Mean-field games with controls submitted to the mean-field interaction

LSE - Workshop 2015 August 24th-28th

François Delarue (Nice – J.-A. Dieudonné)

August 25th, 2015

Basic purpose

- Revisit MFG in case when
  - controls are submitted to the mean-field interaction
- Standard framework $\rightsquigarrow$ interacting particles/players
  - controlled
  - with $\underbrace{\text{mean-field}}_{}$ interaction on the states

    $\Updownarrow$

    interaction of symmetric type
    interaction with the whole population
    no privileged interaction with some particles

  - associate cost functional with each player
  - find equilibria w.r.t. cost functionals
  - shape of the equilibria for a large population?
  - Lasry-Lions, Huang-Caines-Malhamé (2006) $\cdots$
- Here $\rightsquigarrow$ mean-field interaction on the states & controls

1. General set-up

Controlled dynamics

- N interacting players (state in $\mathbb{R}^d$)
  - controlled players with mean-field interaction
  - dynamics of player number $i \in \{1, \dots, N\}$

$$dX_t^i = b(X_t^i, \bar{\nu}_t^N, \alpha_t^i)dt + dW_t^i, \quad X_0^i = x_0, \; t \in [0, T]$$

  - $\perp\!\!\!\perp$ noises $W^1, \dots, W^N$,

$$\bar{\nu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, \alpha_t^j)}, \quad \bar{\mu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}$$

  - choose $\underbrace{\alpha_t^i}_{\text{in convex } A \subset \mathbb{R}^k \text{ at any } t}$ prog. meas. w.r.t. $\sigma(W^1, \dots, W^N)$

Controlled dynamics

- N interacting players (state in $\mathbb{R}^d$)
  - controlled players with mean-field interaction
  - dynamics of player number $i \in \{1, \dots, N\}$

$$dX_t^i = b(X_t^i, \bar{\nu}_t^N, \alpha_t^i)dt + dW_t^i, \quad X_0^i = x_0, \; t \in [0, T]$$

  - $\perp\!\!\!\perp$ noises $W^1, \dots, W^N$, $\bar{\nu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, \alpha_t^j)}$ $\quad \bar{\mu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}$
  - choose $\underbrace{\alpha_t^i}_{\text{in convex } A \subset \mathbb{R}^k}$ prog. meas. w.r.t. $\sigma(W^1, \dots, W^N)$ at any t
- Willing to minimize cost/energy $J^i(\alpha^1, \dots, \alpha^N)$

$$J^i(\dots) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\nu}_t^N, \alpha_t^i)dt\Big]$$

Controlled dynamics

- N interacting players (state in $\mathbb{R}^d$)
  - controlled players with mean-field interaction
  - dynamics of player number $i \in \{1, \ldots, N\}$

$$dX_t^i = b(X_t^i, \bar{\nu}_t^N, \alpha_t^i)dt + dW_t^i, \quad X_0^i = x_0, \; t \in [0, T]$$

  - $\perp\!\!\!\perp$ noises $W^1, \ldots, W^N$, $\bar{\nu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, \alpha_t^j)}$ $\quad \bar{\mu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}$
  - choose α_t^i prog. meas. w.r.t. $\sigma(W^1, \ldots, W^N)$
- Willing to minimize cost/energy $J^i(\alpha^1, \ldots, \alpha^N)$

$$J^i(\ldots) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\nu}_t^N, \alpha_t^i)dt\Big]$$

- $(\alpha^{1,\star}, \ldots, \alpha^{N,\star})$ Nash equilibrium if

$$J^i(\ldots, \alpha^{i-1,\star}, \alpha^i, \alpha^{i+1,\star}, \ldots) \geq J^i(\ldots, \alpha^{i-1,\star}, \alpha^{i,\star}, \alpha^{i+1,\star}, \ldots)$$

Choice of functionals

- Linear dependence upon the control $A = \mathbb{R}^d$

$$dX_t^i = \Big(b(X_t^i, \bar{\nu}_t^N) + \alpha_t^i\Big)dt + dW_t^i$$

- Quadratic cost/energy functionals $J^i(\alpha^1, \dots, \alpha^N)$

$$J^i(\dots) = \mathbb{E}\Big[g(X_T^i, \bar{\nu}_T^N) + \int_0^T \Big(f(X_t^i, \bar{\nu}_t^N) + \frac{1}{2}|\alpha_t^i|^2\Big)dt\Big]$$

- Express the coefficients as

$$b : \mathbb{R}^d \times \underbrace{\mathcal{P}_2(\mathbb{R}^d \times A)}_{\text{square integrable prob. measures on } \mathbb{R}^d \times A} \to \mathbb{R}^d,$$

  - ex 1: $b(x, \nu) = b\Big(x, \int_{\mathbb{R}^d \times A} \varphi d\nu\Big)$ $\quad \begin{array}{l} \varphi = \mathrm{Id} \rightsquigarrow \text{mean} \\ \varphi \approx \delta_{(x,\alpha)} \rightsquigarrow \text{mass at } (x, \alpha) \end{array}$
  - ex 2: $b(x, \nu) = \int_{\mathbb{R}^d \times A} b(x - u, \beta) d\nu(u, \beta) \qquad b(x - u, \beta) = |u| + |\beta| \dots$

Model of exhaustible resources

(Chan-Sircar)

- N producers of oil $\rightsquigarrow X_t^i$ (estimated reserve) at time t

$$dX_t^i = -\alpha_t^i dt + \sigma X_t^i dW_t^i$$

  - ◦ $\alpha_t^i \rightsquigarrow$ instantaneous production rate
  - ◦ σ common volatility for the perception of the reserve
  - ◦ should be a constraint $X_t^i \geq 0$

- Optimize the profit of a producer

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E} \int_0^T (\alpha_t^i P_t - c(\alpha_t^i))dt$$

  - ◦ P_t is selling price
  - ◦ mean-field constraint $\rightsquigarrow$ selling price is a function of the mean-state of the reserves and of the mean-state of the production

$$P_t = P(\frac{1}{N}\sum_{i=1}^{N} X_t^i, \frac{1}{N}\sum_{i=1}^{N} \alpha_t^i)$$

2. Deriving the asymptotic formulation

Asymptotic formulation

• Guess $\rightsquigarrow$ at equilibrium (generalization of the expression of the optimal control in terms of the optimal state)

$$\alpha_t^{i,\star} \approx \alpha(t, X_t^{i,\star}, \bar{\mu}_t^{N,\star})$$

∘ particle system at equilibrium

$$dX_t^{i,\star} \approx \Big(\underbrace{b(X_t^{i,\star}, (id, \alpha(t,\cdot,\bar{\mu}_t^{N,\star}))\sharp\bar{\mu}_t^{N,\star})}_{B(X_t^{i,\star},\bar{\mu}_t^{N,\star})} + \alpha(t, X_t^{i,\star}, \bar{\mu}_t^{N,\star})\Big)dt + dW_t^i$$

∘ particles should decorrelate as $N \nearrow \infty$

∘ $\bar{\mu}_t^N$ and $\mathcal{L}(X_t^{i,\star})$ should tend to the same deterministic limit μ_t

∘ $\bar{\nu}_t^N$ and $\mathcal{L}(X_t^{i,\star}, \alpha_t^{i,\star})$ should tend to $\nu_t = (id, \alpha(t,\cdot,\mu_t))\sharp\mu_t$

• What about an intrinsic interpretation of μ_t?

∘ should describe the global state of the population in equilibrium

∘ asymptotically, expect that any unilateral modification of the control has no influence $\rightsquigarrow$ optimize individually under

$\bar{\nu}_t^N \approx \nu_t = (id, \alpha(t,\cdot,\mu_t))\sharp\mu_t$ law of the (state/control) at equilibrium

Matching problem of MFG

- Revisiting mean-field games theory

Lasry-Lions (2006), Huang-Caines-Malhamé (2006) …

- Define the asymptotic equilibrium state & control of the population as the solution of a fixed point problem

(1) fix a flow of probability measures $(\nu_t)_{0\leq t\leq T}$ (in $\mathcal{P}_2(\mathbb{R}^d \times A)$)

(2) solve the stochastic optimal control problem in the environment $(\nu_t)_{0\leq t\leq T}$

$$dX_t = \big(b(X_t, \nu_t) + \alpha_t\big)dt + dW_t$$

- ∘ with $X_0 \sim m_0 = \mu_0$ (marginal of ν_0 on $\mathbb{R}^d$)
- ∘ with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \nu_T) + \int_0^T \Big(f(X_t, \nu_t) + \frac{1}{2}|\alpha_t|^2\Big)dt\Big]$

(3) let $(X_t^{\star,\nu}, \alpha_t^{\star,\nu})_{0\leq t\leq T}$ be the unique optimal path/optimizer (under nice assumptions) $\rightsquigarrow$ find $(\nu_t)_{0\leq t\leq T}$ such that

$$\nu_t = \mathcal{L}(X_t^{\star,\nu}, \alpha_t^{\star,\nu}), \quad t \in [0, T]$$

- Not a proof of the convergence!

PDE characterization

- Optimize in environment $(\nu_t)_{t\in[0,T]} \rightsquigarrow$ HJB equation for $u(t,x) =$ minimal cost when $X_t = x$

$$\partial_t u + \tfrac{1}{2}\Delta u + b(x,\nu_t)\cdot D_x u + f(x,\nu_t) \underbrace{-\tfrac{1}{2}|D_x u|^2}_{\inf_{z\in\mathbb{R}^d} z\cdot D_x u + \frac{1}{2}|z|^2} = 0 \quad t\in[0,T]$$

$$u(T,x) = g(x,\mu_T),$$

- Optimal feedback function is $-D_x u$
  - optimal control is $(-D_x u(t,X_t))_{0\le t\le T}$ where optimal path is

$$dX_t = \Big(b(X_t,\nu_t) - D_x u(t,X_t)\Big)dt + dW_t$$

  - dynamics of $(\mu_t = \mathcal{L}(X_t))_{0\le t\le T}$

$$\partial_t \mu_t - \tfrac{1}{2}\Delta\mu_t + \mathrm{div}\Big(\mu_t(b(x,\nu_t) - D_x u(t,x))\Big) = 0 \quad t\in[0,T],\ \mu_0 = m_0$$

with

$$\nu_t = \mathcal{L}(X_t, -D_x u(t,X_t)) = (id, -D_x u(t,\cdot))\sharp\mu_t$$

PDE forward-backward system

- Enough to characterize the law of the state $(\mu_t)_{t\in[0,T]}$!
- Forward equation
  - Fokker-Planck equation

$$
\begin{aligned}
&\partial_t \mu_t - \tfrac{1}{2}\Delta \mu_t \\
&\qquad + \operatorname{div}\Big[\mu_t\Big(b\Big(x, (id, -D_x u(t,\cdot))\sharp\mu_t\Big) - D_x u(t,x)\Big)\Big] = 0, \quad t \in [0,T] \\
&\mu_0 = m_0
\end{aligned}
$$

- Backward equation
  - HJB equation

$$
\begin{aligned}
&\partial_t u + \tfrac{1}{2}\Delta u + b\Big(x, (id, -D_x u(t,\cdot))\sharp\mu_t\Big)\cdot D_x u \\
&\qquad + f\Big(x, (id, -D_x u(t,\cdot))\sharp\mu_t\Big) - \tfrac{1}{2}|D_x u|^2 = 0, \quad t \in [0,T] \\
&u(T,x) = g(x, \mu_T),
\end{aligned}
$$

FBSDE formulation

- Standard representation of the value function by means of FBSDE

$$Z_t = D_x u(t, X_t), \quad t \in [0, T]$$

  - FBSDE of the McKean-Vlasov type

$$\begin{aligned}
dX_t &= \left[b(X_t, \mathcal{L}(X_t, -Z_t)) - Z_t\right]dt + dW_t, \\
dY_t &= -\left(f(X_t, \mathcal{L}(X_t, -Z_t)) + \tfrac{1}{2}|Z_t|^2\right)dt + Z_t dW_t, \quad t \in [0, T], \\
Y_T &= g(X_T, \mathcal{L}(X_T))
\end{aligned}$$

- Pontryagin principle

$$Y_t = D_x u(t, X_t), \quad t \in [0, T]$$

  - FBSDE of the McKean-Vlasov type

$$\begin{aligned}
dX_t &= \left[b(X_t, \mathcal{L}(X_t, -Y_t)) - Y_t\right]dt + dW_t, \\
dY_t &= -\partial_x f(X_t, \mathcal{L}(X_t, -Y_t))dt + Z_t dW_t, \quad t \in [0, T], \\
Y_T &= \partial_x g(X_T, \mathcal{L}(X_T))
\end{aligned}$$

3. Existence of a solution

How to attack existence?

• Preliminary remark: as in standard MFG theory, no hope for solving by Picard fixed theorem

∘ at least under classical Lipschitz assumptions only expect small time

• Use Schauder's fixed point theorem

∘ see statement in the next slide

∘ need a structure with a compactness

• Question: what is the unknown in the fixed point?

∘ standard MFG $\rightsquigarrow$ plug the marginal laws of the state as unknown

∘ here $\rightsquigarrow$ plug the marginal laws of the state as unknown plus the feedback function!

∘ prove $\exists$ if bounded and Lipschitz coefficients

Mapping for the fixed point

- Input
  - flow of measures $(\mu_t)_{0\le t\le T}$ on $\mathbb{R}^d$ (marginal laws of the state)
  - Borel function $\varphi : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ (feedback function)
- Let

$$\nu_t = (id, \varphi(t,\cdot))\sharp\mu_t, \quad t\in[0,T]$$

- Solve optimal stochastic control problem

$$dX_t = \big(b(X_t,\nu_t) + \alpha_t\big)dt + dW_t$$

  - with $X_0 \sim \mu_0$ (marginal of ν_0 on $\mathbb{R}^d$)
  - with cost $J(\alpha) = \mathbb{E}\Big[g(X_T,\mu_T) + \int_0^T \Big(f(X_t,\nu_t) + \frac{1}{2}|\alpha_t|^2\Big)dt\Big]$
- Output (if ! solution to the optimization problem)
  - new $\mu_t = \text{law}(X_t)$
  - new φ = optimal feedback

Ψ : input $\mapsto$ output

Use of the Schauder fixed point theorem

- Define $\Psi : E \to E$

  ◦ $E \subset \underbrace{C([0,T],\mathcal{P}_p(\mathbb{R}^d))}_{\text{law of the state}} \times \underbrace{C([0,T)\times\mathbb{R}^d,\mathbb{R}^d)}_{\text{feedback}}$

  ◦ $\mathcal{P}_p(\mathbb{R}^d)$ = set of probability measures on $\mathbb{R}^d$ with finite p-moment equipped with Wasserstein distance $p = 2$

$$W_p(\nu,\nu') = \inf \mathbb{E}\Big[|(X,\alpha) - (X',\alpha')|^p\Big]^{1/p} \qquad \begin{array}{l}(X,\alpha) \sim \nu \\ (X',\alpha') \sim \nu'\end{array}$$

  ◦ $C([0,T],\mathcal{P}_p(\mathbb{R}^d))$ equipped with uniform convergence

  ◦ $C([0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ equipped with uniform convergence on compact subsets

- Choose E in such a way that

  ◦ $\Psi : E \to E$ continuous

  ◦ $\Psi(E)$ is relatively compact

  $\Rightarrow \exists$ fixed point

Compactness I

- Start with optimal feedback = gradient of HJB

$$\partial_t u + \tfrac{1}{2}\Delta u + b\Big(x, (id, \varphi(t,\cdot))\sharp\mu_t\Big) \cdot D_x u$$
$$+ f\Big(x, (id, \varphi(t,\cdot))\sharp\mu_t\Big) - \tfrac{1}{2}|D_x u|^2 = 0, \quad t \in [0,T]$$
$$u(T,x) = g(x, \mu_T),$$

- Same assumption as above $\rightsquigarrow$ $b(x,\nu)$, $f(x,\nu)$ and $g(x,\mu)$ bounded and $g(x,\mu)$ Lipschitz in x
  - estimates for uniformly elliptic HJB equations

$$\sup_{t\in[0,T], x\in\mathbb{R}^d} |D_x u(t,x)| \leq C$$
$$|D_x u(t,x) - D_x u(s,y)| \leq C_\varepsilon(|t-s|^{1/2} + |x-y|), \quad s,t \leq T - \varepsilon$$

  - $D_x u$ in compact subset of $C([0,T) \times \mathbb{R}^d, \mathbb{R}^d)$ equipped with uniform topology on compact subsets (choose the input in the same class)

Compactness II

- What about the flow of measures of diffusion process?

$$dX_t = [b(X_t, \nu_t) + \alpha_t]dt + dW_t$$

- Under the above assumption
  - $b(x, \nu)$ bounded
  - analysis of HJB $\Rightarrow \varphi$ assume the input φ is r bounded by universal bound
- Compactness criterion
  - for any input

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t|^p] \leq C_p, \qquad \sup_{t,s\in[0,T]} \mathbb{E}[|X_t - X_s|^p] \leq C_p|t-s|^{p/2}$$

  - $(\mathcal{L}(X_t))_{0\leq t\leq T} \in$ compact set of $C([0,T], \underbrace{\mathcal{P}_p(\mathbb{R}^d)}_{\text{probabilities with finite } p\text{-moments}})$

Continuity of the mapping

- Assume coefficients are Lipschitz continuous in space and measure arguments $\rightsquigarrow$ prove that Ψ is continuous

  ∘ sequence of inputs of type flow of marginal laws $(\mu_t^n)_{0\le t\le T} \to (\mu_t)_{0\le t\le T}$ converging in Wasserstein unif in t

  ∘ seq. of inputs of type feedback $(\varphi_t^n)_{0\le t<T, x\in\mathbb{R}^d} \to (\varphi_t)_{0\le t<T, x\in\mathbb{R}^d}$ uniformly bounded and converging locally uniformly

- Deduce $\displaystyle\int_0^T \Big[W_2\big(\underbrace{(id,\varphi_t^n)\sharp\mu_t^n}_{\nu_t^n}, \underbrace{(id,\varphi_t)\sharp\mu_t}_{\nu_t}\big)\Big]^2 dt \to 0$

- Stability results for FBSDEs/HJB equations

$$
\begin{aligned}
dX_t^n &= \Big[b(X_t^n,\nu_t^n) - Z_t^n\Big]dt + dW_t \\
dY_t^n &= -\Big[f(X_t^n,\nu_t^n) + \tfrac{1}{2}|Z_t^n|^2\Big]dt + Z_t^n dW_t, \qquad Y_T^n = g(X_T^n,\mu_T^n)
\end{aligned}
$$

  ∘ convergence of $(X_t^n)_{0\le t\le T}$ in $\mathbb{E}[\sup_{0\le t\le T} |\cdot_t|^2]$ norm

  ∘ convergence of $(Z_t^n = D_x u^n(t,X_t^n))_{0\le t\le T}$ in $\mathbb{E}[\int_0^T |\cdot_t|^2 dt]$ norm + compactness $\Rightarrow$ local uniform convergence of $D_x u^n$

4. Uniqueness

Lasry Lions monotonicity condition

- Recall the criterion for the standard case
  - case when coefficients only depend upon μ_t, marginal law of X_t
- Lasry Lions monotonicity condition
  - b, σ do not depend on μ
  - $f(x,\mu,\alpha) = f_0(x,\mu) + f_1(x,\alpha)$ (μ and α are separated)
  - monotonicity property for f_0 and g w.r.t. μ

$$\int_{\mathbb{R}^d} (f_0(x,\mu) - f_0(x,\mu'))d(\mu-\mu')(x) \geq 0$$
$$\int_{\mathbb{R}^d} (g(x,\mu) - g(x,\mu'))d(\mu-\mu')(x) \geq 0$$

- Example

$$h(x,\mu) = \int_{\mathbb{R}^d} f(z, \rho \star \mu(z))\rho(x-z)dz$$

$L \uparrow$ in 2^{nd} variable and ρ even, smooth with compact support

Adapting the Lasry Lions condition

- Generalize the modeling $\rightsquigarrow$ change the cost functional into

$$f(x,\alpha,\nu) \quad \text{instead of} \quad f(x,\nu) + \frac{1}{2}|\alpha|^2$$

  - f strictly convex in α
  - need to change the shape of the optimal feedback function

$$-D_x u \rightsquigarrow \hat{\alpha}(x, D_x u, \nu),$$

where $\hat{\alpha}(x,z,\nu) = \inf_{\alpha \in A}[\alpha \cdot z + f(x,\alpha,\nu)]$, A convex subset of $\mathbb{R}^k$

- Monotonicity is to require the same except for f

$$\int_{\mathbb{R}^d \times A} \Big[f(x,\alpha,\nu) - f(x,\alpha,\nu') \Big] d(\nu - \nu')(x,\alpha) \geq 0$$

- Example

$$f(x,\alpha,\nu) = f_0(x,\nu_x) + f_0'(\nu_\alpha) + \frac{1}{2}|\alpha|^2$$

  - $\nu_x = \mu$ 1st marginal of ν ($\mathcal{L}(X)$), ν_α = 2nd marginal of ν ($\mathcal{L}(\alpha)$)

Monotonicity restores uniqueness

- Assume that for any input $\boldsymbol{\nu} = (\nu_t)_{0\le t\le T}$ unique optimal control $\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}}$

+ existence of an MFG for a given initial condition

- Lasry Lions $\Rightarrow$ if two different $\rightsquigarrow$ $\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}} \neq \boldsymbol{\alpha}^{\star,\boldsymbol{\nu}'}$

$$\underbrace{J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}})}_{\text{cost under } \boldsymbol{\nu}} < J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}'}) \quad \text{and} \quad \underbrace{J^{\boldsymbol{\nu}'}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}'})}_{\text{cost under } \boldsymbol{\nu}'} < J^{\boldsymbol{\nu}'}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}})$$

so that

$$J^{\boldsymbol{\nu}'}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}}) - J^{\boldsymbol{\nu}'}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}'}) + J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}'}) - J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}}) > 0$$
$$J^{\boldsymbol{\nu}'}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}}) - J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}}) - [J^{\boldsymbol{\nu}'}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}'}) - J^{\boldsymbol{\nu}}(\boldsymbol{\alpha}^{\star,\boldsymbol{\nu}'})] > 0$$

$$\int_0^T \underbrace{\mathbb{E}\Big[f(X_t^{\star,\boldsymbol{\nu}}, \alpha_t^{\star,\boldsymbol{\nu}}, \nu_t') - f(X_t^{\star,\boldsymbol{\nu}}, \alpha_t^{\star,\boldsymbol{\nu}}, \nu_t)\Big]}_{\int_{\mathbb{R}^d\times A}\big[f(x,\alpha,\nu_t') - f(x,\alpha,\nu_t)\big]d\nu_t(x,\alpha)} dt$$

$$- \int_0^T \underbrace{\mathbb{E}\Big[f(X_t^{\star,\boldsymbol{\nu}'}, \alpha_t^{\star,\boldsymbol{\nu}'}, \nu_t') - f(X_t^{\star,\boldsymbol{\nu}'}, \alpha_t^{\star,\boldsymbol{\nu}'}, \nu_t)\Big]}_{\int_{\mathbb{R}^d\times A}\big[f(x,\alpha,\nu_t') - f(x,\alpha,\nu_t)\big]d\nu_t'(x,\alpha)} + \cdots > 0$$

  - same for g $\Rightarrow$ LHS must be ≤ 0

Example for Lasry Lions

- 1^{st} example for $f(x, \alpha, \nu)$ satisfying both Lasry-Lions and convexity
  - if $f(x, \alpha, \nu) = f_0(x, \nu_x) + f_1(\alpha, \nu_\alpha)$
  - $A = \mathbb{R}^k$
  - f_0 satisfies standard Lasry Lions
  - $f_1(\alpha, \nu_\alpha) = f_2(\nu_\alpha) + \frac{1}{2}|\alpha - \bar{\nu}_\alpha|^2, \quad \bar{\nu}_\alpha = \int_{\mathbb{R}^k} y d\nu_\alpha(y)$
- 2^{nd} example for $f(x, \alpha, \nu)$ satisfying both Lasry-Lions and convexity

$$f(x, \alpha, \nu) = \int_{\mathbb{R}^d \times A} L(\underbrace{z}_{\zeta_1}, \underbrace{\beta}_{\zeta_2}, \underbrace{\rho \star \nu(z, \beta)}_{\zeta_3})\rho(x - z, \alpha - \beta) dx d\beta$$

  - ρ even, smooth, with compact support
  - L increasing in ζ_3, convex in (ζ_2, ζ_3) with $\partial^2_{\zeta_2\zeta_3} L$ small enough

5. Existence and uniqueness

Motivation

- Find a framework for explaining the passage from N to ∞ players
  - need robustness
  - in particular need existence and uniqueness $\rightsquigarrow$ combine Schauder/compactness and Lasry-Lions
- Recall N-player

$$dX_t^i = b(X_t^i, \bar{\nu}_t^N, \alpha_t^i)dt + dW_t^i, \quad X_0^i = x_0, \ t \in [0, T]$$

  - $(\alpha_t^{1,\star}, \dots, \alpha_t^{N,\star})_{0 \le t \le T}$ Nash equilibrium
- What should be the right object for explaining the approximation

$$\alpha_t^{i,\star} \approx \alpha(t, X_t^{i,\star}, \bar{\mu}_t^{N,\star})$$

- What should be α?

Master field of a MKV FBSDE

- Recall the interpretation of MFG in terms of FBSDE
  - use Pontryagin principle $\rightsquigarrow$ FBSDE of the McKean-Vlasov type

$$
\begin{aligned}
dX_t &= \Big[b(X_t, \mathcal{L}(X_t, -Y_t)) - Y_t\Big]dt + dW_t, \\
dY_t &= -\partial_x f(X_t, \mathcal{L}(X_t, -Y_t))dt + Z_t dW_t, \quad t \in [0, T], \\
Y_T &= \partial_x g(X_T, \mathcal{L}(X_T))
\end{aligned}
$$

- Decoupling field of the FBSDE
  - if no MKV interaction $\rightsquigarrow Y_t = U(t, X_t)$ for some
$U : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ (reminiscent of the Markov property on $\mathbb{R}^d$)
  - if MKV interaction $\rightsquigarrow Y_t = U(t, X_t, \mathcal{L}(X_t))$ for some
$U : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$
- Opt. feedback at equilibrium $(\nu_t)_{0 \le t \le T} \rightsquigarrow -D_x u(t, x) = -U(t, x, \mu_t)$
  - guess is $\alpha_t^{i,\star} \approx -U(t, X_t^{i,\star}, \bar{\mu}_t^{N,\star})$
  - full proof by verification argument: expand $U(t, X_t^{i,\star}, \bar{\mu}_t^{N,\star})$ using
PDE satisfied by U