Equilibrium Strategy of Time-inconsistent Stochastic Control

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joint work with

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Time-consistent control

A standard assumption in the study of stochastic control is the time-consistency: an optimal control viewed from today will remain optimal when viewed from tomorrow. It provides the theoretical foundation of the dynamic programming approach including HJB equation and BSDE.

Time-inconsistent problems

Hyperbolic discounting:

Mean-variance portfolio selection model: Basak

Probability distortion in behavioral finance: Jin and Zhou

Problem Setting

Consider

$$dX_s = [A_s X_s + B_s u_s + b_s]ds + [C_s X_s + D_s u_s + \sigma_s]dW_s; \quad X_0 = x_0, \tag{1}$$

where $u(\cdot)$ is the control, X is the state process. Finally x_0 is the initial state. For any control $u(\cdot)$, there exists a unique solution X. As times evoles, we need to consider the controlled system starting from time $t \in [0, T]$ and state x_t:

$$dX_s = [A_s X_s + B_s u_s + b_s]ds + [C_s X_s + D_s u_s + \sigma_s]dW_s; \quad X_t = x_t. \tag{2}$$

For any control $u(\cdot)$, there exists a unique solution $X^{t,x_t,u}$.

Time-inconsistent control problem

At any time t with the system state $X_t = x_t$, our aim is to minimize

$$\begin{aligned} J(t, x_t, u(\cdot)) \quad := \quad & \frac{1}{2}\mathbb{E}_t \int_t^T [\langle Q_s X_s, X_s\rangle + \langle R_s u_s, u_s\rangle] ds + \frac{1}{2}\mathbb{E}_t \langle G X_T, X_T\rangle \\ & -\frac{1}{2}\langle h\mathbb{E}_t[X_T], \mathbb{E}_t[X_T]\rangle - \langle \mu_1 x_t + \mu_2, \mathbb{E}_t[X_T]\rangle \end{aligned} \tag{3}$$

over u, where $X = X^{t,x_t,u}$, and $\mathbb{E}_t[\cdot] = \mathbb{E}\left[\cdot|\mathcal{F}_t\right]$. $\mu_1 \in \mathbb{R}^+$ and $\mu_2 \in \mathbb{R}^k$ are both constants.

time inconsistency

The first terms in the cost functional are standard in a classical LQ control problem, whereas the last two are unconventional. Specifically, the term $-\frac{1}{2}\langle h\mathbb{E}_t[X_T], \mathbb{E}_t[X_T]\rangle$ is motivated by the variance term in a mean-variance portfolio choice model and the last term $-\langle \mu_1 x_t + \mu_2, \mathbb{E}_t[X_T]\rangle$, which depends on the state x_t at time t, stems from a state-dependent utility function in economics. One way to get around the time-inconsistency issue is to consider only precommitted controls (the cntrols are optimal only when viewd at the initial time), they have not really addresses the time-inconsistency nor provided solutions in a dynamic sense.

definition of equilibrium solution

We adopt the concept of equilibrium solution, which is, for any $t \in [0, T)$, optimal "infinitesimally" via spike variation. Given a control $u^*(\cdot)$. For any $t \in [0, T)$, $\varepsilon > 0$ and v, define

$$u_s^{t,\varepsilon,v} = u_s^* + v\mathbf{1}_{s\in[t,t+\varepsilon)}.$$

Definition

$u^*(\cdot)$ is called an equilibrium if for any $t \in [0, T)$, and v,

$$\lim_{\varepsilon\downarrow 0} \frac{J(t, X_t^*, u^{t,\varepsilon,v}) - J(t, X_t^*, u^*)}{\varepsilon} \geq 0.$$

Comparison with the existing works

The intuition behind this definition is that the controller at any time t is playing a game against all his incarnations in the future. He can commit only for an infinitessimal time ε, so he can only hope to optimize in $[t, t+\epsilon)$.
Remark: We define the equilibrium control in the class of open-loop controls, whereas in the existing works only (Markovian) feedback controls are considered.

Ekeland and Lazrak, 2006, Ekeland and Pirvu, 2008.

Bjork and Murgoci, 2010,

Basak and Chabakauri, 2010.

Comparison with the exsiting works

In the existing works, only existence of equilibrium strategy is treated, including recent works of Czichowsky (2010) and Yong (2011), via discretization procedure.

In Hu, Jin and Zhou (2012), we addressed the existence of equilibrium strategy.

And recently, in Hu, Jin and Zhou (2015), we proved the uniqueness of the equilibrium strategy in the class of open-loop controls.

Main Ideas

I. Applying the perturbation method to establish necessary and sufficient conditions

II. equivalent conditions via Forward-backward stochastic differential equations (FBSDE)

III. Existence of solution via quadratic BSDEs

IV. Uniqueness via difference (BMO martingales and John-Nirenberg inequality)

Maximum principle

Define adjoint processes $(p(\cdot;t),k(\cdot;t))$ and $(P(\cdot;t),K(\cdot;t))$:

$$\begin{aligned} dp(s;t) &= -[A_s'p(s;t)+C_s'k(s;t)+Q_sX_s^*]ds \\ &\quad +k(s;t)dW_s, \\ p(T;t) &= GX_T^*-h\mathbb{E}_t[X_T^*]-\mu_1X_t^*-\mu_2; \end{aligned} \tag{4}$$

$$\begin{aligned} dP(s;t) &= -[A_s'P(s;t)+P(s;t)A+C_s'P(s;t)C_s \\ &\quad +C_s'K(s;t)+K(s;t)C_s+Q_s]ds+K(s;t)dW_s, \\ P(T;t) &= G. \end{aligned} \tag{5}$$

second order expansion

Proposition

$$
\begin{aligned}
& J(t, X_t^*, u^{t,\varepsilon,v}) - J(t, X_t^*, u^*) \\
= \; & \mathbb{E}_t\Big[\int_t^{t+\varepsilon} \{\langle B'p(s;t) + D'k(s;t) + R_s u_s^*, v\rangle \\
& + \frac{1}{2} v'(R + D'P(s;t)D)v\} ds\Big] + o(\varepsilon).
\end{aligned} \tag{6}
$$

Verification theorem

$$
\begin{cases}
dX_s^* = [A_s X_s^* + B_s^* u_s^* + b_s]ds + [C_s X_s^* + D_s^* u_s^* + \sigma_s]dW_s, \\
X_0^* = x_0, \\
dp(s;t) = -[A_s' p(s;t) + C_s' k(s;t) + Q_s X_s^*]ds \\
\qquad\qquad\quad + k(s;t)dW_s, \\
p(T;t) = GX_T^* - hE_t[X_T^*] - \mu_1 X_t^* - \mu_2, \\
B_t' p(t;t) + D_t' k(t;t) + R_t u_t^* = 0
\end{cases}
$$

Verification theorem and necessary condition

Theorem

If the above FBSDE admits a solution, then u^* is the equilibrium solution.

Theorem

If u^* is the equilibrium solution, then FBSDE admits a solution with $k(\cdot, t) = k(\cdot)$.

As in classical LQ control, we attempt to look for linear feedback equilibrium. For this, we consider the following Ansatz:

$$p(s;t) = M_s X_s^* - N_s \mathbb{E}_t[X_s^*] - \Gamma_s^{(1)} X_t^* + \Phi_s$$

where $M, N, \Gamma^{(1)}, \Phi$ are deterministic differential functions with $M' = m, N' = n, (\Gamma^{(1)})'s = \gamma^{(1)}$ and $\Phi' = \phi$.
Applying Ito's formula to $p(s;t)$ and comparing the dW_s terms:

$$k(s;t) = M_s[C_s X_s^* + D_s u_s^* + \sigma_s].$$

System of Riccati equations

$$\begin{cases} \dot{M} + MA + A'M + C'MC + Q \\ \quad -(MB + C'MD)(R + D'MD)^{-1}(B'(M - N - \Gamma^{(1)}) + D'MC) = 0, \\ M_T = G; \end{cases} \tag{7}$$

$$\begin{cases} \dot{N} + NA + A'N - NB(R + D'MD)^{-1}(B'(M - N - \Gamma^{(1)}) + D'MC) = \\ N_T = h; \end{cases}$$

linear equations for $\Gamma^{(1)}$ and Φ

$$\dot{\Gamma^{(1)}} = -A'\Gamma^{(1)}; \Gamma^{(1)}_T = \mu_1 I \tag{9}$$

$$\begin{cases} \dot{\Phi} + (A - [(M+N)B + C'MD + S'](R + D'MD)^{-1}B_s')\Phi \\ \qquad +(M-N)b + C'M\sigma \\ \quad -[(M-N)B + C'MD](R + D'MD)^{-1}D'M\sigma = 0, \\ \Phi_T = -\mu_2. \end{cases} \tag{10}$$

control

$$u_s^* = \alpha_s X_s^* + \beta_s,$$

where

$$\begin{aligned} \alpha_s &= -(R_s + D_s' M_s D_s)^{-1}(B_s'(M_s - N_s - \Gamma_s) + D_s' M_s C_s), \\ \beta_s &= -(R_s + D_s' M_s D_s)^{-1}[B_s' \Phi_s + D_s' M_s \sigma_s]. \end{aligned}$$

Equations (7) and (8) form a system of coupled Riccati equations for (M, N). Once we get the solution for (M, N), equation (10) is a simple ODE.

main result

Theorem

Let (M, N) be a solution of system of Riccati equations (7) and (8), and Φ a solution of ODE (10). Then there exists a unique solution of FBSDE given by the relation:

$$p(s;t) = M_s X_s^* - N_s \mathbb{E}_t[X_s^*] - \Gamma_s^{(1)} X_t^* + \Phi_s;$$

$$k(s;t) = M_s[C_s X_s^* + D_s u_s^* + \sigma_s];$$

and u^* is the unique equilibrium solution.

In order to study (7) and (8), we first note that this system is equivalent to the following system by setting $J = M - N$.

$$\begin{cases} \dot{M} + (2A + C^2)M + Q + \Gamma B(B + CD)(R + D^2M)^{-1}M \\ \quad -(R + D^2M)^{-1}(B + CD)M(BJ + CDM) = 0, \\ M_T = G; \\ \dot{J} + 2AJ + C^2M + Q - (R + D^2M)^{-1}(BJ + CDM)^2 \\ \quad +\Gamma(R + D^2M)^{-1}(B^2J + BCDM) = 0, \\ J_T = G - h. \end{cases} \tag{11}$$

standard case

Theorem

Let us suppose that $R > 0$, $Q \geq 0$, $G \geq 0$ and $G - h \geq 0$. Then if $BCD \geq 0$ or $D > 0$ and $D^2Q + C^2R + \Gamma BCD \geq 0$, then the system (11) admits a unique bounded solution (M, J).

Method: Truncation argument.

singular case

Let us now consider the singular case where $R = 0$. Our idea here is to introduce

$$I = \frac{J}{M},$$

and to find a strictly positive lower bound for I.

Theorem

Let us suppose that $R = 0$, $Q + \frac{\Gamma B(B+CD)}{D^2} \geq 0$ and $Q + \frac{\Gamma BC}{D} \geq 0$. Moreover we suppose that $G \geq h > 0$. Then the system of Riccati equations admits a solution.

mean-variance problem

Let us now consider the mean-variance problem:

$$J(t, x_t, u(\cdot)) = \frac{1}{2}\left[\mathbb{E}_t[X_T^2] - (\mathbb{E}_t[X_T])^2\right] - (\mu_1 x_t + \mu_2)E_t[X_T],$$

where X follows

$$\begin{cases} dX_s = r_s X_s dt + u_s' \theta_s ds + u_s' dW_s \\ X_t = x_t \end{cases} \tag{12}$$

FBSDE

Just as the equilibrium of the LQ problem, we can apply the perturbation method to get the FBSDE system

$$\begin{cases} dX^*(s) = [r(s)X^*(s) + \theta(s)u^*(s)]ds + u^*(s)dW(s) \\ dp(s;t) = -r(s)p(s;t)ds + k(s;t)dW(s) \\ p(T;t) = X^*(T) - E_t[X^*(T)] - \mu_1 X_t^* - \mu_2 \\ \theta(t)p(t;t) + k(t;t) = 0 \end{cases} \quad (13)$$

Ansatz

As before, let us look for a solution in the form:

$$p(s;t) = M_s X_s^* - \Gamma_s^1 X_t^* + \Gamma_s^2 - \mathbb{E}_t[N_s X_s + \Gamma_s^3].$$

Applying Ito's formula and comparing the dW terms:

$$k(s;t) = U_s X_s^* + M_s u_s^* + \gamma_s^2 - \gamma_s^1 X_t^*.$$

Putting the expressions of p and k into the last equation of FBSDE, we get:

$$u_s^* = \alpha_s X_s^* + \beta_s,$$

where

$$\alpha_s = -M^{-1}\left(\theta_s(M - N - \Gamma^1) + U_s - \gamma_s^1\right);$$
$$\beta_s = -M^{-1}\left(\theta(\Gamma^2 - \Gamma^3) + \gamma^2\right).$$

equations

Comparing the corresponding terms, we obtain:

$$\begin{aligned}
dM_s &= -(2rM + (\theta M + U)\alpha)ds + U_s dW_s, M_T = 1; \\
dN_s &= -(2rN + (\theta N + V)\alpha)ds + V_s dW_s, N_T = 1; \\
d\Gamma_s^1 &= -r\Gamma^1 ds, \Gamma_T^1 = \mu_1; \\
d\Gamma_s^2 &= -(r\Gamma^2 + (\theta M + U)\beta)ds + \gamma_s^2 dW_s, \Gamma_T^2 = -\mu_2; \\
d\gamma_s^2 &= -(r\Gamma^3 + (\theta N + V)\beta)ds + \gamma_s^3 dW_s, \Gamma_T^3 = 0.
\end{aligned}$$

To solve the BSDEs, we note that the first two equations are identical, we conclude that

$$M = N, \quad U = V.$$

$$\Gamma_t^1 = \mu_1 e^{\int_t^T r_s ds},$$

and

$$\Gamma_s^2 - \Gamma_s^3 = -\mu_2 e^{\int_s^T r_t dt} = \Gamma_s.$$

Then we obtain the BSDE satisfied by (M, U):

$$\begin{cases} dM_s = -(2rM - \theta U + \theta^2 \Gamma^1 - M^{-1} U_s^2 + \theta \Gamma^1 M^{-1} U) ds + U_s dW_s, \\ M_T = 1. \end{cases}$$

quadratic BSDE: Recall

Linear BSDE: Bismut (1970-1980)
Lipschitz BSDE: Pardoux-Peng (1990)
Quadratic BSDE (or KPZ BSDE): Kobylansky (2000), Briand and Hu (2006,2008), Delbaen, Hu and Richou (2011,2015); Barrieu and El Karoui (2013); Cetin and Danilova (2014); Kramkov and Pulido (2014), Hu and Tang (2014), Kardaras, Xing and Zitkovic (2015)

Proposition

BSDE (14) admits a unique bounded solution. Moreover, there exists a constant $c > 0$ such that $M \geq c$ and the martingale $U \cdot W$ is a BMO martingale.

BSDE Γ^1

Then we consider the BSDE satisfied by (Γ^2, γ^2):

$$\begin{cases} d\Gamma_s^2 = -[r\Gamma^2 - (\theta + M^{-1}U)\gamma^2 - (\theta^2 + \theta M^{-1}U)\Gamma_s]ds + \gamma^2 dW_s, \\ \Gamma_T^2 = -\mu_2, \end{cases} \tag{15}$$

where

$$\Gamma_s = -\mu_2 e^{-\int_s^T r_t dt}.$$

Proposition

BSDE (15) admits a unique bounded solution.

main result

Theorem

$$u_s = -M^{-1}\left[(U_s - \theta\mu_1 e^{\int_s^T r_t dt})X_s^* + \theta\Gamma + \gamma^2\right],$$

is the unique equilibrium strategy.

Proof of Uniqueness 1

Define $\bar{p}(s;t) = p(s;t) - [M_s X_s + \Gamma_s^{(2)} - \mathbb{E}_t M_s X_s + \Gamma_s^{(3)} - \Gamma_s^{(1)} X_t]$
and $\bar{k}(s) = k(s) - [M_s u_s + U_s X_s + \gamma_s^{(2)}]$.
Furthermore,

$$\begin{aligned} 0 &= \theta\bar{p}(s;s) + \bar{k}(s) + \theta[\Gamma_s^{(2)} - \Gamma_s^{(3)} - \Gamma_s^{(1)} X_s] + [M_s u_s + U_s X_s + \gamma_s^{(2)}], \\ u_s &= -M_s^{-1}[(U_s - \theta\Gamma_s^{(1)})X_s + \theta\bar{p}(s;s) + \bar{k}(s) + \theta(\Gamma_s^{(2)} - \Gamma_s^{(3)}) + \gamma_s^{(2)}] \\ &= \alpha_s X_s + \beta_s - M_s^{-1}[\theta\bar{p}(s;s) + \bar{k}(s)], \end{aligned}$$

Proof of Uniqueness 2

Now we plug u_s into the calculation of $d\bar{p}(s;t)$,

$$\begin{cases} d\bar{p}(s;t) = & -\left\{ r_s\bar{p}(s;t) - (\theta_s + U_s M_s^{-1})'[\theta_s\bar{p}(s;s) + \bar{k}(s)] \right. \\ & \left. +\mathbb{E}_t(\theta_s + U_s M_s^{-1})'[\theta_s\bar{p}(s;s) + \bar{k}(s)] \right\} ds \\ & +\bar{k}(s)' dW_s, \\ \bar{p}(T;t) = & 0. \end{cases}$$

Theorem

For any $q \in (1,2)$, the FBSDE admits at most one solution $(\bar{p}, \bar{k})$.

Comparison

Basak and Chabakauri (2010) and Bjork and Murgoci (2010): $\mu_1 = 0$, deterministic coefficients: Existence of Equilibrium Solution. Our result coincides with the one obtained in these papers, althought the definitions are different.
Bjork, Murgoci and Zhou (2012) $\mu_2 = 0$, deterministic coefficients: Existence of Equilibrium Solution The results are different.
Czichowski (2010): $\mu_1 = 0$, Existence of Equilibrium solution in stochastic risk premium by discretization method.

Consider an investor in the financial market with initial wealth x_0 at time $t = 0$, who pursuits high rank-dependent utility. At time t, the rank-dependent utility also depends on the current total wealth. Denoted $X_t = x$, then for any admissible portfolio with terminal wealth X_T, the rank-dependent utility is described in the form

$$\begin{aligned} J(X_T; t, x) &= \int_0^{+\infty} w(t, \mathbb{P}_t(u(X_T) > y))dy \\ &\quad + \int_{-\infty}^{0} (w(t, \mathbb{P}_t(u(X_T) > y)) - 1)dy, \end{aligned}$$

where $w(t, \cdot)$ are probability distortions used at time t, $u(\cdot)$ is the utility/value function, and $\mathbb{P}_t$ means the conditional probability given $\mathcal{F}_t$, which includes the information $X_t = x$.

ODE

For any $t \in [0, T]$, we define a real function by

$$h(x;t) \triangleq \mathbb{E}[w_t' \circ N(\xi)e^{x\xi}],$$

where ξ is a standard Gaussian random variable. In our search for an equilibrium, we take the following ordinary differential equation as the key:

$$\begin{cases} \dot{\Lambda}_t = -\theta_t^2 \left(\frac{h(\sqrt{\Lambda_t};t)}{h'(\sqrt{\Lambda_t};t)} \right)^2 \Lambda_t \\ \Lambda_T = 0. \end{cases} \tag{16}$$

Main Result

Theorem

Suppose equation (16) admits a solution Λ with $\Lambda(t) > 0$ for any $t \in [0, T)$. Denote $\lambda_t := \sqrt{-\Lambda'(t)/|\theta(t)|^2}$. If for any $c \in \mathbb{R}$,

$$0 \leq \int_{-\infty}^{+\infty} w_t' \circ N\left(\frac{c - g(x)}{\sqrt{\Lambda_t}}\right) N'\left(\frac{c - g(x)}{\sqrt{\Lambda_t}}\right)\left(g''(x) + g'(x)^2 \frac{c - g(x)}{\Lambda_t}\right) du \tag{17}$$

then the replicating portfolio for the terminal wealth

$$X_T = I(e^{-\int_0^T \lambda_s \theta_s^\top dW_s})$$

is an equilibrium.

Concluding Remarks

We attempt to formulate and find equilibrium with random parameters. The advantage of our method is that we provide the unique equilibrium solution; whereas in the existing works only the existence of equilibrium solution is addressed up to our best knowledge.

THANK YOU FOR YOUR ATTENTION