al. (2014) maximum long term growth rate**

$$\lim_{t\to\infty}\frac{1}{t}\log({}^{\alpha}S^*_t) = (1-\alpha)\lim_{t\to\infty}\frac{1}{t}\log(S^*_t)$$
$${}^{\alpha}g = (1-\alpha)g$$

restricted drawdown $\Rightarrow$ reduced maximum growth rate
long term view with short term attitude towards risk
realistic alternative to Markowitz mean-variance approach
and utility maximization

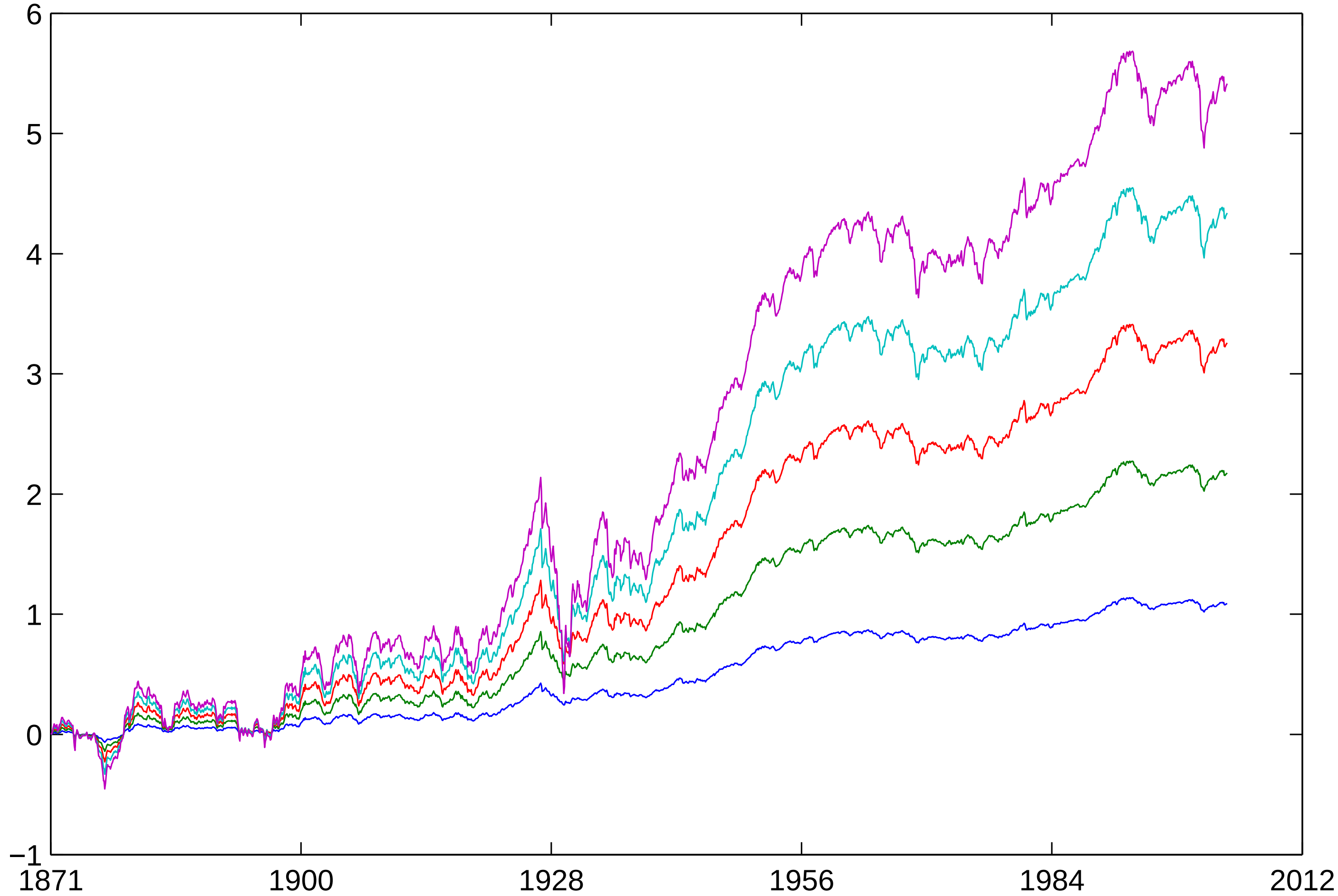

Figure 28: Logarithm of drawdown constrained portfolios

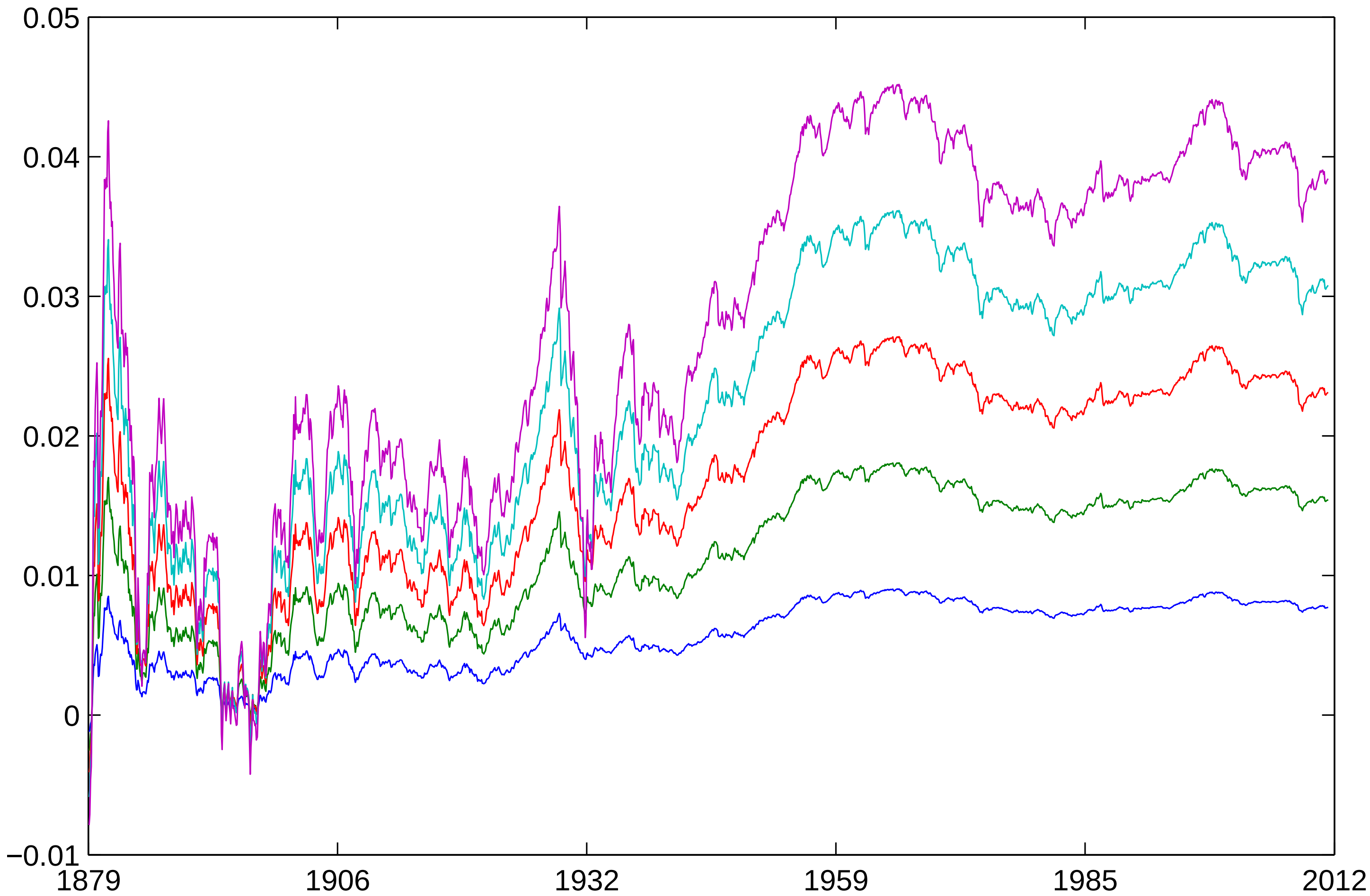

Figure 29: Long term growth of drawdown constrained portfolios

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